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A CHARACTERIZATION OF FGC RINGS

GEORGE IVANOV AND PETER VÁMOS

ABSTRACT. In this note we give a new characterization of commutative rings for which every finitely generated module is a direct sum of cyclics (FGC rings) using only the structure of the injective envelopes of simple modules. Some Baer-Kaplansky categories for FGC rings are studied.

1. Introduction. The structure of commutative rings for which every finitely generated module is a direct sum of cyclics was determined more than twenty-five years ago as the culmination of the work of a number of mathematicians over many years. A self contained exposition of the proof is given in [**14**]. The characterization is internal but it relies on the structure of almost maximal valuation rings for which Gill ([**3**]) obtained a characterization in terms of their indecomposable injectives. An analogous characterization of noncommutative serial rings whose finitely generated modules are direct sums of uniserials (the noncommutative analogue of almost maximal valuation rings) was obtained in [**4**]. It may therefore be useful for the study of noncommutative analogues of FGC rings to have a characterization of FGC rings in terms of their injectives. This note does that. The proof is obtained by reducing to the structure theorem obtained in [**14**]. It would therefore be valuable to have a direct proof that the rings satisfying the characterization are FGC rings.

There has been considerable interest in categories of modules which are determined by their endomorphism rings since the pioneering work of Baer [**1**] and Kaplansky [**6**]. (We refer the reader to [**8**] for a survey of this and related areas and an extensive bibliography of nearly 300 items.) However, it was only recently that Baer-Kaplansky categories for virtually arbitrary rings were shown to exist [**5**]. We obtain some Baer-Kaplansky categories of modules for arbitrary FGC rings and thus show that in this sense, also, FGC rings are generalizations of the ring of integers **Z**.

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2. Main results. Throughout this note all rings are commutative with identity elements and all modules are unital. Every category of modules is assumed to be full.

Definitions. A ring R is *arithmetical* if its lattice of ideals is distributive. This is equivalent to every quotient ring of R having a heterogeneous socle ([**11**]) or to R_M being a valuation ring for every maximal ideal M. A module is *Bezout* if all its finitely generated submodules are cyclic. A module has *finite* (*Goldie*) *dimension* if it contains no infinite direct sum of submodules. A submodule W is a *waist* if every other submodule either contains W or is contained in W. A ring R is a *torch ring* if it has a unique minimal prime ideal P which is uniserial and nonzero; if R/P is an h-local Bezout domain; and if R_M is an almost maximal valuation ring, for every maximal ideal M.

Lemma 1. *A finite dimensional, commutative and indecomposable arithmetical ring R has a unique minimal prime ideal which is a waist in R.*

Proof. Let N be the nilradical of R, and let $\overline{R} = R/N$. Since idempotents lift modulo a nil ideal, $\overline{R}$ is still an indecomposable arithmetical ring. Then by Lemma 7 of [**12**], $\overline{R}$ is a uniform ring with no nilpotent elements. But such a ring is a domain: for if $0 \neq a \in R$ and $\operatorname{Ann} a \neq 0$, then $0 \neq Ra \cap \operatorname{Ann} a$ would be a nonzero nil ideal. Hence N is the unique minimal prime ideal of R. It is contained in every maximal ideal and thus in the Jacobson radical of R. Hence it is a waist in R by [**11**, Proposition 2.1] or [**10**, Lemma 3.1]. □

Lemma 2. *Let R be a commutative arithmetical ring whose Jacobson radical contains a prime ideal P. If S is a simple subfactor of P and M is a maximal ideal of R, then* $\operatorname{Hom}(E(S), E(R/M)) \neq 0$.

Proof. Let S be a simple subfactor of P. Then $S = Ra/Na$ for some $a \in P$ and maximal ideal N. Let L be a complement of Na in P. Since P is a waist [**11**, Proposition 2.1], $R/(L+Na)$ is an essential extension of S and it contains a subfactor isomorphic to R/M, for any maximal ideal M. Hence there is a nonzero homomorphism from a submodule of $R/(L+Na)$ to R/M which can be extended to a homomorphism $E(S) \to E(R/M)$. $\square$

The set of maximal ideals of a ring R will be denoted by $\mathcal{M}(R)$. We will write homomorphisms on the right of their arguments.

Theorem 3. *An indecomposable commutative ring R is an* FGC *ring if and only if all of the following conditions hold.*

(a) *Every indecomposable injective R-module is Bezout.*

(b) *If $M, N \in \mathcal{M}(R)$, $M \neq N$, then* $\mathrm{Hom}_R(E(R/M), E(R/N)) = 0$ *except for at most one such M, in which case* $\mathrm{Hom}_R(E(R/M), E(R/N)) \neq 0$ *for all $N \in \mathcal{M}(R)$.*

(c) $\bigoplus_{M \in \mathcal{M}(R)} E(R/M)$ *is injective.*

Proof. Let R be an indecomposable commutative FGC ring. Then R is one of the following three types of rings: (i) an almost maximal valuation ring; (ii) an h-local Bezout domain which is locally almost maximal; or (iii) a torch ring ([**14**], [**12**]).

(i) In this case there is only one maximal ideal and the result follows from Gill's characterization of these rings ([**3**]).

(ii) An indecomposable injective module E is uniform so any one of its finitely generated submodules must be cyclic. (Note that if R is local, then E must be uniserial.) Therefore (a) holds.

If M is a maximal ideal of R then $E_R(R/M)$ is a torsion module and by Theorem 22 of [**7**] it is isomorphic to $E_{R_M}(R/M)_M$. By Gill's theorem this module is uniserial. Therefore, $\mathrm{Hom}_R(E(R/M), E(R/N)) \neq 0$ if and only if $M = N$. Therefore (b) holds.

