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ON THE SECOND HILBERT 2-CLASS FIELD OF REAL QUADRATIC NUMBER FIELDS WITH 2-CLASS GROUP ISOMORPHIC TO $(2, 2^n)$, $n \geq 2$

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ABSTRACT. Let k be a real quadratic number field with $C_{k,2}$, the 2-Sylow subgroup of its ideal class group, isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2^n\mathbf{Z}$, $n \geq 2$, such that $\mathrm{Gal}\,(k_2/k)$, the galois group over k of the second Hilbert 2-class field of k, is nonabelian. We describe conditions for which we can further refine $\mathrm{Gal}\,(k_2/k)$ in terms of its group structure being modular, metacyclic-nonmodular, or nonmetacyclic, when a prime congruent to 3 mod 4 does not divide the discriminant of k.

1. Preliminaries. Let k be a real quadratic number field with $C_{k,2}$, the 2-Sylow subgroup of its ideal class group, isomorphic to $\mathbf{Z}/2\mathbf{Z}\times\mathbf{Z}/2^n\mathbf{Z}$, $n \geq 2$, which we will denote by $(2, 2^n)$. We let k_1 denote the Hilbert 2-class field of k, i.e., the maximal unramified (including the infinite primes) abelian field extension of k which has degree a power of 2. Then $C_{k,2} \cong \mathrm{Gal}\,(k_1/k)$, the galois group of k_1 over k, and we let $k_2 = (k_1)_1$. In our earlier work we have completely determined when $\mathrm{Gal}\,(k_2/k)$ is abelian, [**1, 2**]. In certain cases, particularly when a prime congruent to 3 mod 4 divides d_k, the discriminant of k, we have utilized information about the capitulation of ideal classes in unramified quadratic extensions of k in order to further classify $\mathrm{Gal}\,(k_2/k$ in terms of its group structure being modular, metacyclic-nonmodular, or nonmetacyclic [**1, 2**]. In the present paper we extend the above classification for nonabelian $\mathrm{Gal}\,(k_2/k)$ to the remaining cases, i.e., when a prime congruent to 3 mod 4 does not divide the discriminant of k.

Recall that a group G is metacyclic if there exists a normal cyclic subgroup, N, of G such that G/N is also cyclic. Finite metacyclic 2-groups, G, for which $G/G' \cong (2, 2^n)$, $n \geq 2$, can be divided into two isomorphism types: the modular groups, $M_{n+2}(2)$, and those which

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are not modular. Recall that $M_n(2)$, $n > 3$ is the group, G, of order 2^n such that $G/G' \cong (2, 2^{n-2})$ and there exists a cyclic subgroup of index 2 in G, cf. [**12**]. We denote $(p_1/p_2)_4$ to be the biquadratic residue symbol of p_1 over p_2.

We begin with the following lemma, cf. [**2**].

Lemma 1. *Let $k = Q(\sqrt{p_1p_2p_3})$ be a real quadratic number field, $p_1 \equiv p_2 \equiv 1 \bmod 4$, $p_3 \equiv 1 \bmod 4$ or $p_3 = 2$, with $C_{k,2} \cong (2, 2^n)$, $n \geq 2$, $(p_i/p_j) = (p_j/p_k) = 1$, $(p_k/p_i) = -1$, for $\{i, j, k\} = \{1, 2, 3\}$. Then $\mathrm{Gal}\,(k_2/k)$ is abelian if and only if $N_{\varepsilon_0} = -1$ and $(p_i/p_j)_4(p_j/p_i)_4 = (p_j/p_k)_4(p_k/p_j)_4 = -1$.*

We now proceed to obtain an equivalent classification of k for $\mathrm{Gal}\,(k_2/k)$ abelian and k as in the conditions of Lemma 1, along with a further refinement of nonabelian $\mathrm{Gal}\,(k_2/k)$ in terms of modular, metacyclic-nonmodular, and nonmetacyclic group structure.

We let $G = \mathrm{Gal}\,(k_2/k)$; then $G' = \mathrm{Gal}\,(k_2/k_1)$ and $G/G' \cong \mathrm{Gal}\,(k_1/k) \cong C_{k,2} \cong (2, 2^n)$, $n \geq 2$, where G' denotes the commutator subgroup, (G, G), of G. We let K be an unramified quadratic extension of k, where k is a real quadratic number field with $C_{k,2} \cong (2, 2^n)$, $n \geq 2$, d_k not divisible by a prime congruent to 3 mod 4, and $N_{\varepsilon_0} = 1$. We let j be the homomorphism from $C_k \to C_K$ where, if $\overline{A}$ denotes the ideal class containing A, then $j(\overline{A}) = \overline{AO_K}$ where O_K is the ring of algebraic integers in K. Thus j is the extension of ideal classes of k in K, and we want to determine the capitulation kernel, $\ker j$, of all possible fields k with k as above.

We collect some known facts concerning $\ker j$. Recall that if T is a cyclic unramified extension of prime degree p over a number field k, then Hilbert's Satz 94 [**13**] guarantees the existence of an ideal class of order p in the ideal class group, C_k, which becomes principal (capitulates) when extended to T. We know that the order of any ideal class in $\ker j$ divides $[K : k] = 2$, see, e.g., [**11**], and $\ker j$ is therefore contained in $C_{k,2}$. We further know that $|\ker j| = [K : k][E_k : N_{K/k}(E_K)] = 2[E_k : N_{K/k}(E_K)]$ where E_k and E_K denote the group of units in k and K, respectively, see, e.g., [**22**].

Since $\ker_j \subseteq C_{k,2} \cong (2, 2^n)$ we know that $|\ker j| = 2$ or $|\ker j| = 4$. Also, following Taussky [**23**], we say the extension K/k satisfies

Condition A, respectively, Condition B, provided $|\ker j \cap N_{K/k}(C_K)| > 1$, respectively, $= 1$. We shall proceed to determine both $|\ker j|$ and whether or not K/k satisfies Condition A in terms of the arithmetic of k.

Let $G = \langle a, b \rangle$ where $\bar{a}^2 = \bar{b}^{2^n} = \bar{I}$, $\bar{x} = xG'$ for any $x \in G$ and I is the identity. Since $G/G' \cong (2, 2^n)$ we note that G contains three subgroups H_1, H_2, H_3 of index 2: $H_1 = \langle b, G' \rangle$, $H_2 = \langle ab, G' \rangle$, and $H_3 = \langle a, b^2, G' \rangle$ with $a^2 \in G'$ and $b^{2^n} \in G'$; it follows that $\overline{H}_1$ and $\overline{H}_2$ are cyclic whereas $\overline{H}_3$ is not ($\overline{H}_i = H_i/G'$, $i = 1, 2, 3$). Let K_i be the subfield of k_2 fixed by H_i. Then $k \subseteq K_i \subseteq k_1$ and K_1, K_2, K_3 are all of the unramified quadratic extensions of k. Let $j_i : C_k \to C_{K_i}$ be the canonical homomorphism described earlier. Since $\ker j_i$ is elementary and $[C_{k,2} : N_{K_i/k}(C_{K_i})] = 2$, if K_i/k satisfies Condition B, then $|\ker j_i| = 2$. Thus, if $|\ker j_i| = 4$, we know that K_i/k satisfies Condition A.

