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INEQUALITIES FOR SOLUTIONS OF MULTIPOINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. The concept of concavity is generalized to functions, y, satisfying nth order differential inequalities, $y^{(n)}(t) \geq 0$, $0 \leq t \leq 1$, and homogeneous multipoint boundary conditions, $y^{(j)}(a_i) = 0$, $j = 0, \dots, n_i$, $i = 1, \dots, k$, where $0 = a_1 < a_2 < \cdots < a_k = 1$ and $\sum_{i=1}^k n_i = n$. A piecewise polynomial, which bounds the function, y, below, is constructed and then is employed to obtain that if $(3a_i + a_{i+1})/4 \leq t \leq (a_i + 3a_{i+1})/4$, then $(-1)^{\alpha_i} y(t) \geq \|y\|(a/4)^m$, $i = 1, \dots, k-1$, where $a = \min_i(a_{i+1} - a_i)$, $\|\cdot\|$ denotes the supremum norm, $m = \max\{n - n_1, n - n_k\}$, and $\alpha_i = \sum_{j=i+1}^k n_j$, $i = 1, \dots, k-1$. An analogous inequality for a related Green's function is also obtained. These inequalities are useful in applications of certain cone theoretic fixed point theorems.

In recent applications of cone theoretic fixed point theorems to boundary value problems (BVPs), inequalities that provide lower bounds for positive functions as a function of the supremum norm have been applied. This type of inequality has been useful in applications to both regular two-point BVPs [**8, 5**] on annular like regions, and singular two-point BVPs [**9, 4**]. This type of inequality is also useful to obtain nonexistence results [**3**]. The particular inequality to which we refer is as follows: if $y''(t) \leq 0$, $0 \leq t \leq 1$ and $y(t) \geq 0$, $0 \leq t \leq 1$, then for $1/4 \leq t \leq 3/4$,

$$y(t) \geq \|y\|/4,$$

where $\|y\| = \sup_{0 \leq t \leq 1} |y(t)|$. An analogous inequality for a Green's function has been employed for regular two-point BVPs [**8**]. Recently, Eloe and Henderson [**6**] showed that if $n \geq 2$ is an integer, $k \in \{1, \dots, n-1\}$, and if

$$(-1)^{n-k} y^{(n)} \geq 0, \quad 0 \leq t \leq 1,$$

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$$
\begin{aligned}
&y^{(j)}(0) = 0, \quad j = 0, \dots, k-1, \\
&y^{(j)}(1) = 0, \quad j = 0, \dots, n-k-1,
\end{aligned} \tag{1}
$$

then, for $1/4 \le t \le 3/4$,

$$
y(t) \ge \|y\|/4^m, \tag{2}
$$

where $m = \max\{k, n-k\}$. We shall refer to the boundary conditions (1) as two-point conjugate boundary conditions [**2**]. The purpose of this paper is to obtain the analogue of (2) for solutions of differential inequalities satisfying multipoint conjugate type boundary conditions.

In particular, we shall study the following problem. Let $n \ge 2$ be an integer, and let $k \in \{2, \dots, n\}$. Let $0 = a_1 < a_2 < \dots < a_k = 1$ be k points, and let $n_i \in \{1, \dots, n-1\}$, $i = 1, \dots, k$, be such that $\sum_{i=1}^k n_i = n$. We shall obtain an analogue of the inequality (2) for solutions, y, for the multipoint conjugate [**2**] boundary value problem (BVP) for the differential inequality

$$
y^{(n)}(t) \ge 0, \quad 0 < t < 1, \tag{3}
$$

$$
y^{(j)}(a_i) = 0, \quad j = 0, \dots, n_i - 1,\ i = 1, \dots, k. \tag{4}
$$

Inequality (2) can be obtained as follows. Assume for simplicity that

$$
(-1)^{n-k} y^{(n)}(t) > 0, \quad 0 < t < 1,
$$

and let $\|y\| = y(t_1)$ for some $t_1 \in (0,1)$. Define the piecewise polynomial, p, by

$$
p(t) = \begin{cases} (\|y\| t^k)/t_1^k, & 0 < t < t_1, \\ (\|y\|(t-1)^{n-k})/(t_1-1)^{n-k} & t_1 < t < 1. \end{cases}
$$

It is then shown [**6**] that $y(t) \ge p(t)$, $0 \le t \le 1$. the righthand side of (2) is obtained by evaluating $\min\{p(1/4), p(3/4)\}$. The argument to obtain the analogue of (2) for solutions of the BVP (3), (4) will be completely analogous.

