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DUAL KRULL DIMENSION AND DUALITY

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Let R be a ring with Krull dimension. Then R has dual Krull dimension. Is there a relationship between the Krull dimension and the dual Krull dimension of R? We shall show that in certain situations the dual Krull dimension is bounded above by the Krull dimension.

All rings are associative with nonzero identity and all modules are unital right modules. The Hopkins-Levitzki theorem asserts that every right Artinian ring is right Noetherian. In other words, every ring with Krull dimension 0 has dual Krull dimension 0. More generally, if R is a right Artinian ring and M an R-module with Krull dimension, then M is Noetherian, i.e., if R has Krull dimension 0. Then any non-zero R-module with Krull dimension has dual Krull dimension 0. It is natural to wonder if this situation is typical, and we are led to ask the following questions.

Question 1. If R is a ring with Krull dimension, is the dual Krull dimension of R bounded above by the Krull dimension?

Question 2. If R is a ring with Krull dimension and M is an R-module with Krull dimension, is the dual Krull dimension of M bounded above by the Krull dimension of R?

As we shall see, these questions are equivalent, i.e., the dual Krull dimension is bounded above by the Krull dimension for *any* ring with Krull dimension if and only if the dual Krull dimension of M is bounded above by the Krull dimension of R for any ring R with Krull dimension and any R-module M with Krull dimension.

Although Questions 1 and 2 are left unanswered in this paper, we do have some information. If R is one of the following types of rings:

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(i) a commutative Noetherian ring, or

(ii) a commutative ring with Krull dimension 1, or

(iii) a maximal valuation domain with Krull dimension, or

(iv) a right Noetherian right V-ring,

and if M is an R-module with Krull dimension, then the dual Krull dimension of M is less than or equal to the Krull dimension of R. In particular, the dual Krull dimension of R is less than or equal to the Krull dimension of R. This latter fact is also true for rings R of the following types:

(v) a commutative domain with Krull dimension 2, or

(vi) a valuation domain.

1. Krull dimension and dual Krull dimension. Let R be a ring, and let M be a (right) R-module. Suppose that M has Krull dimension (for the definition of (Gabriel-Rentschler) Krull dimension see, for example, [**2**] or [**8**] or [**11**]). Lemonnier [**9**, Corollaire 6] showed that in this case M has dual Krull dimension (for the definition, see [**9**] or [**2**]). We shall denote the Krull dimension and the dual Krull dimension of M by $k_R(M)$ and $k_R^0(M)$, respectively. In case $M = R_R$, we write $k_R(R)$ and $k_R^0(R)$ for the Krull dimension and the dual Krull dimension of the right R-module R, if they exist. If there is no ambiguity, we write $k(M)$ for $k_R(M)$, etc.

As we have already remarked, the Hopkins-Levitzki theorem can be restated thus: *if R is a ring with $k(R) = 0$, i.e., R is right Artinian, then $k^0(R) = 0$, i.e., R is right Noetherian, and, more generally, $k^0(M) \leq 0$ for every R-module M with Krull dimension.* For an account of the Hopkins-Levitzki theorem and various generalizations of it, see [**1**] and [**2**]. Gordon and Robson [**8**, Theorem 9.8] prove that there exist commutative Noetherian domains of arbitrary Krull dimension. In other words, for any ordinal $\alpha \geq 0$, there exists a commutative domain R such that $k(R) = \alpha$ and $k^0(R) = 0$. In all these cases, $k^0(R) \leq k(R)$. Is this always true? We suspect not, but so little is known at the present time we are really only guessing.

If R is a ring with $k(R) = 0$, then $k^0(M) \leq 0$ for any R-module with Krull dimension. Also, for any prime p in $\mathbf{Z}$, the ring of rational integers, the Prüfer p-group A satisfies

$$k(A) = 0 \quad \text{and} \quad k^0(A) = 1 = k(\mathbf{Z}).$$

More generally, Lemonnier [**10**, Corollaire 4.5], see also [**6**, Proposition 13], proved that if R is a commutative Noetherian ring and M an R-module with Krull dimension, then $k^0(M) \leq k(R)$. Thus we ask if it is always the case that $k^0(M) \leq k(R)$ for a module M with Krull dimension over a ring R with Krull dimension?