Denoting by C_k^* the narrow class group of k and by $N(\varepsilon_0) = N_{\varepsilon_0} = N_{k/Q}(\varepsilon_0)$ the norm of the fundamental unit ε_0 of k, we refer to the following criteria by Kaplan [**15**] to determine if the 4-rank of C_k equals the 4-rank of C_k^*: when $N(\varepsilon_0) = 1$, the 4-rank of C_k equals the 4-rank of C_k^* if and only if there exists a prime p congruent to 3 mod 4 dividing d or there exist positive integers a, b with the integer a odd such that $d = a^2 + b^2$, and $(a/p) = 1$ for every odd prime p dividing d. Here d is the square-free kernel of the discriminant of k.

We denote by d_k the discriminant of k and define a d_k-splitting of the second kind to be a factorization of $d_k = 1 \cdot d_k$ or $d_k = d_1 \cdot d_2$, $|d_1| \leq |d_2|$, into the product of two fundamental discriminants for which the Kronecker symbols $(d_1/p) = 1$ for all primes $p|d_2$ and $(d_2/p) = 1$ for all primes $p|d_1$. By Redéi and Reichardt [**19, 21**] we know that the number of d_k-splittings of the second kind is 2^{e_2} where e_2 is the 4-rank of C_k^*. It is well-known by a theorem of Gauss that the number of generators of C_k^* is $t - 1$ where t is the number of distinct prime factors of d_k. It is also well-known that the 2-rank of the narrow class group equals the 2-rank of the wider class group if and only if there does not exists a prime congruent to 3 mod 4 dividing d_k, and that the narrow class group is equal to the wider class group if and only if the norm of the fundamental unit of k is -1 [**14**].

We shall use the following notational conventions: p_i, p will denote primes $\equiv 1 \bmod 4$; q_i, q will denote primes $\equiv 3 \bmod 4$; r, r_i will denote any primes; and r^* will denote a fundamental discriminant divisible only by the prime r, i.e., $r^* = (-1)^{(r-1)/2}(r)$ if r is odd and $2^* \in \{8, -8, -4\}$.

Given an unramified quadratic extension K of a real quadratic number field k with discriminant d_k, we know that $K = Q(\sqrt{d_1}, \sqrt{d_2})$ where $d_k = d_1 \cdot d_2$, $d_i > 1$ for $i = 1, 2$, where d_1 and d_2 are fundamental discriminants. Consequently, K contains three real quadratic subfields: $k_0 = k = Q(\sqrt{d_k})$, $F_1 = Q(\sqrt{d_1})$ and $F_2 = Q(\sqrt{d_2})$. We denote by $\varepsilon_0 = \varepsilon$, ε_1 and ε_2 the fundamental units (> 1) of $k_0 = k$, F_1 and F_2, respectively.

Through utilization of the aforementioned 4-rank criteria of Kaplan [**15**] and the Redéi and Reichardt and Gauss formulas [**21**] we obtain the following lemma.

Lemma 2. *All real quadratic number fields k with $C_{k,2} \cong (2, 2^n)$, $n \geq 2$, such that d_k is not divisible by a prime congruent to* $3 \bmod 4$*, are described as follows.*

Case 1. $k = Q(\sqrt{p_1p_2p_3})$, $p_1 \equiv p_2 \equiv p_3 \equiv 1 \bmod 4$, $d_k = p_1p_2p_3$.

A) $(p_1/p_3) = (p_2/p_3) = 1$, $(p_1/p_2) = -1$. $N(\varepsilon_0) = -1$ or [$N(\varepsilon_0) = 1$ and 4-rank of C_k =4-rank of C_k^*] (notice that $C_{k,2}^* \cong (2, 2^n)$, $n \geq 2$).

B) $(p_1/p_3) = (p_1/p_2) = (p_2/p_3) = 1$, $N(\varepsilon_0) = 1$, 4-rank of $C_k \neq$ 4-rank of C_k^* (notice that $C_{k,2}^* \equiv (4, 2^n)$, $n \geq 2$).

Case 2. $k = Q(\sqrt{2p_1p_2})$, $p_1 \equiv p_2 \equiv 1 \bmod 4$, $d_k = 8p_1p_2$.

A) $(p_1/p_2) = -1$, $p_1 \equiv p_2 \equiv 1 \bmod 8$, $N(\varepsilon_0) = -1$ or [$N(\varepsilon_0) = 1$ and 4-rank of C_k =4-rank of C_k^*] (notice that $C_{k,2}^* \cong (2, 2^n)$, $n \geq 2$).

B) $(p_1/p_2) = 1$, $p_1 = 5 \bmod 8$, $p_2 \equiv 1 \bmod 8$, $N(\varepsilon_0) = -1$ or [$N(\varepsilon_0) = 1$ and 4-rank of C_k = 4-rank of C_k^*] (notice that $C_{k,2}^* \cong (2, 2^n)$, $n \geq 2$).

C) (($p_1/p_2) = 1$, $p_1 \equiv p_2 \equiv 1 \bmod 8$, $N(\varepsilon_0) = 1$, 4-rank of $C_k \neq$ 4-rank of C_k^* (notice that $C_{k,2}^* \cong (4, 2^n)$, $n \geq 2$).

We state the following result from our earlier paper [**3**].

Lemma 3. *Let K be an unramified quadratic extension of a real quadratic number field, k. Then $|\ker j| = 2$ if and only if*

a) $N(\varepsilon_i) = -1$ *for* $i = 0, 1, 2$ *and* $\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \in K$

or

b) $N(\varepsilon_0) = 1$ *and* (i) $N(\varepsilon_1) = -1$ *and* ($\sqrt{\varepsilon_0}$ *or* $\sqrt{\varepsilon_0\varepsilon_2} \in K$) *or* (ii) $N(\varepsilon_2) = -1$ *and* ($\sqrt{\varepsilon_0}$ *or* $\sqrt{\varepsilon_0\varepsilon_1} \in K$).

Remark. We note that if $\sqrt{\varepsilon_0\varepsilon_i} \in K$, then $N(\varepsilon_i) = 1$ for $i = 1$ or 2, see [**3**, p. 386].

We introduce some additional notation. Let η be a unit of k. If $N(\eta) = 1$, then denote by δ_η the square-free kernel (Sfk) of $N(1 + \varepsilon)$. If $N(\eta) = -1$, then δ_η is not defined.

In the context of the notation above Lemma 3, we let $\delta, \delta_1, \delta_2$ denote $\delta_{e_0}, \delta_{e_1}, \delta_{e_2}$, respectively, when they are defined. From Lemma 3 and properties of δ, [**4, 18**], we are able to state the following lemma:

Lemma 4. *Let K be an unramified quadratic extension of a real quadratic number field. Assume $N(\varepsilon_0) = 1$. Then $|\ker j| = 2$ if and only if*

(i) $N(\varepsilon_1) = -1$ *and* ($\delta \in K^2$ *or* $\delta\delta_2 \in K^2$)

or

(ii) $N(\varepsilon_2) = -1$ *and* ($\delta \in K^2$ *or* $\delta\delta_1 \in K^2$).

Note. By the comments above this lemma, if δ_i is not defined for $i = 1$ or 2, then "or $\delta\delta_i \in K^2$" is omitted in the lemma. We assume that δ_2 is divisible by two distinct primes.