For notational purposes, set $\alpha_i = \sum_{j=i+1}^{k} n_j$, $i = 1, \ldots, k-1$. It is well-known [**2**] that

$$(5) \qquad \begin{cases} (-1)^{\alpha_i} G(t,s) > 0, & (t,s) \in (a_i, a_{i+1}) \times (0,1), \\ & i = 1, \ldots, k-1, \\ (-1)^{\alpha_i} (\partial^{n_i}/\partial t^{n_i}) G(a_i, s) > 0, & 0 < s < 1, \end{cases}$$

$i = 1, \ldots, k$, where we mean $\alpha_k = 0$, and $G(t,s)$ denotes the Green's function of the BVP, $y^{(n)}(t) = 0$, $0 \le t \le 1$, satisfying (4).

In this paper we shall obtain the following analogue of (2). We shall show that if y satisfies (3), (4), then for $(3a_i + a_{i+1})/4 \le t \le (a_i + 3a_{i+1})/4$,

$$(6) \qquad (-1)^{\alpha_i} y(t) \ge \|y\| (a/4)^m,$$

$i = 1, \ldots, k-1$, where $a = \min_i (a_{i+1} - a_i)$ and $m = \max\{n - n_1, n - n_k\}$.

Assume that y satisfies the differential inequality,

$$(7) \qquad y^{(n)}(t) > 0, \quad 0 < t < 1,$$

and the boundary conditions (4). Since $y(t) = \int_0^1 G(t,s) y^{(n)}(s)\, ds$, it follows that

$$(8) \qquad (-1)^{\alpha_i} y(t) > 0, \quad t \in (a_i, a_{i+1}),\ i = 1, \ldots, k-1,$$

$$(9) \qquad (-1)^{\alpha_i} y^{(n_i)}(a_i) > 0, \quad i = 1, \ldots, k.$$

Let $\|\cdot\|$ denote the supremum norm with respect to continuous functions on $[0,1]$. Let y satisfy the BVP, (7), (4). Let $t_1 \in (a_l, a_{l+1})$ for some $l \in \{1, \ldots, k-1\}$ be such that $\|y\| = (-1)^{\alpha_l} y(t_1)$. Define polynomials p_i, $i = 1, 2$, as follows:

$$(10) \qquad p_1(t) = (-1)^{\alpha_l} \|y\| \prod_{j=1}^{l} (t - a_j)^{n_j} \Big/ \prod_{j=1}^{l} (t_1 - a_j)^{n_j},$$

$$(11) \qquad p_2(t) = (-1)^{\alpha_l} \|y\| \prod_{j=l+1}^{k} (t - a_j)^{n_j} \Big/ \prod_{j=l+1}^{k} (t_1 - a_j)^{n_j}.$$

Finally, define the piecewise polynomial, p, by

$$p(t) = \begin{cases} p_1(t) & 0 < t < t_1, \\ p_2(t) & t_1 < t < 1. \end{cases} \tag{12}$$

Theorem 1. *Let y satisfy the* BVP, (7), (4). *Let $t_1 \in (a_l, a_{l+1})$ for some $l \in \{1, \dots, k-1\}$ be such that $\|y\| = (-1)^{\alpha_l} y(t_1)$. Define polynomials p_i, $i = 1, 2$, by* (10) *and* (11)*, respectively. Then*

$$(-1)^{\alpha_i}(y-p)(t) > 0, \quad a_i < t < a_{i+1},\ i = 1, \dots, k-1, \tag{13}$$

where p is defined by (12).