Let R be a ring. A nonzero R-module N is called *critical* if N has Krull dimension and $k(N/L) < k(N)$ for every nonzero submodule L of N. Dually, N is called *dual critical* if N has Krull dimension and $k^0(L) < k^0(N)$ for any proper submodule L of N. It is well known that any nonzero submodule, respectively nonzero factor module, of a critical, respectively dual critical, module C is again critical, respectively dual critical, with the same Krull dimension, respectively dual Krull dimension, as C (see, for example, [**8**, Proposition 2.3] and [**6**, Proposition 3]).

The first result can be found in [**2**, Corollary 3.14] (or see [**10**, Proposition 4.1] or [**6**, Proposition 10]).

Lemma 1.1. *Let R be any ring, and let M be an R-module with Krull dimension. Then*

(i) $k(M) = \sup\{k(N/L) : L \subseteq N$ *are submodules of* M *and* N/L *is critical*$\}$, *and*

(ii) $k^0(M) = \sup\{k^0(N/L) : L \subseteq N$ *are submodules of* M *and* N/L *is dual critical*$\}$.

The next lemma is surely well known, but we do not have a reference for it. The proof is an easy exercise using [**8**, Lemma 1.1] and is left to the reader.

Lemma 1.2. *Let a ring R be the direct product $R' \times R''$ of rings R', R'', and let M be an R-module. Then the R-module M has* (*dual*) *Krull dimension if and only if the R'-module MR' and the R''-module MR'' have* (*dual*) *Krull dimension, and in this case*

$$k_R(M) = \sup\{k_{R'}(MR'), k_{R''}(MR'')\}$$

and

$$k^0_R(M) = \sup\{k^0_{R'}(MR'), k^0_{R''}(MR'')\}.$$

In particular, $k(R) = \sup\{k(R'), k(R'')\}$ *and* $k^0(R) = \sup\{k^0(R'), k^0(R'')\}$, *if either side exists in each case.*

A nonzero module M is called *subdirectly irreducible* or *cocyclic* (see [**14**]), if the intersection of all nonzero submodules of M is nonzero, in other words M contains an essential simple submodule. It is well known that every nonzero module N is isomorphic to a subdirect product of its subdirectly irreducible factor modules and, in particular, N has a subdirectly irreducible factor module (see, for example, [**4**, Exercise 6.20] or [**14**]).

Theorem 1.3. *The following statements are equivalent for a ring R with Krull dimension ≥ 1.*

(i) $k^0(M) \leq k(R)$ *for every R-module with Krull dimension.*

(ii) $k^0(H) \leq k(R)$ *for every subdirectly irreducible dual critical R-module H such that $HP = 0$ for some prime ideal P of R and $H = HI$ for every ideal I properly containing P.*

(iii) $k^0(S) \leq k(S)$ *for every ring S with Krull dimension such that $S/I \cong R \times \mathbf{Z}$ for some nilpotent ideal I of R.*

Proof. (i) $\Rightarrow$ (ii). Clear.

(ii) $\Rightarrow$ (i). By Lemma 1.1 it is sufficient to prove that $k^0(N) \leq k(R)$ for any dual critical R-module N. Since R has Krull dimension, there exist a positive integer n and prime ideals P_i, $1 \leq i \leq n$, such that $P_1 \cdots P_n = 0$ [**8**, Theorem 7.4]. If $N = NP_i$ for all $1 \leq i \leq n$, then $N = NP_n = NP_{n-1}P_n = \cdots = NP_1 \cdots P_n = 0$, a contradiction. Thus, $N \neq NP_j$ for some $1 \leq j \leq n$. By [**8**, Theorem 7.1], R satisfies the ascending chain condition on prime ideals. Let P be a prime ideal of R maximal with respect to the property $N \neq NP$.

Let I be any ideal properly containing P. There exist a positive integer k and prime ideals Q_i, $1 \leq i \leq k$, such that $Q_1 \cdots Q_k \subseteq I \subseteq Q_1 \cap \cdots \cap Q_k$, again by [**8**, Theorem 7.4]. By the choice of P, $N = NQ_i$, $1 \leq i \leq k$, and hence $N = NI$. There exists a proper submodule L of N such that $NP \subseteq L$ and N/L is subdirectly irreducible. Let

$H = N/L$. Then H is subdirectly irreducible and dual critical, $HP = 0$ and $H = HI$ for every ideal I properly containing P. Moreover, $k^0(N) = k^0(H) \leq k(R)$, by (ii). This proves (i).