We denote by $(e, D)/p$ the Hilbert symbol with respect to p, where $e = SfkN(1 + \eta)$ for $\eta \neq 1$ a unit of norm 1 in a real quadratic field $F = Q(\sqrt{D})$. From [**4**] we can state:

Lemma 5. *For F and $(e, D)/p$ as above, then $(e, D)/p = 1$ for all $p|D$ (see Borevich and Shafarevich [**6**] for definition and properties of $(e, D)/p$.)*

Through Proposition 4 of Benjamin and Snyder [**4**] we make use of criteria for $(\delta_i, D)/p = 1$, $i = 0, 1, 2$, in terms of graph theory as described in the following way, see [**4**].

Let D be a fundamental discriminant, not necessarily positive, of a quadratic field. Let $V(D)$, the set of vertices of D, be the set of primes, r, dividing D. Let R be the subset of $V(D) \times V(D)$ given by

$$R = \left\{ (r_1, r_2) \mid \left(\frac{r_2}{r_1}\right) = -1 \quad \text{or} \quad \left[\left(\frac{r_1}{r_2}\right) = -1, \text{ if } r_1 = 2 \text{ and } r_2 = 3 \bmod 4 \right] \right\}.$$

Then R determines the edges of a graph with vertices $V(D)$. If (r_1, r_2) and (r_2, r_1) are in R, then we say the edge runs both ways between r_1 and r_2 and represent this by r_1——r_2. If, however, $(r_1, r_2) \in R$ but $r_2, r_1) \notin R$, then we say the edge runs from r_1 to r_2 and denote it by $r_1 \longrightarrow r_2$.

If the fundamental unit of a particular quadratic number field needs to be identified, we use subscripts in the following way: $\varepsilon_{p_i p_j}$ is the fundamental unit of $Q(\sqrt{p_i p_j})$; $\varepsilon_{K_{3,i}}$ is the fundamental unit of F_1, where F_1 is described above Lemma 2, for the unramified quadratic extension K_3 that corresponds to the maximal subgroup of G whose factor group is noncyclic, etc.

We utilize the symbol (X_1, X_2, X_3) where $X_i \in \{4, 2, 2A, 2B\}$ for $i = 1, 2, 3$ and $X_i = |\ker j_i|$ with A or B referring to the particular unramified quadratic extension K_i satisfying Taussky's Condition A or B. We note that X_3 always refers to the capitulation in K_3, the unramified quadratic extension of k that corresponds to the noncyclic factor group of the maximal subgroup of $\mathrm{Gal}\,(k_2/k)$. From our earlier work [**5**] we are able to form the following capitulation table to determine the structure of $G = \mathrm{Gal}\,(k_2/k)$. We utilize the following abbreviations: A is abelian, M is modular, MC is metacyclic-nonmodular, NM is nonmetacyclic.

We note that if the capitulation is (2B,2B,2A) then G may be either modular or nonmetacyclic, and if the capitulation is (2A,2A,2A), then G may be either metacyclic-nonmodular or nonmetacyclic. We also not that it is immaterial as to which subgroup we denote as H_1 and H_2 for the two maximal subgroups whose factor groups are cyclic.

TABLE 1. Capitulation and structure of G.

$\ker j_1$	$\ker j_2$	$\ker j_3$	$G = \mathrm{Gal}\,(k_2/k)$
4	4	4	A
2A	2A	4	MC
2B	2B	2A	M or NM
2A	2A	2A	MC or NM
4	4	2A	NM
2A or 2B	4	2A	NM
4	2A or 2B	2A	NM
2A	2B	2A	NM
2B	2A	2A	NM

In order to determine whether Taussky's Condition A or B is satisfied, we make use of the following lemma, which follows directly from Kisilevsky [**17**].

Lemma 6. *Let* $k = Q(\sqrt{p_1p_2p_3})$ *where* $p_1 \equiv p_2 \equiv 1 \bmod 4$, $p_3 \equiv 1 \bmod 4$ *or* $p_3 = 2$. *Let* $K = k(\sqrt{p_i})$, $i = 1, 2$ *or* 3. *Then* K/k *satisfies Condition* A *if and only if* $(p_jp_k/p_i) = 1$.

2. Determination of $\mathrm{Gal}\,(k_2/k)$ **when** $N_{\varepsilon_0} = 1$**.** By the use of Table 1, Lemma 3 and Lemma 4, we are able to determine when $G = \mathrm{Gal}\,(k_2/k)$ is A, M, MC or NM for all capitulation cases except the case where two ideal classes capitulate in each of the three unramified quadratic extensions of k. In this case, G may be M, MC or NM. In order to narrow these possibilities, we utilize Lemma 6 together with the following lemma to determine the Taussky Condition A or B for the case (2,2,2) (k is always a real quadratic number field with $C_{k,2} \cong (2, 2^n)$, $n \geq 2$, and Sfk denotes the square-free kernel).

Lemma 7. *Suppose* $d_k = d_1d_2$ *is a factorization of* d_k *into relatively prime fundamental discriminants* d_1 *and* d_2 *where* $d_1 > 0$ *and* $d_2 > 0$. *Let* $L = k(\sqrt{d_2})$ *and suppose* $A = P_1 \cdots P_s$ *is a nontrivial ideal of* k, *which is a product of distinct prime ideals* P_i *such that the rational* *prime p_i contained in P_i divides d_2. Assuming both δ and δ_2 are defined, i.e., $N(\varepsilon_0) = 1$ and $N(\varepsilon_2) = 1$,*

(i) *If $p_1 \cdots p_2 \neq \delta$ or $Sfk(d_k)$ or $Sfk(d_k/\delta)$, then A is nonprincipal in k. (If δ is not defined and $p_1 \cdots p_s \neq Sfk(d_k)$, then A is nonprincipal in k.)*

(ii) *An ideal $B \subseteq Q(\sqrt{d_2})$, such that $N_{Q(\sqrt{d_2})/Q}(B)$ divides d_2 and $Sfk(N_{Q(\sqrt{d_2})/Q}(B)) = \delta_2$, is a principal ideal in L.*

Proof. By Cohn [**8**] the only principal ambiguous ideals (we identify ambiguous ideals as either the unit ideal or those ideals whose prime factors divide the discriminant) in k are the ideals whose norms from k to Q are 1, d_k, δ or $Sfk(d_k/\delta)$; this proves (i). To prove (ii), notice that $B = A^2(\varepsilon_2 + 1)$ where $N(A)$ divides d_2. Then $\delta_2 = SfkN(\varepsilon_2 + 1)$ and B is a principal ideal in $Q(\sqrt{d_2})$ since A^2 is principal, and therefore B is a principal ideal in L.

Note. We have now narrowed the ambiguous case $(2,2,2)$ to the two remaining ambiguous cases (2B,2B,2A) for which G may be M or NM, and (2A,2A,2A) for which G may be MC or NM.

We distinguish between the capitulations (4,2A,2A) (nonmetacyclic) and (2A,2A,4) (metacyclic-nonmodular) by determining which unramified quadratic extension corresponds to the noncyclic factor group of the maximal subgroup. The following lemma enables us to do this.

We recall the Kaplan criteria for the 4-rank of C_k not being equal to the 4-rank of C_k^*, where d is the square-free kernel of the discriminant of k and d is not divisible by a prime congruent to 3 mod 4. The criteria is $d = a^2 + b^2$ with $0 < a$, a odd, and $(a/p_i) = -1$ for some odd prime dividing d.

