ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 29, Number 3, Fall 1999

CONFORMAL IMAGES OF TANGENTIAL AND NONTANGENTIAL ARCS

JOHN AKEROYD

If f is bounded and analytic in $\mathbf{D} := \{z : |z| < 1\}$ and $\lim_{r\to 1^-} f(re^{i\theta})$ exists for some θ, then, by a normal families argument, $f(z)$ approaches that radial limit as z in $\mathbf{D}$ approaches $e^{i\theta}$ along any nontangential path; see [**1**, Theorem 1.3, p. 6]. In this note we give an analogous result for functions that are analytic and univalent in $\mathbf{D}$; with no loss of generality, we let $e^{i\theta} = 1$ throughout. We first observe that, for any function f that is both analytic and univalent in $\mathbf{D}$, $f([0,1))$ is rectifiable if and only if $f(\gamma\backslash\{1\})$ is rectifiable for each rectifiable Jordan arc γ contained in $\mathbf{D}\cup\{1\}$ that has a nontangential approach in $\mathbf{D}$ to 1 and that satisfies a certain restriction on its "oscillations" near 1 (Theorem 1). We also show that if γ has a tangential approach in $\mathbf{D}$ to 1, then there is a Jordan region Ω and a conformal mapping φ from $\mathbf{D}$ to Ω such that $\varphi([0,1]) = [0,1]$ and yet $\varphi(\gamma)$ is not rectifiable (Theorem 2); for a related result, see [**5**].

To establish the terms of our discussion, let γ be a Jordan arc from $[0,1]$ to the complex plane $\mathbf{C}$ such that $\gamma([0,1))$ is contained in $\mathbf{D}$ and $\gamma(1) = 1$. If the limit as t approaches 1 of $(1-|\gamma(t)|)/(|1-\gamma(t)|)$ exists and is zero, then we say that γ has a *tangential approach* in $\mathbf{D}$ to 1. And, if there exists $\varepsilon > 0$ such that $\varepsilon \le (1-|\gamma(t)|)/(|1-\gamma(t)|)$ whenever $0 \le t < 1$, then we say that γ has a *nontangential approach* in $\mathbf{D}$ to 1. Throughout this paper we let γ denote both the Jordan arc and its *trace* $\gamma([0,1])$. Let $T(z) = (1-z)/(1+z)$ be the Möbius transformation that maps $\{z : \mathrm{Re}\,(z) > 0\}$ onto $\mathbf{D}$, 0 to 1 and 1 to 0. For each nonnegative integer n, let $a_n = T(2^{-n})$ $(= (2^n-1)/(2^n+1))$; notice that $\rho(a_n, a_{n+1}) = (1/3)$ for all n, where $\rho(z,w) := |(z-w)/(1-\bar{w}z)|$ is the *pseudohyperbolic* distance between the points z and w in $\mathbf{D}$. If γ is a rectifiable Jordan arc in $\mathbf{D}\cup\{1\}$, then, for $n = 0,1,2,\dots$, let $\gamma_n = \{z \in \gamma : a_n \le |z| < a_{n+1}\}$ and (with the reference to γ understood), let $M_n = \mathrm{length}\,(\gamma_n)/(a_{n+1}-a_n)$; $\mathrm{length}\,(\gamma_n) := \Lambda_1(\gamma_n)$—the one-dimensional Hausdorff measure of γ_n. For $0 < \varepsilon < 1$ and any

Received by the editors on April 25, 1997, and in revised form on July 31, 1997.

positive integer m, let $\varphi_{m,\varepsilon}(z) = [\log(\log(\cdots(\log(1/z))\cdots))]^{-\varepsilon}$; here, the branch of the logarithm (given by $-\pi < \arg(z) < \pi$) is composed with itself m-times. Notice that for sufficiently small $r > 0$, $\varphi_{m,\varepsilon}$ is a conformal mapping from $\{z : |z| < r \text{ and } \mathrm{Re}\,(z) > 0\}$ onto a bounded subregion of $\{\xi : \mathrm{Re}\,(\xi) > 0\}$ that is symmetric with respect to $\mathbf{R}$ and, for such z,

$$\varphi'_{m,\varepsilon}(z) = \frac{\varepsilon}{(z\log(1/z)\log(\log(1/z))\cdots[\log(\log(\cdots(\log(1/z))\cdots))]^{1+\varepsilon})}.$$

For sufficiently large n, let $\alpha_{m,\varepsilon}(n) = (1/(\varepsilon n^2))\cdot\varphi'_{m,\varepsilon}(1/n)$. Later in this paper we shall concern ourselves with the series $\sum_{n=N}^{\infty}\alpha_{m,\varepsilon}(n)$; a study of this series is made in [**4**, Sections 14 and 37–41].