(i) $\Rightarrow$ (iii). Let S be a ring with Krull dimension such that there exists an isomorphism $\iota : R \times \mathbf{Z} \to S/I$, for some nilpotent ideal I of S. Then $k(S) = k(S/I) = k(R \times \mathbf{Z}) = k(R)$ by Lemma 1.2. There exists a positive integer t such that $I^t = 0$. Consider the chain $S = I^0 \supseteq I \supseteq I^2 \supseteq \cdots \supseteq I^t = 0$. For each $1 \leq j \leq t$, let $A_j = I^{j-1}/I^j$, note that $A_j I = 0$ and hence, by (i), Lemma 1.2 and [**10**, Corollaire 4.5], that

$$\begin{aligned} k_S^0(A_j) = (k_{S/I}^0(A_j) &= \sup\{k_R^0(A_j\iota(R)), k_{\mathbf{Z}}^0(A_j\iota(\mathbf{Z}))\} \\ &\leq \sup\{k(R), 1\} = k(R) = k(S). \end{aligned}$$

Thus $k^0(S) \leq k(S)$ by [**6**, Proposition 5].

(iii) $\Rightarrow$ (i). Let M be any R-module with Krull dimension. Consider the ring S of "matrices" of the form

$$\begin{pmatrix} a & m \\ 0 & r \end{pmatrix}$$

where $a \in \mathbf{Z}$, $m \in M$, $r \in R$, and addition and multiplication are as usual for matrices. Let I denote the set of matrices of the above type with $a = 0$, $r = 0$. Then I is an ideal of S, $I^2 = 0$ and $S/I \cong R \times \mathbf{Z}$. It is clear that the ring S has Krull dimension and $k(S) = k(R)$ [**8**, Lemma 1.1]. By hypothesis, $k^0(S) \leq k(S)$. Now $k_R^0(M) = k_S^0(I) \leq k^0(S) \leq k(S) = k(R)$, as required.

The next result is [**3**, Corollary 4.2], but the proof we now give is more straightforward.

Corollary 1.4. *Let R be a commutative ring with Krull dimension 1. Then $k^0(M) \leq 1$ for every R-module M with Krull dimension.*

Proof. By the theorem it is sufficient to consider an R-module M such that $MP = 0$ for some prime ideal P of R. In this case, R/P is a domain with $k(R/P) \leq 1$, so that R/P is Noetherian. For, if I is any ideal of R which properly contains P,then $k(R/I) < k(R/P)$ by [**8**, Proposition 6.1], so that R/I is Artinian and hence Noetherian. Now [**10**, Corollaire 4.5] gives: $k^0(M) \leq k(R/P) \leq k(R) = 1$.

Corollary 1.5. *Let R be a commutative domain with Krull dimension 2. Then $k^0(R) \leq 2$.*

Proof. For any nonzero ideal I of R, $k(R/I) \leq 1$, by [**8**, Proposition 6.1], and hence $k^0(R/I) \leq 1$ by Corollary 1.4. Thus, $k^0(R) \leq 2$.

Corollary 1.6. *The following statements are equivalent for an ordinal $\alpha \geq 0$.*

(i) *$k^0(M) \leq \alpha$ for all (prime) rings R with Krull dimension α and all R-modules M with Krull dimension.*

(ii) *$k^0(S) \leq \alpha$ for any ring S with Krull dimension α.*

Proof. (i) $\Rightarrow$ (ii). Clear.

(ii) $\Rightarrow$ (i). By the proof of (iii) $\Rightarrow$ (i) in Theorem 1.3.

Note that, in particular, Corollary 1.6 shows that Questions 1 and 2 in the foreword are equivalent.

We now define two ordinals associated with any ring R. Choose representatives U_λ, $\lambda \in \Lambda$, from each of the isomorphism classes of simple R-modules. For each λ in Λ, let E_λ denote the injective hull of U_λ. Consider the collection of submodules with Krull dimension of the E_λ's. Since this collection is a set, the following supremum exists:

$$\delta^0(R) = \sup\{k^0_R(N) : N \text{ is a submodule with Krull dimension of } E_\lambda \text{ for some } \lambda \in \Lambda\}.$$

Proposition 1.7. *For any ring R,*

$$\delta^0(R) = \sup\{k^0_R(M) : M \text{ is an } R\text{-module with Krull dimension}\}.$$

Proof. Let M be an R-module with Krull dimension. Let $L \subseteq N$ be submodules of M such that N/L is dual critical. There exists a proper submodule K of N containing L such that N/K is subdirectly irreducible. There exists λ in Λ such that N/K is isomorphic to a submodule of E_λ. Then $k^0(N/L) = k^0(N/K) \leq \delta^0(R)$. By Lemma 1.1, $k^0(M) \leq \delta^0(R)$. The result follows.

