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PRODUCTS OF FACTORIALS IN BINARY RECURRENCE SEQUENCES

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ABSTRACT. In this paper, we show that every nondegenerate binary recurrence sequence contains only finitely many terms which can be written as products of factorials. Moreover, all such terms can be effectively computed. We also find all the terms of the Fibonacci sequence which are products of factorials.

1. Introduction. Let α and β be nonzero algebraic integers, and let a and b be nonzero algebraic numbers. For any integer $n \geq 0$, let

$$u_n = a\alpha^n + b\beta^n. \tag{1}$$

It is clear that

$$u_{n+2} = ru_{n+1} + su_n \quad \text{for } n = 0, 1, \dots , \tag{2}$$

where $r = \alpha + \beta$ and $s = -\alpha\beta$. We refer to the sequence $(u_n)_{n\geq 0}$ as a *binary recurrence sequence*. If u_0 and u_1 are algebraic integers, then $(u_n)_{n\geq 0}$ is a binary recurrence sequence of algebraic integers. The sequence $(u_n)_{n\geq 0}$ is said to be nondegenerate if α/β is not a root of unity. We refer to the equation

$$x^2 - rx - s = 0$$

as the *characteristic equation* of the sequence $(u_n)_{n\geq 0}$.

Let $\mathcal{PF}$ be the set of all positive integers which can be written as products of factorials; that is,

$$\mathcal{PF} = \left\{ w \mid w = \prod_{j=1}^{k} m_j! \quad \text{for some } m_j \geq 1 \right\}. \tag{3}$$

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For any finite extension $\mathbf{F}$ of $\mathbf{Q}$, let $d_{\mathbf{F}} = [\mathbf{F} : \mathbf{Q}]$. For any algebraic number $\zeta \in \mathbf{F}$, let $N_{\mathbf{F}}(\zeta)$ denote the norm of ζ. For any algebraic integer γ in $\mathbf{F}$, let $[\gamma]_{\mathbf{F}}$ denote the ideal generated by γ in the ring of algebraic integers inside $\mathbf{F}$, and let $\prod_{\mathbf{F}}(\gamma)$ be the set of prime ideals of $\mathbf{F}$ dividing γ.

Now let $t \geq 1$ be an integer. For each $i = 1, \dots, t$, let α_i and β_i be nonzero algebraic integers, and let a_i and b_i be nonzero algebraic numbers. For any integer $n \geq 0$, let

$$u_{i,n} = a_i \alpha_i^n + b_i \beta_i^n. \tag{4}$$

We suppose that each sequence $(u_{i,n})_{n\geq 0}$ is nondegenerate. For each $t = 1, \dots, t$, we denote $r_i = \alpha_i + \beta_i$ and $s_i = -\alpha_i\beta_i$. Finally, let $\mathbf{K} = \mathbf{Q}(\alpha_i, \beta_i, a_i, b_i \mid i = 1, \dots, t)$.

The main result of this article is the following.

Theorem 1. *The equation*

$$\prod_{i=1}^{t} N_{\mathbf{K}}(u_{i,n_i}) \in \mathcal{PF} \tag{5}$$

has finitely many solutions $(n_1, \dots, n_t)$. *Moreover, there exists a computable number* C *depending only on* $d_{\mathbf{K}}$ *and the heights of the numbers* $a_i, b_i, \alpha_i, \beta_i$, *such that every solution of* (5) *satisfies*

$$\max(n_1, \dots, n_t) \leq C. \tag{6}$$

Under some restrictive hypothesis on the sequences $(u_{i,n})_{n\geq 0}$ we obtain inhomogeneous variants of Theorem 1. Let $f_1, \dots, f_t$ be polynomials with algebraic coefficients such that $f = \prod_{i=1}^{t} f_i$ is nonconstant.

Corollary 1. *Suppose that the nondegenerate sequences* $(u_{i,n})_{n\geq 0}$ *given by formula* (4) *satisfy* r_i, $s_i \in \mathbf{Z}$, *and* $|s_i| = 1$ *for* $i = 1, \dots, t$. *Then the equation*

$$\prod_{i=1}^{t} N_{\mathbf{K}}(f_i(u_{i,n_i})) \in \mathcal{PF} \tag{7}$$

has finitely many solutions $(n_1, \dots, n_t)$. *Moreover, there exists a computable number* C *depending only on the numbers* α_i, β_i, a_i, b_i *and the polynomials* f_i, *such that every solution of* (7) *satisfies*

$$\max(n_1, \dots, n_t) \le C. \tag{8}$$

Theorem 2. *Suppose that the sequences* $(u_{i,n})_{n\ge 0}$ *given by formula* (4) *have the same characteristic equation. Let* α, β, r *and* s *denote the common value of the* α_i*'s,* β_i*'s,* r_i*'s and* s_i*'s, respectively. Suppose that* $r, s \in \mathbf{Z}$. *Suppose also that either one of the following holds*:

(i) $\prod_{\mathbf{K}}(\alpha) \ne \prod_{\mathbf{K}}(\beta)$ *and* α *is real.*

(ii) $([\alpha]_{\mathbf{K}}, [\beta]_{\mathbf{K}}) \ne 1$ *and* $f(0) \ne 0$.

Then the equation

$$\prod_{i=1}^{t} N_{\mathbf{K}}(f_i(u_{i,n_i})) \in \mathcal{PF} \tag{9}$$

has finitely many solutions $(n_1, \dots, n_t)$. *Moreover, there exists a computable number* C *depending only on the numbers* α, β, a_i, b_i *and the polynomials* f_i, *such that every solution of* (9) *satisfies*

$$\max(n_1, \dots, n_t) \le C. \tag{10}$$

Our method of proof of Theorem 2 can be used to prove a more general statement. Let r, s, t be rational integers, α, β, γ the zeros of $x^3 - rx^2 - sx - t$, $\mathbf{K} = \mathbf{Q}(\alpha, \beta, \gamma)$, and $(v_{i,n})_{n\ge 0}$ be recursive sequences of order three satisfying the relation:

$$\begin{gathered} v_{i,n+3} = rv_{i,n+2} + sv_{i,n+1} + tv_{i,n} \\ \text{for } n = 0, 1, \dots \quad \text{and} \quad i = 1, 2, \dots, t. \end{gathered} \tag{11}$$

In this case one can generalize Theorem 2 as follows:

Theorem 3. *Assume that either one of the following holds*:

(i) $\prod_{\mathbf{K}}(\alpha) \not\subseteq \prod_{\mathbf{K}}(\beta) \cup \prod_{\mathbf{K}}(\gamma)$ *and the absolute value of one of* α, β, γ *is larger than the other two.*

(ii) $\prod(\alpha) \cap \prod_{\mathbf{K}}(\beta) \not\subseteq \prod(\gamma)$.