Lemma 8. *Let $k = Q(\sqrt{d})$ with $d = p_1p_2p_3$ where $p_1 \equiv p_2 \equiv 1 \bmod 4$ and $p_3 \equiv 1 \bmod 4$ or $p_3 = 2$, such that $(p_i/p_j) = 1$ for all $i \neq j$, $i, j \in \{1, 2, 3\}$. Let $d = a^2 + b^2$ where a is an odd positive integer. Suppose $(a/p_i) = -1$ for some odd prime p_i dividing d. Then there exists a unique $j \in \{1, 2, 3\}$ such that $(p_j/a) = 1$. Moreover, for this j, $\overline{H_{p_j}}$ is noncyclic, where $\overline{H_{p_j}}$ is the class group corresponding to $\mathrm{Gal}\,(k_1/K_{p_j})$ via the Artin map and $K_{p_j} = k(\sqrt{p_j})$.*

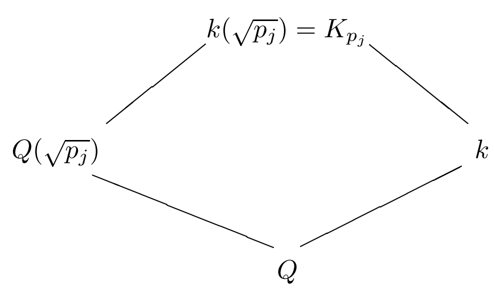
Remark 1. By Kaplan [**16**, p. 323] if $p_3 \equiv 1 \bmod 4$, then $(p_j/a) = (p_i p_l/p_j)_4 (p_j/p_i p_l)_4$ where $\{i,l\} = \{1,2,3\} - \{j\}$ and $(x/y)_4$ represents the biquadratic residue symbol; whereas, if $p_3 = 2$, $(2/a) = \prod_{i=1}^{2}(2p_i/p_j)_4(p_j/2p_i)_4$ with $j \in \{1,2\} - \{i\}$ (note by the hypothesis of Lemma 8 that $(p_j/2p_i)_4$ is always defined).

Remark 2. Notice that the hypotheses of the lemma insure by Lemma 2 that $N_{\varepsilon_0} = 1$ and that the 4-rank of $C_k^* = 2$ whereas the 4-rank of $C_k = 1$.

Proof of Lemma. The existence of a unique $j \in \{1,2,3\}$ such that $(p_j/a) = 1$ follows immediately from Kaplan [**16**, Theorem 1] (the law of biquadratic reciprocity).

Let P_l be the prime ideal of k containing p_l, $l \in \{1,2,3\}$. We claim that exactly one of P_l is principal. This follows immediately from Cohn [**8**], see proof of Lemma 7 above, since $\delta = p_i$ or $\delta = p_j p_k$ for particular values $\{i,j,k\} = \{1,2,3\}$.

Let P_i be one of the nonprincipal ideals. Next we claim that if $M = (a, b+\sqrt{d})$, then the ideal class $\overline{M} \neq \overline{P}_i$, $\overline{M} \neq \overline{I}$ and $\overline{M}^2 = \overline{I}$; by Cohn, $N_{k/Q}(M) = a$ and $M^2 = (b+\sqrt{d})$. This follows immediately because $\overline{M}$ and $\overline{P}_i$ must generate all ambiguous ideal classes, since by Cohn the only ambiguous ideal classes are those containing only ramified primes and $\overline{M}$, and the 2-rank of C_k equals 2. By class field theory, we know that the ideal class $\overline{P}_i \in \overline{H_{p_j}}$ if and only if the Artin symbol $(P_i, K_{p_j}/k) = I$ where $K_{p_j} = k(\sqrt{p_j})$. We consider the following diagram:



By Lang [**19**], $\mathrm{res}_{Q(\sqrt{p_j})}(P_i, K_{p_j}/k) = (p_i, Q(\sqrt{p_j})/Q)^{f(P_i/p_i)} = (p_i, Q(\sqrt{p_j})/Q)$. But $(p_i, Q(\sqrt{p_j})/Q) = (p_j/p_i) = 1$ and $\mathrm{res}_{Q(\sqrt{p_j})}$

$(P_i, K_{p_j}/k) = I$ if and only if $(P_i, K_{p_j}/k) = I$. Similarly, $(M, K_{p_j}/k) = (p_j/a) = 1$. We therefore conclude that $\overline{P}_i \in \overline{H_{p_j}}$ and $\overline{M} \in \overline{H_{p_j}}$, $\overline{P}_i \neq \overline{M}$, and thus $\overline{H_{p_j}}$ is not cyclic.

In Table 2 we list the fields in cases 1 and 2 for which $N(\varepsilon_0) = 1$; δ, δ_1 and δ_2 are defined as in the paragraph above Lemma 4. The table has six columns. The first column gives the number and letter (with lower case index) of the case and subcase, respectively. The second column is divided into two parts: the top part gives the graph associated with the subcase; the plus and minus signs refer to the norms of the fundamental units (+1 or −1) of the real quadratic number field whose discriminant is determined by the product of the two vertices of the graph on either side of the respective plus or minus sign. The bottom part gives the possible values of δ, listed in column format. The third, fourth and fifth columns refer to the three unramified quadratic extensions of k. These three columns are each divided into three parts. The top part determines K_i, $i = A$, B or C, in the form $k(\sqrt{m_i})$ for some integer m_i. The middle part gives the possible values of δ_2 for K_i; a line signifies that δ_2 does not exist. The bottom part lists the capitulation determined by δ (of the second column) and the δ_2's (of the third to fifth columns). We note that since the capitulation is the same for all values of δ_2, it is only listed once, for example, "2A" signifies that all values of δ_2 are 2A. The sixth column lists the possible structures of $\mathrm{Gal}\,(k_2/k)$ for each subcase and value of δ and δ_2's. In a number of subcases we make use of the biquadratic residue symbol, see Remark 1 after Lemma 10, to determine the possible group structures of $\mathrm{Gal}\,(k_2/k)$; the relevant biquadratic residue symbols are also listed in column 6.

Remark. We note that Buell [**7**, p. 167] incorrectly states that for $p_1 \equiv p_2 \equiv p_3 \equiv 1 \bmod 4$, the negative Pell equation is solvable for $k = Q(\sqrt{p_1 p_2 p_3})$, implying $N\varepsilon_0 = -1$, when exactly one of the Kronecker symbols $(p_i/p_j) = -1$, omitting the requirement that $(p_i p_j/p_k)_4 = (p_k/p_i p_j)_4 = -1$, $\{i, j, k\} = \{1, 2, 3\}$. For more errors in Buell, see [**1**].

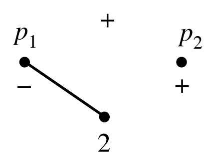
Through an observation of the referee, we make use of the following lemma to eliminate certain values of δ and δ_2 from occurring, see Kubota [**18**], Hiffssatz [**9**].

Lemma 9. *Let* $k = Q(\sqrt{m})$, *m square-free*, $m > 0$, $N\varepsilon_0 = 1$. *Then* $\delta \neq m$.

By way of illustration, we demonstrate how we obtain the data in Table 2 for cases $2B_1$ and $2C_4$; the remaining cases are obtained in a similar manner.

Case $2B_1$. $k = Q(\sqrt{2p_1p_2})$, $p_1 \equiv 5 \bmod 8$, $p_2 \equiv 1 \bmod 8$, $(p_1/p_2) = 1$, $N\varepsilon_0 = 1$, $N_{\varepsilon_{p_1p_2}} = 1$, $N_{\varepsilon_{2p_2}} = 1$, $N_{\varepsilon_{2p_1}} = -1$, by Buell [**7**].