For any ring R we can define a dual to $\delta^0(R)$, namely, $\delta(R) = \sup\{k_R(M) : M$ is a cyclic R-module with Krull dimension$\}$. Using Lemma 1.1 and [**8**, Corollary 4.4], one deduces at once the following result.

Proposition 1.8. *Let R be any ring. Then*

$$\delta(R) = \sup\{k_R(M) : M \textit{ is an } R\textit{-module with Krull dimension}\}.$$

In particular, $\delta(R) = k(R)$ in case R is a ring with Krull dimension.

It is natural to ask if the ordinals $\delta(R)$, $\delta^0(R)$ are accessible, i.e., do there exist R-modules X and Y such that $\delta^0(R) = k_R^0(X)$ and $\delta(R) = k_R(Y)$?

In view of Proposition 1.7, Question 2 in the foreword can be restated thus:

Is $\delta^0(R) \leq k(R)$ for any ring R with Krull dimension?

More generally, we can formulate the following problem. For a given ring R, is there a relationship between the ordinals $\delta(R)$ and $\delta^0(R)$? Note that if R has Krull dimension, then $\delta(R) = k(R)$, so that Question 2 raises a more general question, namely,

Question 3. For which rings R is $\delta^0(R) \leq \delta(R)$?

Recall that a ring R is a right V-ring if every simple right R-module is injective. Let R be a right V-ring. Then $\delta^0(R) = 0$ by definition, and hence $\delta^0(R) \leq \delta(R)$.

2. Duality. Let R be any ring. Let M be an R-module with Krull dimension. Suppose that there exists an R-module M^0 such that the lattice Lat (M) of submodules of M is anti-isomorphic to the lattice Lat (M^0) of submodules of M^0. Then M^0 has Krull dimension by [**9**, Corollaire 6] and

$$k_R^0(M) = k_R(M^0) \quad \text{and} \quad k_R(M) = k_R^0(M^0).$$

If such a lattice anti-isomorphism exists for the module M then, because submodule lattices are upper continuous, the lattice Lat (M) would need to be lower continuous, i.e., M would have to satisfy:

$$N + (\cap_\Lambda L_\lambda) = \cap_\Lambda (N + L_\lambda),$$

for any submodule N and chain of submodules L_λ, $\lambda \in \Lambda$. Such a module M is said to satisfy the property $AB5^*$. Of course, most modules M do not have this property, although linearly compact modules do, see, for example, [**13**, p. 116] or [**14**]. Artinian modules are linearly compact. In view of this fact, it seems sensible to ask:

Question 2*. Let R be any ring. Is $k^0(M) \le \delta(R)$ for every Artinian R-module M?

Suppose that R is a ring such that for every R-module M with Krull dimension there exists an R-module M^0 such that the lattice of submodules of M is anti-isomorphic to the lattice of submodules of M^0. Then $\delta(R) = \delta^0(R)$. This is the motivation for the following discussion.

Throughout the remainder of the paper all rings are *commutative*. What follows now is based on [**12**, Section 5.4]. Let R be a commutative ring, and let E be an injective cogenerator for the category Mod $- R$ of all unital R-modules. For any R-module M, let M^* denote the dual of M relative to E, or the E-dual of M, namely $M^* = \text{Hom}_R(M, E)$. Then M^* is an R-module in the usual way, i.e., $(r\alpha)(m) = r\alpha(m)$ for all $r \in R$, $\alpha \in M^*$, $m \in M$. For any submodule N of M and any submodule H of M^*, we define

$$N^+ = \{\alpha \in M^* : \alpha(n) = 0 \text{ for all } n \in N\},$$
$$H^- = \{m \in M : \alpha(m) = 0 \text{ for all } \alpha \in H\}.$$

Clearly, N^+ is a submodule of M^* and H^- is a submodule of M.