Then the equation

$$\prod_{i=1}^{t} v_{i,n_i} \in \mathcal{PF} \tag{12}$$

has finitely many solutions $(n_1, \dots, n_t)$. *Moreover, there exists a computable number* C *depending only on the sequences* $(v_{i,n})_{n\geq 0}$ *for* $i = 1, \dots, t$ *such that every solution of* (12) *satisfies*

$$\max(n_1, \dots, n_t) \leq C. \tag{13}$$

For any real number x, let $\lfloor x \rfloor$ be the largest integer smaller than or equal to x.

Corollary 2. *Let* $\alpha > 1$ *be a real irrational quadratic unit. Let* $a \in \mathbf{Q}(\alpha)$. *Then the equation*

$$\lfloor a\alpha^n \rfloor \in \mathcal{PF} \tag{14}$$

has finitely many solutions n. *Moreover, there exists a computable number* C *depending only on the numbers* α *and* a *such that every solution of* (14) *satisfies* $n \leq C$.

Corollary 3. *Let* $(u_n)_{n\geq 0}$ *be a nondegenerate binary recurrence sequence of algebraic numbers whose characteristic equation has rational coefficients, and let* f *be a nonconstant polynomial with algebraic coefficients. Let* $k \geq 1$ *be a fixed integer. Then the equation*

$$N_{\mathbf{K}}(f(u_n)) = \prod_{j=1}^{k} m_j! \tag{15}$$

has finitely many solutions n. *Moreover, there exists a computable number* C *depending only on* k, *the sequence* $(u_n)_{n\geq 0}$, *and the polynomial* f, *such that every solution of* (15) *satisfies* $n \leq C$.

Assume now that α and β are the roots of a quadratic equation $x^2 - rx - s = 0$, where r and s are rational integers. Assume also that α/β is not a root of unity. The nondegenerate binary recurrence sequences of integers $(u_n)_{n\geq 0}$

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{for } n = 0, 1, \dots , \tag{16}$$

and

$$u_n = \alpha^n + \beta^n \quad \text{for } n = 0, 1, \dots , \tag{17}$$

are called *Lucas sequences of the first and second kind*, respectively. For a Lucas sequence we give a more precise version of Theorem 1.

Theorem 4. *Let $(u_n)_{n\geq 0}$ be a Lucas sequence. Let α and β denote the two roots of the characteristic equations. Suppose that $|\alpha| \geq |\beta|$. If $|u_n| \in \mathcal{PF}$, then $n \leq \max(12, 2e|\alpha| + 1)$.*

We conclude with the following computational result.

Corollary 4. (i) *If $(L_n)_{n\geq 0}$ is the Lucas sequence $L_0 = 2$, $L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for $n \geq 0$, then the only nontrivial solutions of the equation*

$$L_n = \prod_{j=1}^{k} m_j! \tag{18}$$

are $L_0 = 2!$ and $L_3 = (2!)^2$.

(ii) *If $(F_n)_{n\geq 0}$ is the Fibonacci sequence $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$, then the only nontrivial solutions of the equation*

$$F_n = \prod_{j=1}^{k} m_j! \tag{19}$$

are $F_3 = 2!$, $F_6 = (2!)^3$ and $F_{12} = (2!)^2 \cdot (3!)^2 = 3! \cdot 4!$.

2. Preliminaries. The proofs of Theorems 1, 2 and 3, and of their corollaries use estimations of linear forms in logarithms of algebraic numbers.

Suppose that $\zeta_1, \dots, \zeta_l$ are algebraic numbers, not 0 or 1, of heights not exceeding $A_1, \dots, A_l$, respectively. We assume $A_m \geq e$ for $m = 1, \dots, l$. Put $\Omega = \log A_1 \cdots \log A_l$. Let $\mathbf{F} = \mathbf{Q}(\zeta_1, \dots, \zeta_l)$. Let $n_1, \dots, n_l$ be integers, not all 0, and let $B \geq \max\{|n_m| | m = 1, \dots, l\}$. We assume $B \geq e$. The following result is due to Baker and Wüstholz.

Theorem BW ([**1**]). *If* $\zeta_1^{n_1} \cdots \zeta_l^{n_l} \neq 1$, *then*

$$|\zeta_1^{n_1} \cdots \zeta_l^{n_l} - 1| > \exp(-(17(l+1)d_{\mathbf{F}})^{2l+7}\Omega \log B). \tag{20}$$

In fact, Baker and Wüstholz showed that if $\log \zeta_1 \cdots \log \zeta_l$ are any fixed values of the logarithms, and $\Lambda = n_1 \log \zeta_1 + \cdots + n_l \log \zeta_l \neq 0$, then

$$\log |\Lambda| > -(16 l d_{\mathbf{F}})^{2(l+2)}\Omega \log B. \tag{21}$$

Now (20) follows easily from (21) via an argument similar to the one used by Shorey et al. in their paper [**3**, p. 66].

We also need the following p-adic analogue of Theorem BW which is due to Yu, see [**4**, Theorem 4].

Theorem Y ([**4**])[1]. *Let* π *be a prime ideal of* $\mathbf{F}$ *lying above a prime integer* p. *Assume that* $\mathrm{ord}_\pi \zeta_i = 0$ *for* $i = 1, \dots, l$. *If* $\zeta_1^{n_1} \cdots \zeta_l^{n_l} \neq 1$, *then there exist computable absolute constants* C_1 *and* C_2 *such that*

$$\mathrm{ord}_\pi(\zeta_1^{n_1} \cdots \zeta_l^{n_l} - 1) < (C_1 l d_{\mathbf{F}})^{C_2 l} \frac{p^{d_{\mathbf{F}}}}{\log^2 p} \Omega \log(d_{\mathbf{F}}^2 B). \tag{22}$$

For any rational number $q = k/l$ with k and l integers such that $(k, l) = 1$, let $P(q)$ be the largest prime number P dividing k with the convention that $P(0) = P(\pm 1) = 1$. Let α and β be nonzero algebraic integers, and let a and b be nonzero algebraic numbers. Put

$$u_n = a\alpha^n + b\beta^n \quad \text{for } n = 0, 1, \dots. \tag{23}$$

Assume that $(u_n)_{n\geq 0}$ is nondegenerate. The following result is due to Yu and Hung.

Theorem YH ([**5**]). *Let* $\mathbf{L} = \mathbf{Q}(\alpha, \beta, a, b)$. *There exist computable positive numbers* C_3 *and* C_4 *depending only on* $d_{\mathbf{L}}$ *and on the heights of the numbers* α, β, a, b *such that*

$$P(N_{\mathbf{L}}(u_n)) > C_3 n^{1/(d_{\mathbf{L}}+1)}, \tag{24}$$

whenever $n > C_4$.

The proof of Theorem 3 uses the following result of Carmichael.

Theorem C ([**2**]). *Let* $(u_n)_{n\geq 0}$ *be a Lucas sequence. Then* $P(u_n) \geq n-1$ *for* $n \geq 12$.

We also use the following estimations.

Lemma. *Let* $n \geq 2$ *be an integer, and let* $p \leq n$ *be a prime number. Then*

(i)

$$n^{n/2} \leq n! \leq n^n. \tag{25}$$

(ii)

$$\frac{n}{4(p-1)} \leq \operatorname{ord}_p(n!) \leq \frac{n}{p-1}. \tag{26}$$

Proof. (i) Obvious.