Graph:



Since $d_k = 8p_1p_2$ and $\delta | d_k$, $\delta \neq 1$, $\delta \neq d_k$, see Kubota [**17**], and $\delta \neq 2p_1p_2$, cf. Lemma 9, we find that our initial possible values for δ are $p_1, p_2, 2, 2p_1, 2p_2$.

By Lemma 5 we know that $(\delta, d_k)/r = 1$ for all primes $r | d_k$.

We make use of the following graph theory techniques from Benjamin and Snyder [**4**] to narrow our possible values of δ that satisfy Lemma 5.

The particular criteria we are using in this case is the following: $(\delta, d_k)/r = 1$ if and only if

(i) the number of edges from r into $V(\delta)$ is even if r does not divide δ,

(ii) the number of edges from r into $V(Sfk(d_k/\delta))$ is

$$\begin{cases} \text{even if } r \mid \delta \text{ and } (r \equiv 1 \bmod 4 \text{ or } r = 2) \\ \text{odd if } r \mid \delta \text{ and } r \equiv 3 \bmod 4. \end{cases}$$

By the above criteria, we find that there are two possible values of δ: p_2 or $2p_1$, as listed in Table 2.

In a similar manner, we make use of the above graph theory criteria and Lemma 9 to determine the possible values of δ_2 for the unramified quadratic extensions $Q(\sqrt{2}, \sqrt{p_1p_2})$ where $d_2 = p_1p_2$ and $Q(\sqrt{p_1}, \sqrt{2p_2})$ where $d_2 = 8p_2$ (we note that δ_2 is not defined for $Q(\sqrt{p_2}, \sqrt{2p_1})$, where $d_2 = 8p_1$, since $N_{\varepsilon_{2p_1}} = -1$).

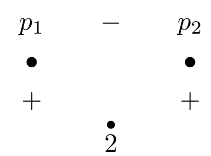
We find that, for $Q(\sqrt{2}, \sqrt{p_1p_2})$, $\delta_2 = p_1$ or p_2, and for $Q(\sqrt{p_1}, \sqrt{2p_2})$, $\delta_2 = p_2$ or 2. We now apply Lemma 4 to obtain the capitulation (2,2,2) for the three unramified quadratic extensions of k. In order to distinguish between Taussky's Conditions A or B, we apply Lemmas 6 and 7. We find that $Q(\sqrt{2}, \sqrt{p_1p_2})$ and $Q(\sqrt{p_1}, \sqrt{2p_2})$ satisfy Condition B and $Q(\sqrt{p_2}, \sqrt{2p_1})$ satisfies Condition A.

From Table 1 we are able to conclude that the capitulation is (2B,2B,2A) and Gal$\,(k_2/k)$ may be either modular or nonmetacyclic.

Case $2C_4$. $k = Q(\sqrt{2p_1p_2})$, $p_1 \equiv p_2 \equiv 1 \bmod 8$, $(p_1/p_2) = 1$,

$$N_{\varepsilon_0} = 1, \quad N_{\varepsilon_{p_1p_2}} = -1, \quad N_{\varepsilon_{2p_1}} = N_{\varepsilon_{2p_2}} = 1.$$

Graph:



In a similar manner to that of our example worked out for case $2B_1$, we find that there are six possible values of δ: 2, p_1p_2, p_1, $2p_2$, p_2, $2p_1$; for $Q(\sqrt{p_1}, \sqrt{2p_2})$, $\delta_2 = 2$ or p_2, and for $Q(\sqrt{p_2}, \sqrt{2p_1})$, $\delta_2 = 2$ or p_1 (we note that δ_2 is not defined for $Q(\sqrt{2}, \sqrt{p_1p_2})$ since $N_{\varepsilon_{p_1p_2}} = -1$). The δ values 2 and p_1p_2 have the same possible capitulations, as do the δ values p_1 and $2p_2$, and p_2 and $2p_1$, all of which are listed in Table 2. Applying Lemma 4 and Lemma 6, we find that for all values of δ and δ_2, either 4 ideal classes capitulate, or 2 ideal classes capitulate satisfying Taussky's Condition A, as described in Table 2. However, unlike our previous example for case $2B_1$, Table 1 does not suffice to always determine Gal$\,(k_2/k)$ based upon these capitulations. We need to make use of the biquadratic residue symbol, see Remark 1 after Lemma 8, to determine which unramified quadratic extension corresponds to the noncyclic factor group of the maximal subgroup of Gal$\,(k_2/k)$. For instance, when δ takes on the values p_1, $2p_2$, p_2 or $2p_1$, 4 ideal classes capitulate in $Q(\sqrt{2}, \sqrt{p_1p_2})$. If $\prod(2p_i/p_j)_4(p_j/2p_i)_4 = 1$

(we let $\prod = \prod_{i=1}^{2}$, $j \in \{1,2\} - \{i\}$) then we are able to specify our capitulation as (2A,2A,4) (denoting $\ker j_1$, $\ker j_2$, $\ker j_3$, respectively) by making use of Table 1 to obtain that $\mathrm{Gal}\,(k_2/k)$ is metacyclic-nonmodular. On the other hand, if $\prod(2p_i/p_j)_4(p_j/2p_i)_4 = -1$, then our capitulation is either (2A,4,2A) or (4,2A,2A), and from Table 1 we obtain that $\mathrm{Gal}\,(k_2/k)$ is nonmetacyclic. These biquadratic residue symbols, with all possible corresponding group structures for $\mathrm{Gal}\,(k_2/k)$ are listed in the last column of Table 2.

TABLE 2.

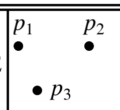
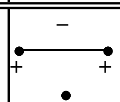
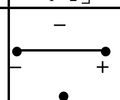
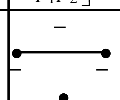
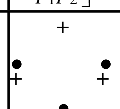
CASE		$\frac{K_A}{\delta_2}$	$\frac{K_B}{\delta_2}$	$\frac{K_C}{\delta_2}$	$\mathrm{Gal}(k_2/k)$
	δ	CAPI	TULA	TION	
$1A_1$		$k(\sqrt{p_1})$ (p_2, p_3)	$k(\sqrt{p_2})$ (p_1, p_3)	$k(\sqrt{p_3})$ ———	
	$\begin{matrix}p_3\\p_1p_2\end{matrix}\Big]$	2B	2B	2A	M or NM
$1A_2$		$k(\sqrt{p_1})$ (p_2, p_3)	$k(\sqrt{p_2})$ ———	$k(\sqrt{p_3})$ ———	
	$\begin{matrix}p_3\\p_1p_2\end{matrix}\Big]$	2B	4	2A	NM
$1A_3$		$k(\sqrt{p_1})$ ———	$k(\sqrt{p_2})$ ———	$k(\sqrt{p_3})$ ———	
	$\begin{matrix}p_3\\p_1p_2\end{matrix}\Big]$	4	4	2A	NM
$1B_1$		$k(\sqrt{p_1})$ (p_2, p_3)	$k(\sqrt{p_2})$ (p_1, p_3)	$k(\sqrt{p_3})$ (p_1, p_2)	
	$\begin{matrix}p_3\\p_1p_2\end{matrix}\Big]$	2A	2A	2A	MC or NM
	$\begin{matrix}p_1\\p_2p_3\end{matrix}\Big]$	2A	2A	2A	MC or NM
	$\begin{matrix}p_2\\p_1p_3\end{matrix}\Big]$	2A	2A	2A	MC or NM

TABLE 2. Continued.