For each R-module M, denote $M^{**} = (M^*)^*$ and define $\varepsilon_M : M \to M^{**}$ by

$$\varepsilon_M(m)(\alpha) = \alpha(m), \quad m \in M, \alpha \in M^*.$$

Then ε_M is an R-homomorphism, called the *evaluation map*. The module M is said to be *E-reflexive* in case ε_M is an isomorphims and is said to be *E-torsionless* in case ε_M is a monomorphism.

The next lemma collects some known facts, see, for example, [**4**].

Lemma 2.1. *The following statements hold for an R-module M.*

(i) $(N_1 + N_2)^+ = N_1^+ \cap N_2^+$ *for any submodules N_1 and N_2 of M.*

(ii) *If $L \subseteq N$ are submodules of M, then $N^+ \subseteq L^+$ and $L^+ = N^+$ if and only if $L = N$.*

(iii) $N = N^{+-}$ *for any submodule N of M.*

(iv) *M is E-torsionless.*

Lemma 2.2. *Let M be an E-reflexive module. With the above notation, one has*:

(i) $\varepsilon_M(H^-) = \{\alpha \in M^{**} : \alpha(H) = 0\}$ *for any submodule H of M^*.*

(ii) *The mapping $N \mapsto N^+$ is an anti-isomorphism from* $\mathrm{Lat}\,(M)$ *to* $\mathrm{Lat}\,(M^*)$.

Proof. (i) is easy and is left to the reader.

(ii) Let $H \subseteq K$ be distinct submodules of M^*. By Lemma 2.1, $K^\times \subseteq H^\times$ are distinct submodules, where $H^\times = \{\alpha \in M^{**} : \alpha(H) = 0\}$. By (i), $\varepsilon_A(H^-) = H^\times \supseteq K^\times = \varepsilon_M(K^-)$, so that $H^- \supseteq K^-$ are distinct. If now H and K are arbitrary submodules of M^* such that $H^- = K^-$, then $(H+K)^- = H^- \cap K^- = H^- = K^-$, and so $H + K = H = K$. Thus the mapping $H \mapsto H^-$ from $\mathrm{Lat}\,(M^*)$ to $\mathrm{Lat}\,(M)$ is injective, and, by Lemma 2.1 (iii), it is also onto. It follows that $H \mapsto H^-$ is a lattice anti-isomorphism from $\mathrm{Lat}\,(M^*)$ to $\mathrm{Lat}\,(M)$ with inverse $N \mapsto N^+$ from $\mathrm{Lat}\,(M)$ to $\mathrm{Lat}\,(M^*)$.

Combining Lemma 2.2 (ii) with the remarks at the beginning of this section, we have at once:

Proposition 2.3. *Let R be a commutative ring, and let E be an injective cogenerator for* $\mathrm{Mod} - R$. *Then $k^0(M) \leq \delta(R)$ for any E-reflexive R-module M with Krull dimension.*

Next we show that if $\varepsilon_E : E \to E^{**}$ is an isomorphism then so too is ε_M for any module M with Krull dimension.

Theorem 2.4. *Let R be a commutative ring such that there exists an injective cogenerator E for $Mod - R$ which is E-reflexive. Then $\delta^0(R) \leq \delta(R)$.*

Proof. Let M be any subdirectly irreducible R-module with Krull dimension, and let U denote the essential simple socle of M. Clearly U, and hence M, embeds in E. There exists an exact sequence $0 \to M \to E \to X \to 0$, for a suitable module X, and we can form the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varepsilon_M} & & \downarrow{\scriptstyle \varepsilon_E} & & \downarrow{\scriptstyle \varepsilon_X} & & \\ 0 & \longrightarrow & M^{**} & \longrightarrow & E^{**} & \longrightarrow & X^{**} & \longrightarrow & 0 \end{array}$$

By Lemma 2.1 (iv), the mappings ε_M and ε_X are monomorphisms and, by hypothesis, ε_E is an isomorphism. By [**4**, Lemma 3.14], ε_M is an isomorphism. By Proposition 2.3, $k^0(M) \leq \delta(R)$. Hence $\delta^0(R) \leq \delta(R)$. □

The next result is surely known but we include a proof for completeness.