(ii) Let $\sigma_p(n)$ be the sum of the digits of the integer n written in the base p. We use the fact that

$$\operatorname{ord}_p(n!) = \frac{n - \sigma_p(n)}{p-1}. \tag{27}$$

From formula (27), it follows immediately that $\mathrm{ord}_p(n!) \le (n/(p-1))$. For the other inequality we distinguish two cases.

Case 1. Suppose that $p^2 \le n$. Notice that

$$\sigma_p(n) \le (p-1)(\lfloor \log_p(n) \rfloor + 1) \le (p-1)\frac{\log(np)}{\log p}.$$

Hence, it is enough to show that

$$(p-1)\frac{\log(np)}{n\log p} \le 3/4. \tag{28}$$

One can check easily that, if $p \ge 2$ is fixed, then the function

$$f(x) = \frac{\log(xp)}{x\log p}$$

is decreasing for $x \ge p$. Since $n \ge p^2$, it follows that

$$(p-1)\frac{\log(np)}{n\log p} \le (p-1)\frac{\log(p^3)}{p^2\log p} = \frac{3(p-1)}{p^2} \le 3/4 \quad \text{for } p \ge 2.$$

Case 2. Suppose that $n < p^2$. It follows that $n = ap + b$ for some integers a and b such that $1 \le a \le p-1$, and $0 \le b \le p-1$. Then

$$n - \sigma_p(n) = ap + b - (a+b) = ap - a.$$

Hence, it is enough to show that $4(ap - a) \ge ap + b$. But this is equivalent to $3ap \ge 4a + b$. If $a \ge 2$, this follows because $3ap \ge 6p > 4(p-1) + (p-1) \ge 4a + b$. If $a = 1$, this follows because the inequality $3p \ge 4 + (p-1)$ is equivalent to $2p \ge 3$, which is obvious. □

3. The proofs.

The proof of Theorem 1. By $C_1, C_2, \dots,$ we shall denote computable positive numbers depending only on the numbers α_i, β_i, a_i, b_i

for $i = 1, \dots, t$. Let $d = d_{\mathbf{K}}$. Let Δ be a common denominator of all numbers a_i, b_i for $i = 1, \dots, t$. From equation (5) it follows that

$$\prod_{i=1}^{t} N_{\mathbf{K}}(\Delta u_{i,n_i}) = \Delta^{td} \prod_{j=1}^{k} m_j! \tag{29}$$

For simplicity, denote

$$w_{i,n} = \Delta u_{i,n} = c_i \alpha_i^n + d_i \beta_i^n \quad \text{for } n = 0, 1, \dots ,$$

where $c_i = \Delta a_i$ and $d_i = \Delta b_i$. Notice that c_i and d_i are algebraic integers for all $i = 1, \dots, t$. Equation (29) can be rewritten as

$$\prod_{i=1}^{t} N_{\mathbf{K}}(w_{i,n_i}) = \Delta^{td} \prod_{j=1}^{k} m_j!. \tag{30}$$

For any $i = 1, \dots, t$, let $\mathbf{K_i} = \mathbf{Q}(\alpha_i, \beta_i, a_i, b_i)$. By Theorem YH we know that there exist positive numbers $C_{i,1}$ and $C_{i,2}$ such that

$$P(N_{\mathbf{K}_i}(w_{i,n})) > C_{i,1} n_i^{1/(d_{\mathbf{K}_i}+1)}, \tag{31}$$

whenever $n > C_{i,2}$. Let $C_1 = \min(C_{1,1}, \dots, C_{t,1})$, and let $C_2 = \max(C_{1,2}, \dots, C_{t,2})$. Let p be the largest prime number dividing

$$\Delta \prod_{i=1}^{t} N_{\mathbf{K}}(c_i d_i \alpha_i \beta_i).$$

Let C_3 be the smallest positive integer such that $C_1 n^{1/(d+1)} > p$ whenever $n > C_3$. Let $C_4 = \max(C_2, C_3)$. Finally, let q be the first prime number greater than p.

Assume that $1 \le n_1 \le \dots \le n_t$ and that $m_1 \le \dots \le m_k$. Suppose that $n_t > C_4$. Then

$$P(N_{\mathbf{K}_t}(w_{t,n_t})) > C_1 n_t^{1/(d_{\mathbf{K}_t}+1)} \ge C_1 n_t^{1/(d+1)} > p. \tag{32}$$

Since $N_{\mathbf{K}_t}(w_{t,n_t}) \mid N_{\mathbf{K}}(w_{t,n_t})$, it follows, by formula (30) and inequality (32), that

$$m_k \ge P(N_{\mathbf{K}_t}(w_{t,n_t})) > C_1 n_t^{1/(d_{\mathbf{K}_t}+1)} \ge C_1 n_t^{1/(d+1)} > p. \tag{33}$$

Hence, $m_k \geq q$. Since $q \nmid \Delta$, it follows that

$$\begin{aligned} \text{ord}_q\left(\Delta^{td}\prod_{j=1}^{k} m_j!\right) &= \sum_{m_j\geq q} \text{ord}_q(m_j!) \\ &\geq \frac{1}{4(q-1)}\sum_{m_j\geq q} m_j \\ &\geq \frac{C_1}{4(q-1)} n_t^{1/(d+1)}. \end{aligned} \tag{34}$$

Denote $(C_1/(4(q-1)))$ by C_5.

Now let π be an arbitrary prime ideal of $\mathbf{K}$ lying above q. Let A be an upper bound for the heights of all the numbers α_i/β_i and a_i/b_i. We assume that $A \geq e$. Let $\Omega = (\log A)^2$. It follows, by Theorem Y, that

$$\begin{aligned} \text{ord}_\pi w_{i,n_i} &= \text{ord}_\pi(b_i\beta_i^{n_i}) + \text{ord}_\pi\left(\left(-\frac{a_i}{b_i}\right)\left(\frac{\alpha_i}{\beta_i}\right)^{n_i} - 1\right) \\ &\leq (C_6 d_{\mathbf{K}_i})^{C_7}\frac{q^{d_{\mathbf{K}_i}}}{\log^2 q}\Omega\log(d_{\mathbf{K}_i}^2 n_i), \end{aligned}$$

for some absolute constants C_6 and C_7. Let

$$C_8 = (C_6 d)^{C_7}\frac{q^d}{\log^2 q}\Omega.$$

Then

$$\text{ord}_\pi w_{i,n_i} \leq C_8 \log(d^2 n_t).$$

If $w_{i,n_i}^{(\mu)}$ is any conjugate of w_{i,n_i} in $\mathbf{K}$, then, by a similar argument, it follows that

$$\text{ord}_\pi w_{i,n_i}^{(\mu)} \leq C_8 \log(d^2 n_t).$$

Hence,

$$\begin{aligned} \text{ord}_q\left(\Delta^{td}\prod_{j=1}^{k} m_j!\right) &= \text{ord}_q\left(\prod_{j=1}^{t} N_{\mathbf{K}}(w_{i,n_i})\right) \leq \text{ord}_\pi\left(\prod_{j=1}^{t}\prod_{\mu=1}^{d} w_{i,n_i}^{(\mu)}\right) \\ &\leq C_8 td\log(d^2 n_t). \end{aligned} \tag{35}$$