Proof. There are two cases to consider. These are $\alpha_l = 1$ or $\alpha_l = n-1$ and $\alpha_l \in \{2, \dots, n-2\}$. We first consider the easier case, $\alpha_l = 1$ or $\alpha_l = n-1$. We shall address the case $\alpha_l = 1$ as the case $\alpha_l = n-1$ is addressed analogously. We point out that the proof in the case $\alpha_l = n-1$ when $k = 3$ appears in [**7**]. Note that $t_1 \in (a_{k-1}, a_k)$. On $[0, t_1]$, set $h = y - p$. Then $h^{(n)}(t) > 0$, $0 < t < t_1$, and that $t_1 \in (a_{k-1}, a_k)$. On $[0, t_1]$, set $h = y - p$. Then $h^{(n)}(t) > 0$, $0 < t < t_1$, and $h^{(j)}(a_i) = 0$, $j = 0, \dots, n_i - 1$, $i = 1, \dots, k-1$, $h(t_1) = 0$. Then h satisfies (8) on $[0, t_1]$ where $a_k = t_1$. In particular, $y - p$ satisfies (8) on $[0, t_1]$ where $a_k = t_1$. To analyze $h = y - p$ on $[t_1, 1]$, apply Rolle's theorem to y. Since y satisfies (7), y'' does not vanish on $[t_1, 1]$. Since $\alpha_l = 1$, $y''(t_1) > 0$ and so y'' is positive on $(t_1, 1)$. Thus, h satisfies $h''(t) > 0$, $t_1 < t < 1$, $h(t_1) = h(1) = 0$ and so $h(t) < 0$, $t_1 < t < 1$. The proof that $y - p$ satisfies (13) is complete in the case $\alpha_l = 1$ or $\alpha_l = n-1$.

Now assume that $k \in \{2, \dots, n-2\}$. We shall show the details that $y - p$ satisfies (13) for $t \in (t_1, a_{l+1}) \cup \cup_{i=l+1}^{k-1}(a_i, a_{i+1})$ and then briefly outline the similar details for $t \in \cup_{i=1}^{l-1}(a_i, a_{i+1}) \cup (a_l, t_l)$. Apply Rolle's theorem repeatedly to $y(t)$. Note that, under condition (7), if $t \in (0,1)$ is a root of $y^{(j)}$, $j < n$, and $y^{(j)}(t)$ is not specified in the boundary conditions, (4), then t is a simple root of $y^{(j)}$. In particular, $y^{(j)}$ changes sign at t.

We shall now label specific consecutive roots of $y^{(j)}$ and determine the sign of $y^{(j)}$ interior to these consecutive roots. Let $t_{2,0}$ and

$t_{2,1}$ denote roots of y'' satisfying $t_{2,0} < t_1 < t_{2,1}$ and y'' does not vanish on $(t_{2,0}, t_{2,1})$. Inductively, let $t_{j+1,0}$ and $t_{j+1,1}$ denote roots of $y^{(j+1)}$ satisfying $t_{j+1,0} < t_{j,1} < t_{j+1,1}$ and $y^{(j+1)}$ does not vanish on $(t_{j+1,0}, t_{j+1,1})$, $j = 1, \dots, \alpha_l$.

We shall now determine the sign of $y^{(j)}(t)$ for $t_{j,0} < t < t_{j,1}$, $j = 2, \dots, \alpha_l$. Since $(-1)^{\alpha_l} y(t_1) > 0$ is an extreme point, and t_1 is a simple root of y', then $(-1)^{\alpha_l} y''(t_1) < 0$. In particular, $(-1)^{\alpha_l} y''(t) < 0$, $t_{2,0} < t < t_{2,1}$. $t_{2,1}$ denotes a simple root of y''; in particular, y'' is increasing at $t_{2,1}$ and so, $(-1)^{\alpha_l} y'''(t_{2,1}) > 0$. Thus, $(-1)^{\alpha_l} y'''(t) > 0$, $t_{3,0} < t < t_{3,1}$. It now follows readily by induction that

$$(-1)^{\alpha_l + j} y^{(j)}(t) < 0, \quad t_{j,0} < t < t_{j,1} < j = 2, \dots, \alpha_l. \tag{14}$$