Proposition 2.5. *Let R be a commutative ring, and let E be an injective cogenerator for* $\mathrm{Mod} - R$ *such that every endomorphism of E is given by multiplication by an element of R. Then E is E-reflexive. Moreover, the converse holds if R embeds in E.*

Proof. Suppose that every endomorphism of E is given by multiplication by an element of R. Let $\alpha \in E^{**}$. Let $\iota : E \to E$ be the identity mapping, and let $e = \alpha(\iota) \in E$. We claim that $\alpha = \varepsilon_E(e)$. Let $\beta \in E^*$. Then there exists $r \in R$ such that $\beta(x) = rx$ for all x in E. Thus, $\beta = r\iota$. Now

$$\varepsilon_E(e)(\beta) = \beta(e) = re = r\alpha(\iota) = \alpha(r\iota) = \alpha(\beta).$$

It follows that $\alpha = \varepsilon_E(e)$. Hence ε_E is onto and, by Lemma 2.1 (iv), ε_E is an isomorphism.

Now suppose that R embeds in E and that ε_E is an isomorphism. By the proof of Theorem 2.4, $\varepsilon_R : R \to R^{**}$ is an isomorphism. Let θ be any endomorphism of E. Note that $R^* = \mathrm{Hom}_R(R, E) \cong E$, and we let $\eta : R^* \to E$ denote the canonical isomorphism. Then $\theta\eta \in R^{**} = \varepsilon_R(R)$. There exists $a \in R$ such that $\theta\eta = \varepsilon_R(a)$. For any $e \in E$, $\theta(e) = \theta\eta(\gamma)$, where $\gamma \in R^*$ satisfies $\gamma(1) = e$, so that $\theta(e) = \theta\eta(\gamma) = \varepsilon_R(a)(\gamma) = \gamma(a) = a\gamma(1) = ae$. Thus, $\theta(e) = ae$ for all e in E.

In [**13**] Vamos defines a commutative ring R to be *classical* if the injective hull of every simple R-module is linearly compact. He proves that if R is a complete local, not necessarily Noetherian, ring with E the injective hull of the unique simple R-module and R is classical, then every endomorphism of E consists of multiplication by an element of R [**13**, Theorem 3.1]. Combining this fact with Theorem 2.4 and Proposition 2.5 we have at once:

Theorem 2.6. *Let R be a complete local classical ring. Then $\delta^0(R) \leq \delta(R)$.*

Vamos [**13**, Corollary 4.2 and Proposition 4.3] points out that Noetherian rings are classical and maximal valuation rings are complete local and classical. Recall that R is called a *valuation ring* if, for all a, b in R, either $Ra \subseteq Rb$ or $Rb \subseteq Ra$. A valuation ring R is called *maximal* provided the R-module R is linearly compact. Thus $\delta^0(R) \leq \delta(R)$ for every complete local Noetherian ring R and every maximal valuation ring R, by Theorem 2.6.

Corollary 2.7. *Let R be a valuation domain with Krull dimension. Then $k^0(R) \leq k(R)$.*

Proof. By [**7**, Theorems 1.1.11 and 1.5.10] there exist a maximal valuation domain S and a monomorphism $\varphi : R \to S$ such that the mapping $I \mapsto \varphi(I)S$ from $\mathrm{Lat}\,(R_R)$ to $\mathrm{Lat}\,(S_S)$ is an isomorphism with inverse $J \mapsto \varphi^{-1}(J \cap \varphi(R))$. By Theorem 2.6, $k^0(R) = k^0(S) \le k(S) = k(R)$, as required.

In [**5**], a *Baer duality* is defined to be a triple $(R, {}_R U_S, S)$ consisting of rings R, S and a bimodule ${}_R U_S$, faithful on both sides, such that $\mathrm{Lat}\,({}_R R)$ and $\mathrm{Lat}\,(U_S)$ are anti-isomorphic, as are $\mathrm{Lat}\,({}_R U)$ and $\mathrm{Lat}\,(S_S)$. In this case, if S has Krull dimension, then $k^0(S) = k({}_R U) \le \delta({}_R R)$ and $k(S) = k^0({}_R U) \le \delta^0({}_R R)$. If R and S are isomorphic rings, then R is said to be *Baer selfdual* and, in case R has Krull dimension, $k^0(R) \le \delta({}_R R)$ and $k(R) \le \delta^0({}_R R)$. In particular, any commutative Baer selfdual ring R with Krull dimension satisfies $k^0(R) \le \delta(R) = k(R) \le \delta^0(R)$. Every valuation domain is Baer selfdual with respect to K/R, where K is the field of fractions of R [**5**, p. 7]. This gives another proof of Corollary 2.7. In [**5**] Anh and Menini conjecture that every valuation ring is Baer selfdual.

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