From inequalities (34) and (35) it follows that

$$C_8 td \log(d^2 n_t) > C_5 n_t^{1/(d+1)}. \tag{36}$$

Inequality (36) clearly shows that $n_t < C_9$. □

Proof of Corollary 1. We first show that we may assume that all polynomials f_i are monic and linear for $i = 1, \dots, t$. Indeed, let

$$f_i(X) = c_i(X - \zeta_{i,1})(X - \zeta_{i,2}) \cdots (X - \zeta_{i,j_i})$$
$$\text{where } j_i = \deg(f_i).$$

By replacing $\mathbf{K}$ with $\mathbf{K}(\zeta_{i,1}, \dots, \zeta_{i,j_i})$, one may assume that the f_i's are linear. If

$$f_i(X) = c_i X + d_i$$

for some i, then one may replace f_i by

$$\tilde{f}_i = X + d_i,$$

and the sequences $(u_{i,n})_{n \geq 0}$ by

$$\widetilde{u_{i,n}} = c_i u_{i,n}$$

and notice that with these notations $\tilde{f}_i(\widetilde{u_{i,n}}) = f_i(u_{i,n})$. Thus, we may assume that all polynomials f_i are monic and linear for $i = 1, \dots, t$. Denote

$$w_{i,n} = f_i(u_{i,n}) = a_i \alpha_i^n + b_i \beta_i^n + \zeta_i \quad \text{for } n = 0, 1, \dots .$$

We may assume that $s_i = -1$. Indeed, if $s_i = 1$, then we can replace the sequence $(u_{i,n})_{n \geq 0}$ by the two sequences

$$\hat{u}_{i,n} = u_{i,2n} = a_i(\alpha_i^2)^n + b_i(\beta_i^2)^n \quad \text{for } n = 0, 1, \dots$$

and

$$\tilde{u}_{i,n} = u_{i,2n+1} = (a_i \alpha_i)(\alpha_i^2)^n + (b_i \beta)(\beta_i^2)^n \quad \text{for } n = 0, 1, \dots$$

which have the companion polynomial $x^2 - (r_i^2 + 2s_i)x + s_i^2 = x^2 - (r_i^2 + 2)x + 1$. Since $s_i = -1$, it follows that $\beta_i = \alpha_i^{-1}$. Since $(u_{i,n})_{n \geq 0}$

is nondegenerate for every $i = 1, \dots, t$, it follows that α_i is a real quadratic unit, not 1 or -1. Notice that

$$\begin{aligned} w_{i,n} &= a_i\alpha_i^n + b_i\alpha_i^{-n} + \zeta_i = a_i\alpha_i^{-n}(\alpha_i^n - z_{i,1})(\alpha_i^n - z_{i,2}) \\ &= \alpha_i^{-n}(a_i\alpha_i^n - a_i z_{i,1})(\alpha_i^n - z_{i,2}) \end{aligned}$$

where $z_{i,1}$ and $z_{i,2}$ are the roots of the polynomial

$$z^2 + \frac{b_i}{a_i}z + \frac{\zeta_i}{a_i}.$$

Let $\mathbf{K}' = \mathbf{K}[z_{i,1}, z_{i,2} \mid i = 1, \dots, t]$. Equation (7) implies that

$$\begin{aligned} \prod_{i=1}^{t} N_{\mathbf{K}'}(w_{i,n_i}) &= N_{\mathbf{K}'}\bigg(\prod_{i=1}^{t} \alpha_i^{-n_i}\bigg) \\ &\quad \cdot \bigg(\prod_{i=1}^{t} N_{\mathbf{K}'}(a_i\alpha_i^n - a_i z_{i,1}) N_{\mathbf{K}'}(\alpha_i^n - z_{i,2})\bigg) \\ &= \bigg(\prod_{j=1}^{k} m_j!\bigg)^{d'}, \end{aligned}$$

where $d' = [\mathbf{K}' : \mathbf{K}]$. Notice that $\prod_{i=1}^{t} \alpha_i^{-n_i}$ is a unit. Hence,

$$N_{\mathbf{K}'}\bigg(\prod_{i=1}^{t} \alpha_i^{-n_i}\bigg) = 1.$$

Corollary 1 follows now from Theorem 1 because

$$\bigg(\prod_{j=1}^{k} m_j!\bigg)^{d'} \in \mathcal{PF}. \qquad \square$$

Proof of Theorem 2. We may again assume that all polynomials f_i are monic and linear for $i = 1, \dots, t$. Let $f_i(X) = X + \zeta_i$ for $i = 1, \dots, t$. We assume that at least one of the numbers ζ_i is nonzero (otherwise the conclusion of Theorem 2 follows from Theorem 1). Let $d = d_{\mathbf{K}}$.

We assume that $|s| \neq 1$ (otherwise the theorem follows from Corollary 1). Since $|s| \neq 1$ it follows that none of the numbers α and β is a root of unity.

Let Δ be a common denominator of all the numbers a_i, b_i, ζ_i for $i = 1, \dots, t$. Let c_i, d_i and η_i denote Δa_i, Δb_i and $\Delta \zeta_i$, respectively. Assume that $\mathbf{K}$ contains all the roots ζ_i for $i = 1, \dots, t$ (otherwise we can just replace $\mathbf{K}$ by a larger field containing all these elements). Equation (9) implies that

$$\prod_{i=1}^{t} N_{\mathbf{K}}(w_{i,n_i}) = \Delta^{td} \prod_{j=1}^{k} m_j!, \tag{37}$$

where

$$w_{i,n} = c_i \alpha^n + d_i \beta^n + \eta_i \quad \text{for } n = 0, 1, \dots .$$

It is clear that c_i, d_i and η_i are algebraic integers for $i = 1, \dots, t$.

(i) By $C_1, C_2, \dots,$ we shall denote computable positive numbers depending only on the numbers $\alpha, \beta,$ and a_i, b_i, ζ_i for $i = 1, \dots, t$. It is clear that both α and β are real. By an argument similar to the one used at the beginning of the proof of Corollary 1, we conclude that we can assume that both α and β are positive. It is clear that $\alpha \neq \beta$.