We must also count the number of roots of $y^{(j)}$ to the right of $t_{j,1}$, $j = 2, \dots, \alpha_l$. Note that, due to (7) and due to the boundary conditions, (4), y has precisely α_l roots, counting multiplicities in $(t_1, 1]$. By Rolle's theorem, y' has at least $\alpha_l - 1$ roots, counting multiplicities in $(t_{2,1}, 1]$. Again, by (7), y' has precisely $\alpha_l - 1$ roots, counting multiplicities in $(t_{2,1}, 1]$. Inductively, it follows that $y^{(j)}$ has precisely $\alpha_l - j$ roots, counting multiplicities in $(t_{j+1,1}, 1]$, $j = 1, \dots, \alpha_l$. In particular,

$$y^{(\alpha_l)}(t) \neq 0, \quad t_{\alpha_l,1} < t \le 1. \tag{15}$$

We now label the smallest root, r_j, of $p_2^{(j)}$, $j = 0, \dots, \alpha_l - 1$, and determine the sign of $p_2^{(j)}$ to the left of r_j. Since $p_2^{(\alpha_l)}$ is a positive constant, it follows that

$$\alpha_{l+1} = r_0 = \cdots = r_{n_{l+1}-1} < r_{n_{l+1}} < \cdots < r_{\alpha_l - 1} < 1,$$

and each root $r_{n_{l+1}}, \dots, r_{\alpha_l - 1}$ is simple. $(-1)^{\alpha_l} p_2(t) > 0$, $t < r_0$. As a_{l+1} is a root of order n_{l+1}, it follows that $(-1)^{\alpha_l + j} p_2^{(j)}(t) > 0$, $t < r_j$, $j = 0, \dots, n_{l+1} - 1$, and $(-1)^{\alpha_l + n_l + 1} p_2^{(n_{l+1})}(a_{l+1})) > 0$. Thus, $(-1)^{\alpha_l + j} p_2^{(j)}(t) > 0$, $t < r_j$, $j = n_{l+1}$. Since $r_{n_{l+1}}$ is a simple root of $p_2^{(n_{l+1})}$, it follows that $(-1)^{\alpha_l + n_l + 1} p_2^{(n_l + 1)}$ is decreasing at $r_{n_{l+1}}$ and $(-1)^{\alpha_l + n_{l+1} + 1} p_2^{(n_{l+1}+1)}(r_{n_{l+1}}) > 0$. In particular, $(-1)^{\alpha_l + j} p_2^{(j)}(t) > 0$, $t < r_j$, $j = n_{l+1} + 1$. It again follows readily by induction that

$$(-1)^{\alpha_l + j} p_2^{(j)}(t) > 0, \quad t < r_j, \ j = 2, \dots, \alpha_l - 1. \tag{16}$$

Set $h(t) = y - p_2(t)$, $t_1 \le t \le 1$. Because of the boundary conditions (4) and the construction of p_2, h has at least $\alpha_l + 1$ roots in $[t_1, 1]$. We first argue that h has precisely $\alpha_l + 1$ roots in $[t_1, 1]$. Assume, for the sake of contradiction, that h has at least $\alpha_l + 2$ roots in $[t_1, 1]$. Then h' has at least $\alpha_l + 1$ roots in $(t_1, 1]$. Note that if $h^{(j)}$ has a root, then $y^{(j)}$ and $p_2^{(j)}$ have the same sign. h'' has at least α_l roots in $(t_1, 1]$. By (14) and (16) it follows that h'' has at least α_l roots in $[t_{2,1}, 1]$. Inductively, apply Rolle's theorem, (14) and (16), and obtain that $h^{(j)}$ has at least $\alpha_l + 2 - j$ roots in $[t_{j,1}, 1]$, $j = 2, \dots, \alpha_l$. In particular, $y^{(\alpha_l)} = h^{(\alpha_l)}$ vanishes in $(t_{\alpha_l,1}, 1]$. This contradicts (15). Thus we have shown that h has precisely $\alpha_l + 1$ roots in $[t_1, 1]$.

To argue that y satisfies (13) when $t > t_1$, note that

$$(-1)^{\alpha_l} h'(t_1^+) = (-1)^{\alpha_l + 1} p_2'(t_1^+) > 0$$

by (16). In particular, $h(t_1) = 0$ and $(-1)^{\alpha_l} h'(t_1) > 0$ and so, $(-1)^{\alpha_l} h(t) > 0$, $t_1 < t < \alpha_{l+1}$. h has precisely $\alpha_l + 1$ roots in $[t_1, 1]$ and so h has a root of order n_{l+1} at a_{l+1}. Since $\alpha_l + n_{l+1}$ and $\alpha_{l+1} = \alpha_l - n_{l+1}$ have the same parity, $(-1)^{\alpha_{l+1}} h(t) > 0$, $a_{l+1} < t < a_{l+2}$. Since we know precisely the order of each root of h at each a_i, $i = l+1, \dots, k-1$, (13) follows on $(t_1, a_{l+1}) \cup \cup_{i=l+1}^{k-1} (a_i, a_{i+1})$.