CASE	p_1 • p_2 • • p_3 δ	$\frac{K_A}{\delta_2}$	$\frac{K_B}{\delta_2}$	$\frac{K_C}{\delta_2}$	$\mathrm{Gal}(k_2/k)$
		CAPI	TULA	TION	
$1B_2$	− • − • − •	$k(\sqrt{p_1})$ ———	$k(\sqrt{p_2})$ ———	$k(\sqrt{p_3})$ ———	
	p_3, p_1p_2	4	4	2A	NM
	p_1, p_2p_3	2A	4	4	NM
	p_2, p_1p_3	4	2A	4	NM
$1B_3$	− • + • + •	$k(\sqrt{p_1})$ (p_2, p_3)	$k(\sqrt{p_2})$ (p_1, p_3)	$k(\sqrt{p_3})$ ———	
	p_3, p_1p_2	2A	2A	2A	MC or NM
	p_1, p_2p_3	2A	2A	4	$\left[\left(\frac{p_1p_2}{p_3}\right)_4\left(\frac{p_3}{p_1p_2}\right)_4 = \begin{cases} 1: MC \\ -1: NM \end{cases}\right]$
	p_2, p_1p_3	2A	2A	4	
$1B_4$	− • + • − •	$k(\sqrt{p_1})$ ———	$k(\sqrt{p_2})$ (p_1, p_3)	$k(\sqrt{p_3})$ ———	
	p_3, p_1p_2	4	2A	2A	$\left[\left(\frac{p_2p_3}{p_1}\right)_4\left(\frac{p_1}{p_2p_3}\right)_4 = \begin{cases} 1: MC \\ -1: NM \end{cases}\right]$
	p_1, p_2p_3	2A	2A	4	$\left[\left(\frac{p_1p_2}{p_3}\right)_4\left(\frac{p_3}{p_1p_2}\right)_4 = \begin{cases} 1: MC \\ -1: NM \end{cases}\right]$
	p_2, p_1p_3	4	2A	4	NM

TABLE 2. Continued.

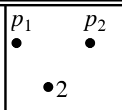
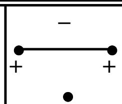
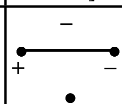
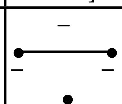
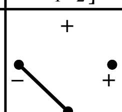
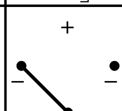
CASE		$\frac{K_A}{\delta_2}$	$\frac{K_B}{\delta_2}$	$\frac{K_C}{\delta_2}$	$\mathrm{Gal}(k_2/k)$
	δ	CAPI	TULA	TION	
$2A_1$		$k(\sqrt{p_1})$ $(p_2, 2)$	$k(\sqrt{p_2})$ $(p_1, 2)$	$k(\sqrt{2})$ ———	
	$\begin{matrix}2\\p_1p_2\end{matrix}\Big]$	2B	2B	2A	M or NM
$2A_2$		$k(\sqrt{p_1})$ ———	$k(\sqrt{p_2})$ $(p_1, 2)$	$k(\sqrt{2})$ ———	
	$\begin{matrix}2\\p_1p_2\end{matrix}\Big]$	4	2B	2A	NM
$2A_3$		$k(\sqrt{p_1})$ ———	$k(\sqrt{p_2})$ ———	$k(\sqrt{2})$ ———	
	$\begin{matrix}2\\p_1p_2\end{matrix}\Big]$	4	4	2A	NM
$2B_1$		$k(\sqrt{p_1})$ $(p_2, 2)$	$k(\sqrt{p_2})$ ———	$k(\sqrt{2})$ (p_1, p_2)	
	$\begin{matrix}p_2\\2p_1\end{matrix}\Big]$	2B	2A	2B	M or NM
$2B_2$		$k(\sqrt{p_1})$ ———	$k(\sqrt{p_2})$ ———	$k(\sqrt{2})$ (p_1, p_2)	
	$\begin{matrix}p_2\\2p_1\end{matrix}\Big]$	4	2A	2B	NM

TABLE 2. Continued.

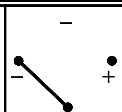
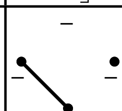
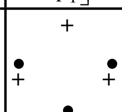
CASE	p_1 p_2 $\bullet 2$	$\frac{K_A}{\delta_2}$	$\frac{K_B}{\delta_2}$	$\frac{K_C}{\delta_2}$	$\mathrm{Gal}(k_2/k)$
	δ	CAPI	TULA	TION	
$2B_3$		$k(\sqrt{p_1})$ $(p_2, 2)$	$k(\sqrt{p_2})$ ——	$k(\sqrt{2})$ ——	
	2, $2p_1$	2B	2A	4	NM
$2B_4$		$k(\sqrt{p_1})$ ——	$k(\sqrt{p_2})$ ——	$k(\sqrt{2})$ ——	
	p_2, $2p_1$	4	2A	4	NM
$2C_1$		$k(\sqrt{p_1})$ $(p_2, 2)$	$k(\sqrt{p_2})$ $(p_1, 2)$	$k(\sqrt{2})$ (p_1, p_2)	
	2, p_1p_2	2A	2A	2A	MC or NM
	p_1, $2p_2$	2A	2A	2A	MC or NM
	p_2, $2p_1$	2A	2A	2A	MC or NM

TABLE 2. Continued.

CASE	p_1 p_2 • • •2	$\frac{K_A}{\delta_2}$	$\frac{K_B}{\delta_2}$	$\frac{K_C}{\delta_2}$	$\mathrm{Gal}(k_2/k)$
	δ	CAPI	TULA	TION	
$2C_2$	− + −	$k(\sqrt{p_1})$ ______	$k(\sqrt{p_2})$ $(p_1, 2)$	$k(\sqrt{2})$ ______	
	$\left.\begin{matrix}2\\ p_1p_2\end{matrix}\right]$	4	2A	2A	$\left(\frac{2p_2}{p_1}\right)_4\left(\frac{p_1}{2p_2}\right)_4 = \begin{cases}1: MC\\ -1: NM\end{cases}$
	$\left.\begin{matrix}p_1\\ 2p_2\end{matrix}\right]$	2A	2A	4	$\prod\left(\frac{2p_i}{p_j}\right)_4\left(\frac{p_j}{2p_i}\right)_4 = \begin{cases}1: MC\\ -1: NM\end{cases}$
	$\left.\begin{matrix}p_2\\ 2p_1\end{matrix}\right]$	4	2A	4	NM
$2C_3$	+ − −	$k(\sqrt{p_1})$ ______	$k(\sqrt{p_2})$ ______	$k(\sqrt{2})$ (p_1, p_2)	
	$\left.\begin{matrix}2\\ p_1p_2\end{matrix}\right]$	4	4	2A	NM
	$\left.\begin{matrix}p_1\\ 2p_2\end{matrix}\right]$	2A	4	2A	$\left(\frac{2p_1}{p_2}\right)_4\left(\frac{p_2}{2p_1}\right)_4 = \begin{cases}1: MC\\ -1: NM\end{cases}$
	$\left.\begin{matrix}p_2\\ 2p_1\end{matrix}\right]$	4	2A	2A	$\left(\frac{2p_2}{p_1}\right)_4\left(\frac{p_1}{2p_2}\right)_4 = \begin{cases}1: MC\\ -1: NM\end{cases}$

TABLE 2. Continued.