Suppose, for example, that π is a prime ideal in $\mathbf{K}$ such that $\pi \in \prod_{\mathbf{K}}(\alpha) - \prod_{\mathbf{K}}(\beta)$. Let $p = N_{\mathbf{K}}(\pi)$. Let

$$C_1 = \max(\mathrm{ord}_\pi d_i \mid i = 1, \dots, t).$$

Let A be an upper bound for the heights of all numbers $1/\beta$ and η_i/d_i for $i = 1, \dots, t$. Suppose that $A \geq e$. Let $\Omega = (\log A)^2$. Since

$$\mathrm{ord}_\pi(d_i\beta^n + \eta_i) = \mathrm{ord}_\pi(-d_i\beta^n) + \mathrm{ord}_\pi\left(\left(-\frac{\eta_i}{d_i}\right)(\beta^{-1})^n - 1\right),$$

it follows, by Theorem Y, that

$$\mathrm{ord}_\pi(d_i\beta^n + \eta_i) < \begin{cases} C_1 & \text{if } \eta_i = 0, \\ C_1 + (C_2 d)^{C_3}(p^d/\log^2 p)\Omega \log(d^2 n) & \text{if } \eta_i \neq 0, \end{cases} \tag{38}$$

for some absolute constants C_2 and C_3. Let $C_4 = (C_2 d)^{C_3} \Omega p^d / \log^2 p$. Let C_5 be a positive integer such that

$$n > 2C_4 \log(d^2 n) > 2C_1 \tag{39}$$

whenever $n > C_5$. Suppose now that $n_1 \leq \cdots \leq n_t$ and that $m_1 \leq \cdots \leq m_k$.

Assume first that $n_1 > C_5$. Since

$$\mathrm{ord}_\pi(c_i\alpha^{n_i}) \geq n_i > C_1 + C_4\log(d^2n_i) > \mathrm{ord}_\pi(d_i\beta^n + \eta_i),$$

it follows that

$$\begin{aligned}\mathrm{ord}_\pi(w_{i,n_i}) &= \mathrm{ord}_\pi(c_i\alpha^{n_i} + d_i\beta^{n_i} + \eta_i) \\ &< C_1 + C_4\log(d^2n_t) < 2C_4\log(d^2n_t).\end{aligned} \tag{40}$$

Assume that $w_{i,n_i}^{(\mu)} = c_i^{(\mu)}(\alpha^{(\mu)})^{n_i} + d_i^{(\mu)}(\beta^{(\mu)})^{n_i} + \eta_i^{(\mu)}$ is a conjugate of w_{i,n_i} in $\mathbf{K}$. Notice that $\{\alpha^{(\mu)}, \beta^{(\mu)}\} = \{\alpha, \beta\}$. It follows, by a similar argument, that

$$\mathrm{ord}_\pi(w_{i,n_i}^{(\mu)}) < C_1 + C_4\log(d^2n_t) < 2C_4\log(d^2n_t).$$

Hence,

$$\mathrm{ord}_\pi\left(\prod_{i=1}^{t} N_{\mathbf{K}}(w_{i,n_i})\right) = \mathrm{ord}_\pi\left(\prod_{i=1}^{t}\prod_{\mu=1}^{d} w_{i,n_i}^{(\mu)}\right) < 2tdC_4\log(d^2n_t). \tag{41}$$

On the other hand,

$$\begin{aligned}\mathrm{ord}_\pi\left(\Delta^{td}\prod_{j=1}^{k} m_j!\right) &\geq \sum_{j=1}^{k}\mathrm{ord}_\pi(m_j!) \geq \sum_{j=1}^{k}\mathrm{ord}_p(m_j!) \\ &\geq \frac{1}{4(p-1)}\sum_{m_j\geq p} m_j.\end{aligned}$$

Hence,

$$\sum_{m_j\geq p} m_j < C_6\log(d^2n_t),$$

where $C_6 = 8td(p-1)C_4$. Since $\log x > x$ for any real number $x > 1$, it follows that

$$\sum_{m_j\geq p} m_j\log m_j < \left(\sum_{m_j\geq p} m_j\right)^2 < C_6^2(\log(d^2n_t))^2. \tag{42}$$

From the lemma, it follows that

$$\Delta^{td} \prod_{m_j \geq p} m_j! < \exp(C_7 + C_8(\log(d^2 n_t)^2) \tag{43}$$

where $C_7 = td \log \Delta$ and $C_8 = C_6^2$.

Assume that $\alpha > \beta$. Rewrite equation (37) as

$$\prod_{i=1}^{t} N_{\mathbf{K}}(w_{i,n_i}) = \Big(\prod_{m_j < p} m_j!\Big)\Big(\Delta^{td} \prod_{m_j \geq p} m_j!\Big). \tag{44}$$

Let $p_1 < \cdots < p_l$ be all the prime numbers less than p. For each $m = 1, \ldots, l$, let $q_m = \mathrm{ord}_{p_m}(\prod_{m_j < p} m_j!)$. Then

$$q_m \leq \mathrm{ord}_{p_m}\Big(\Delta^{td} \prod_{j=1}^{k} m_j!\Big) = \mathrm{ord}_{p_m}\Big(\prod_{i=1}^{t} N_{\mathbf{K}}(w_{i,n_i})\Big).$$

Hence,

$$\begin{aligned} p_m^{q_m} &\leq \prod_{i=1}^{t} N_{\mathbf{K}}(w_{i,n_i}) \\ &= \alpha^{\sum_{i=1}^{t}\sum_{\mu=1}^{d} n_i} \prod_{i=1}^{t}\prod_{\mu=1}^{d}\left(u_{i,\mu} + v_{i,\mu}\left(\frac{\beta}{\alpha}\right)^{n_i} + \eta_i^{(\mu)}\alpha^{-n_i}\right) \\ &\leq C_9 \alpha^{tdn_t}, \end{aligned}$$

where $u_{i,\mu}, v_{i,\mu}$ are equal to $c_i^{(\mu)}, d_i^{(\mu)}$ or $d_i^{(\mu)}$, $c_i^{(\mu)}$ according to whether $\alpha^{(\mu)} = \alpha$ or $\alpha^{(\mu)} = \beta$. In the above inequality

$$C_9 = \prod_{i=1}^{t}\prod_{\mu=1}^{d}(|c_i^{(\mu)}| + |d_i^{(\mu)}| + \eta_i^{(\mu)}|).$$

It follows that

$$q_m \leq \frac{1}{\log p_m}(\log C_9 + tdn_t \log \alpha) \leq C_{10} + C_{11} n_t,$$

where $C_{10} = (\log C_9/\log 2)$ and $C_{11} = (td\log\alpha/\log 2)$. From equation (44), it follows that

$$N_{\mathbf{K}}(w_{t,n_t}) = Q\prod_{m=1}^{l} p_m^{\delta_m}, \tag{45}$$

where Q is an integer which divides $(\Delta^{td}\prod_{m_j\geq p} m_j!)$ and $\delta_m \geq 0$ are integers such that

$$\delta_m \leq q_m \leq C_{10} + C_{11}n_t \quad \text{for } m = 1, \ldots, l. \tag{46}$$