Theorem 1. *Let γ be a rectifiable Jordan arc in $\mathbf{D}\cup\{1\}$ that has a nontangential approach in $\mathbf{D}$ to $\{1\}$.*

(i) *If the sequence $\{M_n\}_{n=0}^{\infty}$ is bounded, then, for any conformal mapping φ defined on $\mathbf{D}$, $\varphi(\gamma\backslash\{1\})$ is rectifiable if and only if $\varphi([0,1))$ is rectifiable.*

(ii) *If the sequence $\{M_n\}_{n=0}^{\infty}$ is not bounded and there exist m, ε and a subsequence $\{M_{n_k}\}$ of $\{M_n\}$ such that $\{(n_{k+1}-n_k)\}$ is bounded and, for sufficiently large n_k,*

$$M_{n_k} \geq \varepsilon\Big/\bigg(\sum_{n=n_k}^{\infty}\alpha_{m,\varepsilon}(n)\bigg),$$

then there is a conformal mapping φ on $\mathbf{D}$ such that $\varphi([0,1))$ is rectifiable and yet $\varphi(\gamma\backslash\{1\})$ is not rectifiable.

Remark. The condition in Theorem 1(i), that $\{M_n\}_{n=0}^{\infty}$ is bounded, is almost certainly not "sharp." However, the slowness of the growth that is permitted of $\{M_{n_k}\}$ in (ii) indicates that the condition in (i) is nearly sharp in certain settings. The guiding principle is that a given rate of growth of $\{(n_{k+1}-n_k)\}$ requires a commensurate rate of growth of $\{M_{n_k}\}$ in order to insure the result of Theorem 1(ii). A precise understanding of this interplay between the rates of growth of $\{(n_{k+1}-n_k)\}$ and $\{M_{n_k}\}$ seems inaccessible since it requires a thorough understanding of what $|\varphi'|\big|_{[0,1)}$ can look like, where φ is a conformal mapping on $\mathbf{D}$ such that $\varphi([0,1))$ is rectifiable.

Proof (of Theorem 1). (i) In this setting $M := \sup_n M_n < \infty$. By our choice of $\{a_n\}_{n=0}^{\infty}$ and the assumption that γ has a nontangential approach in $\mathbf{D}$ to 1, we can apply [**3**, Theorem 4, p. 52] along with the chain rule (or we can apply [**6**, Lemma 2.2, p. 130]) and find a positive constant C, independent of φ and n, such that $|\varphi'(z)| \leq C \cdot \min\{|\varphi'(t)| : a_n \leq t \leq a_{n+1}\}$ whenever $z \in \gamma_n$. Therefore,

$$\begin{aligned}
\text{length}\,(\varphi(\gamma\backslash\{1\})) &= \sum_{n=0}^{\infty} \text{length}\,(\varphi(\gamma_n)) \\
&= \sum_{n=0}^{\infty} \int_{\gamma_n} |\varphi'(z)|\,|dz| \\
&\leq CM \cdot \sum_{n=0}^{\infty} \min\{|\varphi'(t)| : a_n \leq t \leq a_{n+1}\} \cdot (a_{n+1} - a_n) \\
&\leq CM \cdot \sum_{n=0}^{\infty} \text{length}\,(\varphi([a_n, a_{n+1}])) \\
&= CM \cdot \text{length}\,(\varphi([0,1))).
\end{aligned}$$

So, if $\varphi([0,1))$ is rectifiable, then so is $\varphi(\gamma\backslash\{1\})$. The converse holds similarly.

(ii) By our hypothesis, there exist m and ε such that, for sufficiently large n_k,

$$M_{n_k} \geq \varepsilon \Big/ \left(\sum_{n=n_k}^{\infty} \alpha_{m,\varepsilon}(n) \right).$$

For w in $\mathbf{D}$, let $T(w) = (1-w)/(1+w)$ $(= T^{-1})$, let $\Gamma = T(\gamma)$ and, for $n = 0, 1, 2, \ldots$, let $\Gamma_n = T(\gamma_n)$ and let $M_n^* = 2^{(n+1)} \cdot \text{length}\,(\Gamma_n)$. By a routine conformal mapping argument, we need only produce a conformal mapping φ on $\{z : |z| < r \text{ and } \operatorname{Re}(z) > 0\}$, for some $r > 0$, such that $\varphi((0,r))$ is rectifiable and yet $\varphi(\{z \in \Gamma : 0 < |z| < r\})$ is not. Now since $|T'(w)|$ is near $1/2$ when w is near 1, we can make a smaller choice of $\varepsilon > 0$ if necessary and get (from our hypothesis) that

$$M_{n_k}^* \geq \varepsilon \Big/ \left(\sum_{n=n_k}^{\infty} \alpha_{m,\varepsilon}(n) \right),$$

if n_k is sufficiently large. For fixed $r > 0$ sufficiently small, consider the conformal mapping $\varphi_{m+1,\varepsilon}$ defined on $\{z : |z| < r \text{ and } \operatorname{Re}(z) > 0\}$. By