▯ CASE	p_1 • p_2 • • 2 δ	$\frac{K_A}{\delta_2}$	$\frac{K_B}{\delta_2}$	$\frac{K_C}{\delta_2}$	$\mathrm{Gal}\left(k_2/k\right)$
		C A P I	T U L A	T I O N	
$2C_4$	$-$ • • $+$ $+$ •	$k\left(\sqrt{p_1}\right)$ $(2,\ p_2)$	$k\left(\sqrt{p_2}\right)$ $(p_1,\ 2)$	$k\left(\sqrt{2}\right)$ _____	
	2 p_1p_2]	2A	2A	2A	MC or NM
	p_1 $2p_2$]	2A	2A	4	$\left[\prod\left(\frac{2p_i}{p_j}\right)_4\left(\frac{p_j}{2p_i}\right)_4 = \begin{cases} 1: MC \\ -1: NM \end{cases}\right]$
	p_2 $2p_1$]	2A	2A	4	
$2C_5$	$+$ • • $-$ $+$ •	$k\left(\sqrt{p_1}\right)$ $(2,\ p_2)$	$k\left(\sqrt{p_2}\right)$ _____	$k\left(\sqrt{2}\right)$ $(p_1,\ p_2)$	
	2 p_1p_2]	2A	4	2A	$\left(\frac{2p_1}{p_2}\right)_4\left(\frac{p_2}{2p_1}\right)_4 = \begin{cases} 1: MC \\ -1: NM \end{cases}$
	p_1 $2p_2$]	2A	4	2A	$\left(\frac{2p_2}{p_1}\right)_4\left(\frac{p_1}{2p_2}\right)_4 = \begin{cases} 1: MC \\ -1: NM \end{cases}$
	p_2 $2p_1$]	2A	2A	2A	MC or NM

TABLE 2. Continued.

CASE	p_1 • p_2 • • 2	$\frac{K_A}{\delta_2}$	$\frac{K_B}{\delta_2}$	$\frac{K_C}{\delta_2}$	$\mathrm{Gal}(k_2/k)$
	δ	CAPITULATION			
$2C_6$	− (top); • − , • − ; •	$k(\sqrt{p_1})$	$k(\sqrt{p_2})$	$k(\sqrt{2})$	
	2, p_1p_2	4	4	2A	NM
	p_1, $2p_2$	2A	4	4	NM
	p_2, $2p_1$	4	2A	4	NM

From Table 2 and our previous lemmas, we can now state the following theorem.

Theorem 1. *Let* $k = Q(\sqrt{p_1p_2p_3})$ *be a real quadratic number field,* $p_1 \equiv p_2 \equiv 1 \bmod 4$*,* $p_3 \equiv 1 \bmod 4$ *or* $p_3 = 2$*, with* $C_{k,2} \cong (2, 2^n)$*,* $n \geq 2$*,* $N\varepsilon_0 = 1$*. If the* 4*-rank of* C_k *is equal to the* 4*-rank of* C_k^**, then* $\mathrm{Gal}\,(k_2/k)$ *is either modular or nonmetacyclic. If the* 4*-rank of* C_k *is not equal to the* 4*-rank of* C_k^**, then* G *is either metacylic-nonmodular or nonmetacyclic.*

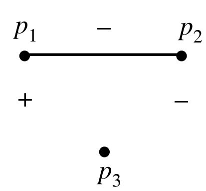
3. Determination of $\mathrm{Gal}\,(k_2/k)$ **when** $N_{\varepsilon_0} = -1$**.** We now assume k is a real quadratic number field with $C_{k,2} \cong (2, 2^n)$, $n \geq 2$, δ_k not divisible by a prime congruent to $3 \bmod 4$, and $N_{\varepsilon_0} = -1$. Since δ is not defined in this case, we must rely on Lemma 3 directly to determine the order of the capitulation kernel.

In Table 3, we list the fields in cases 1 and 2 for which $N(\varepsilon_0) = -1$; therefore, δ is not defined for the fields in this table. This table has eight columns. We let subscripts added to the general cases 1A, 2A and 2B represent all possibilities for the norms of the fundamental units of the respective real quadratic number fields; these are listed in column 1. In column 2 we list the norm of the fundamental unit of the base field, N_{ε_0}. In columns 3, 4 and 5 we list the norms of the fundamental units of the three real quadratic subfields. In column 6 we list the relevant fundamental unit criteria to determine the capitulation, as described in Lemma 3. In column 7 we list the capitulation determined by the fundamental unit criteria in column 6. In column 8 we list the possible group structures for each subcase, related to the fundamental unit criteria in column 6. We illustrate the technique by showing how we obtain the data in Table 3 for case $1A_2$.

Case $1A_2$. $k = Q(\sqrt{p_1p_2p_3})$, $p_1 \equiv p_2 \equiv p_3 \equiv 1 \bmod 4$,

$$\left(\frac{p_1}{p_3}\right) = \left(\frac{p_2}{p_3}\right) = 1, \qquad \left(\frac{p_1}{p_2}\right) = -1$$
$$N_{\varepsilon_0} = N_{\varepsilon_{p_1p_2}} = -1 \quad \text{(by Buell [}\mathbf{7}\text{])}$$
$$N_{\varepsilon_{p_1p_3}} = 1, N_{\varepsilon_{p_2p_3}} = -1 \quad \text{(without loss of generality).}$$

Graph:



By Lemma 3, we see that for $Q(\sqrt{p_2}, \sqrt{p_1p_3})$, four ideal classes capitulate.

In order to determine how many ideal classes capitulate for $Q(\sqrt{p_3}, \sqrt{p_1p_2})$ and $Q(\sqrt{p_1}, \sqrt{p_2p_3})$ we must make use of the condition $\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \in K$ as stated in Lemma 3. The three possibilities for $\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \in K$ or $\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \notin K$, where $K = Q(\sqrt{p_1}, \sqrt{p_2p_3})$ or $Q(\sqrt{p_3}, \sqrt{p_1p_2})$, are listed in Table 3. In the case where $\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \in K$, we must make use of Lemma 8 to determine if K satisfies the Taussky Condition A or B. From Table 1 we are able to conclude that the three possible capitulations are (2B,4,2A), (4,4,2A), (4,4,4), and the possibilities for Gal(k_2/k) are respectively nonmetacyclic, nonmetacyclic and abelian.

Note. In Table 3 we notice that, for $p_1 \equiv p_2 \equiv p_3 \equiv 1 \bmod 4$, the case $N\varepsilon_{p_1p_3} = -1$, $N\varepsilon_{p_2p_3} = 1$ is symmetrical to the case $N\varepsilon_{p_1p_3} = 1$, $N\varepsilon_{p_2p_3} = -1$, and the case $N\varepsilon_{2p_1} = -1$, $N\varepsilon_{2p_2} = 1$ is symmetrical to the case $N\varepsilon_{2p_1} = 1$, $N\varepsilon_{2p_2} = -1$; therefore, these cases are not included as separate cases. Similarly the capitulation (2B,4,2A) is symmetrical to the capitulation (4,2B,2A) and is therefore not included as a separate capitulation.

To simplify our categories we now define $k = Q(\sqrt{p_1p_2p_3})$ with $p_1, p_2, p_3 \equiv 1$ or $2 \pmod 4$.