From inequality (43), it follows that

$$\log Q < C_7 + C_8(\log(d^2 n_t))^2. \tag{47}$$

Write

$$N_{\mathbf{K}}(w_{t,n_t}) = \prod_{\mu=1}^{d}(u_\mu\alpha^{n_t} + v_\mu\beta^{n_t} + \eta^{(\mu)}),$$

where both u_μ and v_μ are some conjugates of either c_t or d_t. Let $G = \prod_{\mu=1}^{d} u_\mu$. Write

$$N_{\mathbf{K}}(w_{t,n_t}) = G\alpha^{dn_t} + F. \tag{48}$$

Combining equations (45) and (48), we obtain

$$1 - \frac{1}{G}\alpha^{-dn_t}\prod_{m=1}^{l} p_m^{\delta_m} Q = -\frac{F}{G\alpha^{dn_t}}. \tag{49}$$

Let $\gamma = \max(\alpha^{-1}, \beta/\alpha)$. Notice that $0 < \gamma < 1$. Then

$$\begin{aligned} \left|\frac{F}{G\alpha^{dn_t}}\right| &= \left|1 - \prod_{\mu=1}^{d}\left(1 + \left(\frac{v_\mu}{u_\mu}\right)\left(\frac{\beta}{\alpha}\right)^{n_t} + \left(\frac{\eta^{(\mu)}}{u_\mu}\right)\alpha^{-n_t}\right)\right| \\ &\leq \gamma^{n_t} C_{12}, \end{aligned} \tag{50}$$

where

$$C_{12} = \left|1 - \prod_{\mu=1}^{d}\left(1 + \left|\frac{v_\mu}{u_\mu}\right| + \left|\frac{\eta^{(\mu)}}{u_\mu}\right|\right)\right|.$$

Let $C_{13} = \log C_{12}$, and let $C_{14} = -\log\gamma > 0$. From formulae (49) and (50), it follows that

$$(51)\qquad \left|1 - \frac{1}{G}\alpha^{-dn_t}\prod_{m=1}^{l} p_m^{\delta_m} Q\right| = \left|\frac{F}{G\alpha^{dn_t}}\right| \le \exp(C_{13} - C_{14}n_t).$$

Let A_1 be an upper bound for the heights of all the numbers $1/G$, α^{-1} and p_m for $m = 1, \dots, l$. Suppose that $A_1 > e$. Let $\Omega_1 = (\log A_1)^{l+2}$. Let $C_{15} = \max(C_{11}, 1)$. It follows, by Theorem BW, equation (49) and inequality (46), that

$$(52)\qquad \left|1 - \frac{1}{G}\alpha^{-dn_t}\prod_{m=1}^{l} p_m^{\delta_m} Q\right| > \exp(-(17(l+4)d)^{2l+13}\Omega_1 \log Q \log(C_{10} + C_{15}n_t)).$$

From inequalities (51) and (52), it follows that

$$(53)\qquad C_{14}n_t - C_{13} < C_{16}\log Q \log(C_{10} + C_{15}n_t),$$

where

$$C_{16} = (17(l+4)d)^{2l+13}\Omega_1.$$

From formulae (53) and (47), we conclude that

$$(54)\qquad C_{14}n_t < C_{13} + C_{16}(C_7 + C_8(\log(d^2n_t))^2)\log(C_{10} + C_{15}n_t).$$

Inequality (54) implies that $n_t < C_{17}$.

Notice that the conclusion $n_t < C_{17}$ came from the fact that we assumed $n_1 = \min(n_i \mid i = 1, \dots, t) > C_5$. This concludes the proof of the theorem if $t = 1$. On the other hand, if $t > 1$, but $0 \le n_1 \le C_5$, then w_{1,n_1} can take only finitely many values. Hence, equation (37) reduces to finitely many equations of the same type as (37) with only $t-1$ factors, namely $w_{1,n_1} \cdot w_{2,n_2}$ and w_{i,n_i} for $i = 3, \dots, t$. Assertion (i) follows now by induction on t.

The case $\alpha < \beta$ can be treated similarly.

(ii) By $C_1, C_2, \dots,$ we shall denote computable positive numbers depending only on the numbers α, β and a_i, b_i, ζ_i for $i = 1, \dots, t$. Suppose that $f = \prod_{i=1}^{t} f_i$ has the property that $f(0) \ne 0$. With our notation, this amounts to $\zeta_i \neq 0$ for any $i = 1, \dots, t$. Let π be a prime ideal in $\mathbf{K}$ such that $\pi \mid ([\alpha]_{\mathbf{K}}, [\beta]_{\mathbf{K}})$. Let $p = N_{\mathbf{K}}(\pi)$.

Suppose that $n_1 \leq \cdots \leq n_t$ and that $m_1 \leq \cdots \leq m_k$.

Let $\eta = \prod_{i=1}^{t} \eta_i$. Let $C_{1,\mu} = \mathrm{ord}_{\pi}(\eta^{(\mu)})$, where $\eta^{(\mu)}$ is a conjugate of η in $\mathbf{K}$. Let

$$C_1 = \sum_{\mu=1}^{d} C_{1,\mu}.$$

Notice that

$$\prod_{i=1}^{t} w_{i,n_i}^{(\mu)} \equiv \eta^{(\mu)} \pmod{\pi^{n_1}}. \tag{55}$$

Suppose that $n_1 > C_1$. From congruences (55), it follows that

$$\mathrm{ord}_{\pi}\biggl(\prod_{i=1}^{t} N_{\mathbf{K}}(w_{i,n_i})\biggr) = C_1.$$

Therefore,

$$\begin{aligned} C_1 &= \mathrm{ord}_{\pi}\biggl(\Delta^{td} \prod_{j=1}^{k} m_j!\biggr) \geq \mathrm{ord}_{p}\biggl(\Delta^{td} \prod_{j=1}^{k} m_j!\biggr) \\ &\geq \mathrm{ord}_{p}(m_k!) \geq \frac{m_k}{4(p-1)}. \end{aligned} \tag{56}$$

It follows that

$$m_k \leq C_2, \tag{57}$$

where $C_2 = 4(p-1)C_1$. Let $p_1 < p_2 \cdots < p_l$ be all the prime numbers which are less than $\Delta \cdot C_2$. From equation (37) and formula (57), it follows that

$$\prod_{i=1}^{t} N_{\mathbf{K}}(w_{i,n_i}) = \prod_{m=1}^{l} p_m^{q_m} \quad \text{for some integers } q_m \geq 0. \tag{58}$$

By an argument similar to the one used at (i), it follows easily that

$$q_m < C_3 + C_4 n_t \quad \text{for } m = 1, \dots, l, \tag{59}$$

where C_3 and C_4 are some computable constants depending only on the numbers α, β, c_i, d_i and η_i for $i = 1, \dots, t$. We may assume that $\min(C_3, C_4) \geq 1$. By equation (58), it follows that

$$N_{\mathbf{K}}(w_{t,n_t}) \mid \prod_{i=1}^{t} N_{\mathbf{K}}(w_{i,n_i}) = \prod_{m=1}^{l} p_m^{q_m}.$$

Hence,

$$N_{\mathbf{K}}(w_{t,n_t}) = \prod_{m=1}^{l} p_m^{\delta_m}, \tag{60}$$

for some integers $\delta_m \geq 0$, such that

$$\delta_m \leq q_m < C_3 + C_4 n_t \quad \text{for } m = 1, \dots, l. \tag{61}$$