[**3**, Theorem 4, p. 52] along with the chain rule, or by [**6**, Lemma 2.2, p. 130], there are positive constants C_1 and C_2 and, by the boundedness of $\{(n_{k+1} - n_k)\}$, there is a positive constant C_3 such that, for any k_0 sufficiently large,

$$
\begin{aligned}
&\sum_{k=k_0}^{\infty} \text{length}\,(\varphi_{m+1,\varepsilon}(\Gamma_{n_k})) \\
&= \sum_{k=k_0}^{\infty} \int_{\Gamma_{n_k}} |\varphi'_{m+1,\varepsilon}(z)|\,|dz| \\
&\geq \sum_{k=k_0}^{\infty} C_1 \cdot \max\{|\varphi'_{m+1,\varepsilon}(t)| : 2^{-(n_k+1)} \leq t \leq 2^{-n_k}\} \cdot M^*_{n_k} \cdot 2^{-(n_k+1)} \\
&\geq \left(\varepsilon C_1 \cdot \sum_{k=k_0}^{\infty} |\varphi'_{m+1,\varepsilon}(2^{-n_k})| 2^{-(n_k+1)}\right) \Big/ \left(\sum_{n=n_{k_0}}^{\infty} \alpha_{m,\varepsilon}(n)\right) \\
&\geq C_2 \cdot \left(\sum_{k=k_0}^{\infty} \alpha_{m,\varepsilon}(n_k)\right) \Big/ \left(\sum_{n=n_{k_0}}^{\infty} \alpha_{m,\varepsilon}(n)\right) \\
&\geq C_3.
\end{aligned}
$$

So, $\varphi_{m+1,\varepsilon}(\{z \in \Gamma : 0 < |z| < r\})$ is not rectifiable, though $\varphi_{m+1,\varepsilon}((0,r))$ is. $\square$

Theorem 2. *Let γ be a Jordan arc in $\mathbf{D} \cup \{1\}$ that has a tangential approach in $\mathbf{D}$ to 1. Then there is a Jordan region Ω and a conformal mapping φ from $\mathbf{D}$ onto Ω such that $\phi([0,1]) = [0,1]$ and yet $\varphi(\gamma)$ is not rectifiable.*

Proof. Let $\{a_n\}_{n=2}^{\infty}$ be a decreasing sequence of positive real numbers such that $a_n < (1/n^2)$. For $n = 2, 3, 4, \ldots$, let

$$
A_n = \left\{z : \frac{1}{n} - \frac{1}{n^3} \leq |z-1| \leq \frac{1}{n}\right\} \setminus \{z = x + iy : 0 < x < 1 \text{ and } -a_n < y < a_n\}.
$$

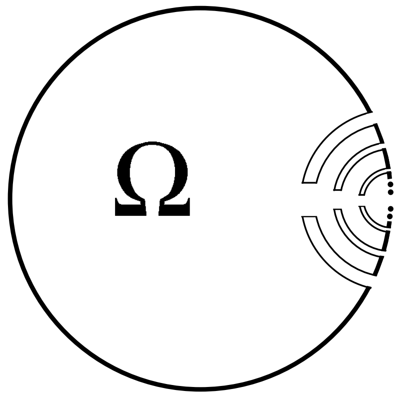

FIGURE 1.

Claim. *For an appropriate choice of* $\{a_n\}_{n=2}^{\infty}$*, the Jordan region*

$$\Omega := \mathbf{D} \backslash \Big(\bigcup_{n=2}^{\infty} A_n \Big),$$

(see Figure 1*) satisfies Theorem* 2*.*

Since Ω is symmetric with respect to the real line and is a Jordan region, there is a conformal mapping ϕ from $\mathbf{D}$ onto Ω, that extends to a homeomorphism between $\overline{\mathbf{D}}$ and $\overline{\Omega}$ such that $\phi([0,1]) = [0,1]$.