To obtain (13) on $\cup_{i=0}^{l-1} (a_i, a_{i+1}) \cup (a_l, t_1)$, apply a similar argument to $h = y - p_1$ on $(0, t_1)$. We omit the details and the proof of Theorem 1 is complete.

Corollary 2. *Assume that y satisfies the* BVP, (3), (4). *Then y satisfies* (6).

Proof. First assume that y satisfies the BVP, (7), (4), and employ Theorem 1. Assume $\|y\| = (-1)^{\alpha_l} y(t_1)$ for $t_1 \in (a_l, a_{l+1})$ for some $l \in \{1, \dots, k-1\}$.

Note that p_1 has precisely one critical point in each subinterval (a_i, a_{i+1}), $i = 1, \dots, l-1$. Thus, for $i = 1, \dots, l-1$, if $t \in \cup_{i=1}^{l-1} [(3a_i + a_{i+1})/4,\ (a_i + 3a_{i+1})/4]$, then

$$\begin{aligned}(-1)^{\alpha_i} y(t) &\ge (-1)^{\alpha_i} p_1(t) \\ &\ge \min\{(-1)^{\alpha_i} p_1(3a_i + a_{i+1})/4,\ (-1)^{\alpha_i} p_1(a_i + 3a_{i+1})/4\}.\end{aligned}$$

Thus,
$$(-1)^{\alpha_i} y(t) \geq \|y\|(a/4)^{n-\alpha_l} \geq \|y\|(a/4)^{n-n_k}.$$
Similarly, for $i = l+1, \dots, k-1$, if $t \in \cup_{i=l+1}^{k-1}[(3a_i + a_{i+1})/4,\ (a_i + 3a_{i+1})/4]$, then
$$(-1)^{\alpha_i} y(t) \geq \|y\|(a/4)^{n-n_1}.$$

Now assume that $t \in (a_l, a_{l+1})$. There are three cases to consider. $t_1 < (3a_l + a_{l+1})/4$, $(3a_l + a_{l+1})/4 \leq t_1 \leq (a_l + 3a_{l+1})/4$, and $(a_l + 3a_{l+1})/4 < t_1$. We consider the case $(3a_l + a_{l+1})/4 \leq t_1 \leq (a_l + 3a_{l+1})/4$ in detail; the details for the remaining two cases are similar. Suppose $(3a_l + a_{l+1})/4 \leq t \leq t_1$. Then

$$(-1)^{\alpha_l} y(t) \geq (-1)^{\alpha_l} p_1(3a_l + a_{l+1})/4 \geq \|y\|(a/4)^{n-n_k}.$$

If $t_1 \leq t \leq (a_l + 3a_{l+1})/4$, then

$$(-1)^{\alpha_l} y(t) \geq (-1)^{\alpha_l} p_2(a_l + 3a_{l+1})/4 \geq \|y\|(a/4)^{n-n_1}.$$

The proof that if y satisfies the BVP, (7), (4), then y satisfies (6) is complete.

Suppose that y satisfies the BVP, (3), (4). For $\varepsilon > 0$, define

$$y(\varepsilon, t) = y(t) + \varepsilon \prod_{j=1}^{k} (t - a_j)^{n_j}.$$

For each $\varepsilon > 0$, $y(\varepsilon, t)$ satisfies (7), (4) and hence (6). By continuity in ε, (6) holds for $\varepsilon = 0$; in particular, y satisfies (6) and Corollary 2 is proved.

For our final result, we shall obtain the analogue of (6) for the Green's function, $G(t,s)$, of the BVP, $y^{(n)}(t) = 0$, $0 < t < 1$, satisfying (4). We shall refer to this result as a corollary since, if we employ standard properties of $G(t,s)$ in place of (7), the argument proceeds precisely as in the proof of Theorem 1 and as in the proof of Corollary 2 which relates to y satisfying (7).