However, it is easy to see that

$$N_{\mathbf{K}}(w_{t,n_t}) \equiv N_{\mathbf{K}}(\eta_t) \pmod{\pi^{d n_t}}. \tag{62}$$

From equation (60) and congruence (62), it follows that

$$\operatorname{ord}_\pi\left(\prod_{m=1}^{l} p_m^{\delta_m} - N_{\mathbf{K}}(\eta_t) \right) \geq d n_t. \tag{63}$$

Let $C_5 = \operatorname{ord}_\pi(N_{\mathbf{K}}(\eta_t))$. Let A be an upper bound for the heights of all numbers $N_{\mathbf{K}}(\eta_t)$, $p_1, \dots, p_l$. Suppose that $A \geq e$. Let $\Omega = (\log A)^{l+1}$. It follows, by Theorem Y and inequality (61), that

$$\begin{aligned} &\operatorname{ord}_\pi\left(\prod_{m=1}^{l} p_m^{\delta_m} - N_{\mathbf{K}}(\eta_t) \right) \\ &\quad = \operatorname{ord}_\pi(N_{\mathbf{K}}(\eta_t)) + \operatorname{ord}_\pi\left(\frac{1}{N_{\mathbf{K}}(\eta_t)} \prod_{m=1}^{l} p_m^{\delta_m} - 1 \right) \\ &\quad < C_5 + (C_6(l+1)d)^{C_7(l+1)} \frac{p^d}{\log^2 p} \Omega \log(d^2 C_3 + d^2 C_4 n_t). \end{aligned} \tag{64}$$

Let

$$C_8 = (C_6(l+1)d)^{C_7(l+1)} \frac{p^d}{\log^2 p} \Omega.$$

From inequalities (63) and (64), it follows that

$$dn_t < C_5 + C_8 \log(d^2 C_3 + d^2 C_4 n_t). \tag{65}$$

Inequality (65) shows that $n_t < C_9$.

Notice that the conclusion $n_t < C_9$ came from the fact that we assumed $n_1 = \min(n_i \mid i = 1, \dots, t) > C_1$. This concludes from the proof of the theorem if $t = 1$. On the other hand, if $t > 1$, but $0 \le n_1 \le C_1$, then w_{1,n_1} can take only finitely many values. Hence, equation (37) reduces to finitely many equations of the same type as (37) with only $t-1$ factors, namely $w_{1,n_1} \cdot w_{2,n_2}$ and w_{i,n_i} for $i = 3, \dots, t$. The assertion (ii) follows now by induction on t. □

Proof of Theorem 3. Follows from arguments similar to the ones employed in the proof of Theorem 2.

Proof of Corollary 2. Let $\alpha > 1$ be a real irrational quadratic unit. Let $\mathbf{K} = \mathbf{Q}(\alpha)$. Let $d > 1$ be a square free positive integer such that $\mathbf{K} = \mathbf{Q}(\sqrt{d})$. Let $\sigma \in \mathrm{Gal}\,(\mathbf{K}/\mathbf{Q})$ be such that $\sigma(\sqrt{d}) = -\sqrt{d}$. Finally, let $\Delta > 0$ be a positive integer such that $\Delta \cdot a$ is an algebraic integer. Let

$$u_n = a\alpha^n + \sigma(a)\sigma(\alpha)^n \quad \text{for } n = 0, 1, \dots . \tag{66}$$

It is clear that $(u_n)_{n \ge 0}$ is a binary recurrence sequence of rational numbers. Moreover, $\Delta \cdot u_n \in \mathbf{Z}$ for all $n \ge 0$. Since α is a unit, it follows that $|\sigma(\alpha)| = 1/\alpha < 1$. Let $C_1 > 0$ be such that $|\sigma(a)\sigma(\alpha)^n| < 1$ for $n > C_1$. It follows, by equation (66), that

$$\lfloor a\alpha^n \rfloor = u_n + r_n \quad \text{for } n > C_1$$

where $\Delta \cdot r_n$ is an integer in the interval $[-\Delta + 1, \Delta - 1]$. Hence, all solutions of equation $\lfloor a\alpha^n \rfloor \in \mathcal{PF}$ can be found among the solutions of finitely many equations of the form

$$(\Delta u_n) + (\Delta c) \in \mathcal{PF}$$

where $(\Delta c) \in \{-\Delta + 1, -\Delta + 2, \dots, \Delta - 1\}$. The conclusion of the corollary follows now from Theorems 1 and 2. □

Proof of Corollary 3. By $C_1, C_2, \dots,$ we shall denote computable positive numbers depending only on the numbers α, β, a, b and f. Suppose that

$$f(X) = a_0 X^t + a_1 X^{t-1} + \cdots + a_t, \quad \text{for some } a_0, \dots, a_t \text{ with } a_0 \neq 0. \tag{67}$$

We write

$$f(X) = a_0 \prod_{i=1}^{t} (X + \zeta_i).$$

Equation (15) implies that

$$\prod_{i=1}^{t} N_{\mathbf{K}}(w_{i,n}) = \Delta^{(t+1)d} \prod_{j=1}^{k} m_j!, \tag{68}$$

where

$$w_{i,n} = c_i \alpha^n + d_i \beta^n + \eta_i \quad \text{for } n = 0, 1, \dots .$$

Here Δ is a common denominator of a, b, a_0 and the roots ζ_i of f, $c_1 = \Delta^2 a_0 a$, $d_1 = \Delta^2 a_0 b$ and $c_i = \Delta a$, $d_i = \Delta b$ for $i = 2, \dots, t$. We assume again that $\mathbf{K}$ contains all the numbers ζ_i for $i = 1, \dots, t$.

We distinguish two cases.

Case 1. Suppose that $([\alpha]_{\mathbf{K}}, [\beta]_{\mathbf{K}}) \neq 1$. It follows, by Theorem 2, that we may assume that $f(0) = 0$. Hence, we assume that $\eta_1 = 0$. By replacing the sequence u_n with the two sequences

$$\hat{u}_n = u_{2n} = a(\alpha^2)^n + b(\beta^2)^n \quad \text{for } n = 0, 1, \dots$$

and

$$\tilde{u}_n = u_{2n+1} = (a\alpha)(\alpha^2)^n + (b\beta)(\beta^2)^n \quad \text{for } n = 0, 1, \dots ,$$

it follows that we may assume that $\alpha\beta > 0$. Assume that $\pi \mid ([\alpha]_{\mathbf{K}}, [\beta]_{\mathbf{K}})$ for some prime ideal π in $\mathbf{K}$. Let $p = N_{\mathbf{K}}(\pi)$. Since $\eta_1 = 0$, it follows that

$$\text{ord}_\pi\left(\prod_{i=1}^{t} N_{\mathbf{K}}(w_{i,n})\right) \geq dn. \tag{69}$$