If $0 < s < 1$, then $\gamma([0,s])$ is a compact subset of $\mathbf{D}$ and so we can apply Harnack's Inequality and find a constant $c > 1$ such that

$$\frac{1}{c} \cdot \omega(\cdot, \mathbf{D}, z) \le \omega(\cdot, \mathbf{D}, 0) \le c \cdot \omega(\cdot, \mathbf{D}, z)$$

whenever $z \in \gamma([0,s])$; if E is a bounded Dirichlet region and $z_0 \in E$, then $\omega(\cdot, E, z_0)$ denotes harmonic measure on ∂E evaluated z_0. By elementary methods involving the Maximum Principle or by standard estimates derived from the theory of extremal length, see [**2**, Proposition 7.2, p. 102], we have that, for $\varepsilon > 0$,

$$\omega(\{w : |w| = 1 \text{ and } \operatorname{Re}(w) \le 0\}, \Omega, \zeta) < \varepsilon$$

whenever $\zeta \in \Omega$ and $|\zeta - 1| \leq 1/2$, provided a_2 is sufficiently small. However, since $G := \mathbf{D}\backslash\{z : |z-1| \leq 1/2\}$ is contained in Ω independent of a_2, the Maximum Principle tells us that

$$\omega(\{w : |w| = 1 \text{ and } \mathrm{Re}\,(w) \leq 0\}, \Omega, 0) \\ \geq \omega(\{w : |w| = 1 \text{ and } \mathrm{Re}\,(w) \leq 0\}, G, 0) > 0$$

independent of a_2. Since $\omega(\cdot, \Omega, \zeta) = \omega(\phi^{-1}(\cdot), \mathbf{D}, \phi^{-1}(\zeta))$, and $\phi(0) = 0$, we can now conclude that $|1 - \phi(z)| > 1/2$ for all z in $\gamma([0,s])$, provided a_2 is sufficiently small.

For z in $\mathbf{D}$, let $\rho(z) = \inf\{\rho(z,r) : 0 \leq r < 1\}$—the *pseudohyperbolic* distance from z to $[0,1)$—and let $\varphi_z(w) = (w-z)/(1-\bar{z}w)$. Since γ has a tangential approach in $\mathbf{D}$ to 1, $\rho(z)$ approaches 1 as z in $\gamma\backslash\{1\}$ approaches 1. Let $K_2 = \{z : |z-1| = 17/48 \text{ and } |\arg(1-z)| \leq \pi/4\}$, and let $r_2 = \phi^{-1}(17/48)$. Notice that $K_2 \subseteq \Omega$ and $\mathrm{dist}\,(K_2, \partial\Omega) = 1/48$ independent of a_n, $n = 2, 3, 4, \ldots$. So, by Harnack's Inequality, there is a constant $d > 1$ independent of a_n, $n = 2, 3, 4, \ldots$, such that

$$\frac{1}{d} \cdot \omega(\cdot, \Omega, \zeta) \leq \omega(\cdot, \Omega, \zeta') \leq d \cdot \omega(\cdot, \Omega, \zeta)$$

for any ζ and ζ' in K_2. So there exists R, $0 < R < 1$, such that $(\phi_{r_2} \circ \phi^{-1})(K_2) \subseteq \{w : |w| \leq R\}$ independent of a_n, $n = 2, 3, 4, \ldots$. Consequently, $\rho(z) \leq R$ for all z in $\phi^{-1}(K_2)$ independent of a_n, $n = 2, 3, 4, \ldots$. So there exists s, $0 < s < 1$, such that $\phi^{-1}(K_2) \cap \gamma([s,1]) = \varnothing$ independent of a_n, $n = 2, 3, 4, \ldots$. Moreover, by our earlier work, $K_2 \cap \phi(\gamma([0,s])) = \varnothing$ provided a_2 is sufficiently small. Consequently, $K_2 \cap \phi(\gamma([0,1])) = \varnothing$, provided a_2 is sufficiently small. In the same way we can choose a_n, for $n = 3, 4, 5, \ldots$, sufficiently small so that $K_n \cap \phi(\gamma([0,1])) = \varnothing$, where $K_n = \{z : |z-1| = ((1/n) + (1/(n+1)) - (1/n^3))/2 \text{ and } |\arg(1-z)| \leq \pi/4\}$, and our choice of a_n is not affected by our choice of a_k for $k > n$. Since $a_n < 1/n^2$, this forces the length of $\phi(\gamma)$ to be infinite. $\square$

Acknowledgment. The author is grateful to the referee for helpful suggestions.

REFERENCES

1. P.L. Duren, *Theory of H^p spaces*, Academic Press, New York, 1970.

2. W.H.J. Fuchs, *Topics in the theory of functions of one complex variable*, D. Van Nostrand Company, Princeton, New Jersey, 1967.

3. G.M. Goluzin, *Geometric theory of functions of a complex variable*, American Mathematical Society, New York, 1969.

4. K. Knopp, *Theory and application of infinite series*, Dover Publications, New York, 1990.

5. A.J. Lohwater, G. Piranian and W. Rudin, *The derivative of a Schlicht function*, Math. Scand. **3** (1955), 103–106.

6. C. Pommerenke, *Linear-invariante Familien analytischer Funktionen* I, Math. Ann. **155** (1964), 108–154.

Department of Mathematics, University of Arkansas, 301 Science-Engineering Building, Fayetteville, AR 72701
E-mail address: akeroyd@comp.uark.edu