The purpose of (7) is so that we can count precisely the number of roots of $y^{(j)}$ and, hence, determine the simplicity of appropriate roots. Recall [**2**] that $G(t,s)$ is C^{n-2} on $[0,1] \times [0,1]$,and for each $s \in (0,1)$,

G satisfies (4) as a function of t. Moreover, for $0 < s < t$, $G(t,s)$ is C^{n-1} as a function of t and is, in fact, an $n-1$ order polynomial in t. Similarly, for $0 < t < s$, $G(t,s)$ is C^{n-1} as a function of t and is an $n-1$ order polynomial in t. Let $s \in (0,1)$ be fixed. By (4) and by Rolle's theorem, $(\partial^{n-2}/\partial t^{n-2})G(t,s)$ has at least two roots in $(0,1)$. It is the case that $(\partial^{n-2}/\partial t^{n-2})G(t,s)$ has precisely two roots $t_{n-2,0}, t_{n-2,1}$ in $(0,1)$ and $t_{n-2,0} < s < t_{n-2,1}$. Suppose, for the sake of contradiction, that $(\partial^{n-2}/\partial t^{n-2})G(t,s)$ has at least two roots in $(0,s)$. Then $(\partial^{n-1}/\partial t^{n-1})G(t,s)$ vanishes in $(0,s)$. In particular, $(\partial^j/\partial t^j)G(t,s)$ vanishes in $(0,s)$, $j = 0,\dots,n-1$. As G is an $n-1$ order polynomial in t, G is identically zero on $(0,s)$. This contradicts (5) and so $(\partial^{n-2}/\partial t^{n-2})G(t,s)$ has precisely one root in $(0,s)$. Similarly, $(\partial^{n-2}/\partial t^{n-2})G(t,s)$ has precisely one root in $(s,1)$.

This argument that G does not vanish identically on triangles, $t < s$ or $s < t$, also implies that if t is a root of $(\partial^j/\partial t^j)G(t,s)$ which is not specified in the boundary conditions (4), then the root is simple.

Now the arguments in the case, $\alpha_l \in \{2,\dots,n-2\}$ apply to $G(t,s)$ and we obtain the following result.

Corollary 3. *For each $s \in (0,1)$, let $\|G(\cdot,s)\| = \sup_{0\le t\le 1}|G(t,s)|$. Then, for $(3a_i + a_{i+1})/4 \le t \le (a_i + 3a_{i+1})/4$,*

$$(-1)^{\alpha_i}G(t,s) \ge \|G(\cdot,s)\|(a/4)^m, \quad 0 < s < 1,$$

$i = 1,\dots,k-1$.

To obtain Corollary 3 in the case $\alpha_l = 1$, define the piecewise polynomial, p, in the obvious way. As before, on $(t_1,1)$, $G - p$ is concave up and vanishes at t_1 and at 1. On $(0,t_1)$, $\hat{G} = G - p$ is the Green's function of a BVP, $y^{(n)}(t) = 0$, $0 < t < t_1$, satisfying (4) with $a_k = t_1$. Thus, $\hat{G}$ satisfies (5) with $a_k = 1$. Details of this argument are provided in [**6**].

REFERENCES

1. E. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.

2. W. Coppel, *Disconjugacy*, Springer-Verlag, Berlin, 1971.

3. H. Dang, K. Schmmitt and R. Shivaji, *On the number of solutions of boundary value problems involving the p-Laplacian*, Electronic J. Differential Equations 1996, No. 1 (1996), 1–9.

4. P.W. Eloe and J. Henderson, *Singular nonlinear* $(k, n-k)$ *conjugate boundary value problems*, J. Differential Equations **133** (1997), 136–151.

5. ———, *Positive solutions for higher order ordinary differential equations*, Electronic J. Differential Equations 1995, No. 3 (1995), 1–8.

6. ———, *Inequalities based on a generalization of concavity*, Proc. Amer. Math. Soc. **125** (1997), 2103–2108.

7. P.W. Eloe and J. McKelvey, *Positive solutions of three point boundary value problems*, Comm. Appl. Nonlinear Anal. **4** (1997), 45–54.

8. L.H. Erbe and H. Wang, *On the existence of positive solutions of ordinary differential equations*, Proc. Amer. Math. Soc. **120** (1994), 734–748.

9. J.A. Gatica, V. Oliker and P. Waltman, *Singular nonlinear boundary value problems for second-order ordinary differential equations*, J. Differential Equations **79** (1989), 62–78.

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