Let $C_1 = d\text{ord}_p(\Delta)$. Suppose that $m_1 \leq m_2 \leq \cdots m_k$. From equation (68), inequality (69) and the lemma, we conclude that

$$\begin{aligned} n &\leq \text{ord}_p\left(\Delta^{(t+1)d}\prod_{j=1}^{k} m_j!\right) \\ &\leq (t+1)C_1 + k\cdot\text{ord}_p(m_k!) \\ &\leq (t+1)C_1 + \frac{k}{p-1}m_k. \end{aligned}$$

Therefore,

$$m_k \geq C_2 n - C_3, \tag{70}$$

where $C_2 = (p-1)/k$ and $C_3 = (p-1)(t+1)C_1/k$. Suppose that $|\alpha| \geq |\beta|$. It follows that

$$\left|\prod_{i=1}^{t} N_{\mathbf{K}}(w_{i,n})\right| \leq |\alpha|^{tdn} C_4$$

where

$$C_4 = \prod_{i=1}^{t}\prod_{\mu=1}^{d}(|c_i^{(\mu)}| + |d_i^{(\mu)}| + |\eta_i^{(\mu)}|).$$

From the lemma, we conclude that

$$\exp\left(\frac{1}{2}m_k \log m_k\right) \leq m_k! \leq \Delta^{(t+1)d}\prod_{j=1}^{k} m_j! \leq |\alpha|^{tdn} C_4. \tag{71}$$

Hence,

$$m_k \log m_k \leq C_5 + C_6 n, \tag{72}$$

where $C_5 = 2\log C_4$ and $C_6 = 2td\log|\alpha|$. Combining inequalities (71) and (72) we obtain

$$(C_2n - C_3)\log(C_2n - C_3) \leq C_5 + C_6n. \tag{73}$$

Formula (73) shows that $n < C_7$.

Case 2. Suppose that $([\alpha]_{\mathbf{K}}, [\beta]_{\mathbf{K}}) = 1$. It follows, by Theorem 1, that we may assume that at least one of the numbers α and β is not a unit. From Theorem 2, we conclude that we may assume that α and β are complex conjugate. In this case, none of the numbers α and β is a unit. Let $\pi \in \prod_{\mathbf{K}}(\alpha) - \prod_{\mathbf{K}}(\beta)$.

Suppose that $m_1 \leq m_2 \leq \cdots \leq m_k$. By an argument similar to the one used in the proof of Theorem 2(i), one concludes that

$$m_k! \leq \exp(C_8 + C_9(\log(d^2n))^2),$$

for some constants C_8, C_9 depending only on π, the sequence $(u_n)_{n\geq 0}$, and the polynomial f. It follows that

$$\prod_{j=1}^{k} m_j! \leq (m_k!)^k \leq \exp(C_{10} + C_{11}(\log(d^2n)^2), \tag{74}$$

where $C_{10} = kC_8$ and $C_{11} = kC_9$. On the other hand, notice that

$$\begin{aligned} |f(X)| &\geq |a_0||X^t|\left(1 - \left|\frac{a_1}{a_0}\right| \cdot \frac{1}{|X|} - \cdots - \left|\frac{a_t}{a_0}\right| \cdot \frac{1}{|X|^t}\right) \\ &\geq \frac{|a_0|}{2} \cdot |X|, \end{aligned} \tag{75}$$

for $|X| \geq C_{12} = \max((|a_1| + \cdots + |a_t|)/(2|a_0|), 1)$.

Let A be an upper bound for the heights of the numbers α/β and a/b. Assume that $A \geq e$. Let $\Omega = (\log A)^2$. By Theorem BW, it follows that, for $n \geq 3$,

$$\begin{aligned} |u_n| = |a\alpha^n + b\beta^n| &= |a||\alpha|^n\left|1 - \left(-\frac{b}{a}\right)\left(\frac{\beta}{\alpha}\right)^n\right| \\ &\geq |a||\alpha|^n \exp(-68^{11}\Omega\log n) \\ &= \exp(C_{13} + C_{14}n - C_{15}\log n), \end{aligned} \tag{76}$$

where $C_{13} = \log |a|$, $C_{14} = \log |\alpha|$ and $C_{15} = 68^{11}\Omega$. Let C_{16} be such that

$$C_{13} + nC_{14} - C_{15} \log n > \log C_{12} \quad \text{for } n > C_{16}.$$

From inequality (75), it follows that

$$(77) \quad |f(u_n)| \geq \frac{|a_0|}{2}|u_n| \geq \exp(C_{17} + C_{14}n - C_{15}\log n) \quad \text{for } n > C_{16},$$

where $C_{17} = C_{13} + \log(|a_0|/2)$. From equation (15) and inequalities (74) and (77), we obtain

$$(78) \qquad C_{17} + C_{14}n - C_{15}\log n \leq C_{10} + C_{11}(\log(d^2 n))^2.$$

Inequality (78) clearly shows that $n < C_{18}$. $\square$

Proof of Theorem 4. Let $(u_n)_{n\geq 0}$ be a Lucas sequence. We first show that $2|\alpha|^n \geq u_n$ for $n \geq 1$. This is clear if $(u_n)_{n\geq 0}$ is a Lucas sequence of the second kind. Suppose that $(u_n)_{n\geq 0}$ is a Lucas sequence of the first kind. Since α and β are the two roots of the equation $x^2 - rx - s = 0$, it follows that $|\alpha - \beta|^2 = |r^2 + 4s| \geq 1$. Hence, $|\alpha - \beta| \geq 1$. It follows that

$$2|\alpha|^n \geq |\alpha^n - \beta^n| \geq \frac{\alpha^n - \beta^n}{\alpha - \beta} = u_n.$$

Suppose now that $u_n = \prod_{j=1}^k m_j!$ for some $n > \max(12, 2|\alpha| + 1)$. Assume that $m_1 \leq m_2 \leq \cdots \leq m_k$. It follows, by Theorem C, that $P(u_n) \geq n - 1$. Hence, $m_k \geq n - 1$. It follows that

$$(79) \qquad 2|\alpha|^n \geq u_n \geq m_k! \geq (n-1)! \geq \left(\frac{n-1}{e}\right)^{n-1},$$

where the last inequality above follows from Stirling's formula. We conclude that

$$(80) \qquad e|\alpha|(2|\alpha|)^{(1/(n-1))} > n - 1.$$

It is enough to show that $2 > (2|\alpha|)^{(1/(n-1))}$ or that $2^{n-1} > 2|\alpha|$. But this is immediate because $n - 1 > 2|\alpha|$ and because $2^x > x$ for $x > 0$. $\square$

Proof of Corollary 4. Straightforward consequence of Theorem 4. □

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ENDNOTES

1. Here we use Theorem 4 on page 275 in [**4**]. However, in [**4**] the bound is quadratic in $\log(d_{\mathbf{F}}^2 B)$. Kunrui Yu has informed us that the dependence of the bound is, in fact, linear in $\log(d_{\mathbf{F}}^2 B)$ and that the apparent quadratic dependence of the bound in [**4**] on this term is just a misprint.

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