From Table 3 and our previous lemmas we are able to state the following theorem.

Theorem 2. *Let k be a real quadratic number field, $p_1 \equiv p_2 \equiv 1 \bmod 4$, $p_3 \equiv 1 \bmod 4$ or $p_3 = 2$, with $C_{k,2} \cong (2, 2^n)$, $n \geq 2$, $N\varepsilon_0 = -1$. Then* $\mathrm{Gal}\,(k_2/k)$ *is nonabelian if and only if there exists an unramified quadratic extension K of k for which $N_{\varepsilon_2} = -1$ and $\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \in K$. If* $\mathrm{Gal}\,(k_2/k)$ *is nonabelian, then* $\mathrm{Gal}\,(k_2/k)$ *must be either modular or nonmetacyclic.*

By combining Lemma 1 and Theorem 2 we are able to state the following corollary which relates our unit roots, see Cohn [**9**], to biquadratic residue symbols.

Corollary 1. *Let k be a real quadratic number field, $p_1 \equiv p_2 \equiv 1 \bmod 4$, $p_3 \equiv 1 \bmod 4$ or $p_3 = 2$, with $C_{k,2} \cong (2, 2^n)$, $n \geq 2$, $N\varepsilon_0 = -1$, $(p_i/p_j) = (p_j/p_k) = 1$, $(p_k/p_i) = -1$ for $\{i, j, k\} = \{1, 2, 3\}$. Then there exists an unramified quadratic extension K of k for which $N_{\varepsilon_2} = -1$ and $\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \in K$, if and only if either $(p_i/p_j)_4(p_j/p_i)_4 = 1$ or $(p_j/p_k)_4(p_k/p_j)_4 = 1$.*

We conclude with a numerical example that illustrates how Table 3 can be used to determine $\mathrm{Gal}\,(k_2/k)$ for a particular field.

Let $k = Q(\sqrt{2 \cdot 13 \cdot 17} = Q(\sqrt{442})$; $(17/13) = 1$, $13 \equiv 5 \bmod 8$, $17 \equiv 1 \bmod 8$, $N(\varepsilon_0) = -1$, $N(\varepsilon_{13 \cdot 17}) = 1$, $N(\varepsilon_{2 \cdot 13}) = -1$, $N(\varepsilon_{2 \cdot 17}) = 1$. From Table 3, case $2B_2$, we see that G is either abelian or

TABLE 3.

Case	N_{ε_0}	$N_{\varepsilon_{p_1p_2}}$	$N_{\varepsilon_{p_1p_3}}$	$N_{\varepsilon_{p_2p_3}}$	Fundamental Unit Criteria	Capitulation	$\mathrm{Gal}\,(k_2/k) = G$
$1A_1$ $2A_1$ $2B_1$	-1	-1	1	1	$\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \in Q(\sqrt{p_3}, \sqrt{p_1p_2})$	(4,4,2A)	NM
					$\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \notin Q(\sqrt{p_3}, \sqrt{p_1p_2})$	(4,4,4)	A
$1A_2$ $2A_2$ $2B_2$	-1	-1	1	-1	$\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \in Q(\sqrt{p_3}, \sqrt{p_1p_2})$ and $\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \in Q(\sqrt{p_1}, \sqrt{p_2p_3})$	(2B,4,2A)	NM
					$\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \in Q(\sqrt{p_3}, \sqrt{p_1p_2})$ and $\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \notin Q(\sqrt{p_1}, \sqrt{p_2p_3})$	(4,4,2A)	NM
					$\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \notin Q(\sqrt{p_3}, \sqrt{p_1p_2})$ and $\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \notin Q(\sqrt{p_1}, \sqrt{p_2p_3})$	$(4, 4, 4)$	A

TABLE 3. Continued.

Case	N_{ε_0}	$N_{\varepsilon_{p_1p_2}}$	$N_{\varepsilon_{p_1p_3}}$	$N_{\varepsilon_{p_2p_3}}$	Fundamental Unit Criteria	Capitulation	Gal $(k_2/k) = G$
$1A_3$ $2A_3$ $2B_3$	-1	-1	-1	-1	$\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \in Q(\sqrt{p_i}, \sqrt{p_jp_k})$ for all i, j, k such that $\{i, j, k\} = \{1, 2, 3\}$	(2B,2B,2A)	M or NM
					$\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \notin Q(\sqrt{p_i}, \sqrt{p_jp_k})$ for all i, j, k such that $\{i, j, k\} = \{1, 2, 3\}$	(4,4,4)	A
					$\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \in Q(\sqrt{p_3}, \sqrt{p_1p_2})$ and $\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \in Q(\sqrt{p_1}, \sqrt{p_2p_3})$ and $\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \notin Q(\sqrt{p_2}, \sqrt{p_1p_3})$	(2B,4,2A)	NM
					$\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \in Q(\sqrt{p_3}, \sqrt{p_1p_2})$ and $\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \notin Q(\sqrt{p_1}, \sqrt{p_2p_3})$ and $\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \notin Q(\sqrt{p_2}, \sqrt{p_1p_3})$	$(4, 4, 2A)$	NM

nonmetacyclic. We check further to see if $\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \in Q(\sqrt{17}, \sqrt{2\cdot 13})$

$$\begin{aligned}\varepsilon_0 &= 21 + \sqrt{442}\\ \varepsilon_1 &= 4 + \sqrt{17}\\ \varepsilon_2 &= 5 + \sqrt{26}\\ \varepsilon_0\varepsilon_1\varepsilon_2 &= 862 + 209\sqrt{17} + 169\sqrt{26} + 41\sqrt{442}.\end{aligned}$$

Let $K = Q(\sqrt{17}, \sqrt{2\cdot 13})$. Set

$$\begin{aligned}C_0 &= Tr_{K/Q}(\varepsilon_0\varepsilon_1\varepsilon_2 + \varepsilon_0 + \varepsilon_1 - \varepsilon_2) = 3528\\ C_1 &= Tr_{K/Q}(\varepsilon_0\varepsilon_1\varepsilon_2 + \varepsilon_0 - \varepsilon_1 + \varepsilon_2) = 3536\\ C_2 &= Tr_{K/Q}(\varepsilon_0\varepsilon_1\varepsilon_2 - \varepsilon_0 + \varepsilon_1 + \varepsilon_2) = 3400\\ C_3 &= Tr_{K/Q}(\varepsilon_0\varepsilon_2\varepsilon_2 - \varepsilon_0 - \varepsilon_1 - \varepsilon_2) = 3328\end{aligned}$$

$$\begin{aligned}\sqrt{C_0} &= 42\sqrt{2}\\ \sqrt{C_1} &= 4\sqrt{13\cdot 17}\\ \sqrt{C_2} &= 10\sqrt{2\cdot 17}\\ \sqrt{C_3} &= 16\sqrt{13}.\end{aligned}$$

Since $\sqrt{C_j} \notin K$ for $j = 0, 1, 2, 3$, we have $\sqrt{\varepsilon_0\varepsilon_1\varepsilon_2} \notin K$ by Kubota [**18**].

Therefore, by Table 3, G is abelian.

We note that our earlier work, as stated in Lemma 1, also demonstrates that $\mathrm{Gal}\,(k_2/k)$ is abelian for $k = Q(\sqrt{2\cdot 13\cdot 17})$, since $(17/13)_4 = (2/17)_4 = -1$ and $(13/17)_4 = (17/2)_4 = 1$.

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