Let $E = \bigoplus_M E(R/M)$ and let $\phi : L \to E$ be a homomorphism from an ideal L of R. By [**12**] and [**2**] every factor ring of R has finite dimension so if $K = \ker\phi$ then L/K has finite socle. Therefore $\mathrm{Soc}\,(L/K)\phi$ is contained in a finite subsum $\bigoplus_{i=1}^{n} E(R/M_i)$ which is injective since each summand is injective. It follows that ϕ can be extended to a homomorphism $\Phi : R \to E$ which means that E is injective and so (c) holds.

(iii) Both (a) and (c) are proved as for (ii).

If R is a torch ring, then it has a unique minimal nonzero prime ideal P, which is uniserial and all of its simple subfactors are isomorphic to R/M for a maximal ideal M. Moreover P is a waist [**12**], R/P is a h-local Bezout domain which is locally almost maximal and R has at least two maximal ideals M and N, say. Since R/M and R/N are not isomorphic, $PL = 0$ for any submodule L of $E(R/N)$ and so $E(R/N)$ is an R/P module. It follows from (ii)(b) above that $\mathrm{Hom}_R(E(R/N'), E(R/N)) = 0$ when both $N, N' \neq M$. If $0 \neq a \in P$, then $Ra/Ma \cong R/M$ and so $R/Ma \subseteq E(R/M)$. But $Ma \subseteq N$ so there is a projection $R/Ma \to R/N$ and it lifts to a nonzero map $E(R/M) \to E(R/N)$. Therefore $\mathrm{Hom}_R(E(R/M), E(R/N)) \neq 0$ and so (b) holds.

Now consider the converse and let R be a ring satisfying (a)–(c). Let N be a maximal ideal of R. Then it is well known (e.g., [**9**, Proposition 5.6]) that $E(R/N)$ is in a natural way an injective R_N-module with simple socle. By (a) it is Bezout as an R-module and thus as an R_N-module. Since R_N is local, $E(R/N)$ is uniserial as an R_N-module. By [**3**], R_N is an almost maximal valuation ring. Since this is true for all maximal ideals, it follows ([**12**]) that R is arithmetical.

As R is arithmetical all its ideals and their quotient modules have distributive submodule lattices and so their socles, if they exist, are heterogeneous. From (c) it follows that the socle, if it exists, of every factor module of every ideal L has a finite number of nonisomorphic simple modules. Otherwise there would be a homomorphism from the socle of L to $E = \bigoplus_{M \in \mathcal{M}(R)} E(R/M)$ whose image is in an infinite number of the $E(R/M)$'s, and that is impossible since the homomorphism extends to $R \to E$ and the image of the identity of R lies in only a finite number of $E(R/M)$'s. Since the socle has to be heterogeneous, it must be finite. Hence R has finite dimension and the same is true for every quotient ring of R. By Lemma 1, R has a unique minimal prime ideal which we will denote by P. If R has a unique maximal ideal, then it is an almost maximal valuation ring and thus an FGC ring.

So we can assume that R has at least two maximal ideals. First we assume that $P = 0$, that is to say, R is a domain. Let P_1 be a nonzero prime ideal of R and assume that it is contained in two maximal ideals M and N. Then localizing at $T = R \setminus (M \cup N)$ produces a ring $R' = T^{-1}R$ with two maximal ideals $M' = T^{-1}M$ and

$N' = T^{-1}N$ and prime ideal $P' = T^{-1}P_1$. We may assume that P' has a simple subfactor isomorphic to R'/N'. Since P' is in the Jacobson radical of $T^{-1}R$, Lemma 2 shows that $\mathrm{Hom}_{R'}(E(R'/N'), E(R'/M')) \neq 0$. Proposition 5.6 of [**9**] and a standard argument now shows that $\mathrm{Hom}_R(E(R/N), E(R/M)) \neq 0$. But this is impossible since, by the above argument, $E(R/M)$ and $E(R/N)$ are uniserial R_M and R_N modules with no common subfactors. Hence every nonzero prime ideal of R is contained in only one maximal ideal.

Let I be a nonzero ideal of R which is not prime. Then R/I is a (finite) direct sum of indecomposable rings R_i each of which has a unique minimal prime N_i, by the argument above. Each N_i gives rise to a unique prime in R/I which comes from a prime $P_i \supseteq I$ in R. But each P_i is contained in only one maximal ideal, so I is contained in only a finite number of maximal ideals (since there are only a finite number of P_i's). Hence R is an h-local Bezout domain.

Now consider the case when $P \neq 0$ and let $S = R/M$, $M \in \mathcal{M}$ be a simple subfactor of P. By Lemma 2, $\mathrm{Hom}\,(E(R/M), E(R/N)) \neq 0$ for all maximal ideals N and from (b) it follows that M is unique. Since the submodule lattice of P is distributive, it follows that P is uniserial whose simple subfactors are all isomorphic to S. Consequently R is a torch ring and is therefore an FGC ring ([**12**]). □

We now turn to the question of endomorphism rings of modules over FGC rings. The celebrated Baer-Kaplansky theorem tells us that abelian torsion groups are determined by their endomorphism rings. Since FGC rings are generalizations of the ring **Z** of integers, we are interested in discovering how much of that theorem can be applied to modules over such rings.

Definition. A (full) category of R-modules is said to be a *Baer-Kaplansky* (*or* $B-K$) *category* or to have the *Baer-Kaplansky* (*or* $B-K$) *property* if any two of its modules are isomorphic whenever their endomorphism rings are isomorphic (as rings).

Theorem 4. *Let R be an indecomposable* FGC *ring and let $\mathcal{C}$ be a category of finitely generated R-modules each of which has a copy of R as a summand. Then $\mathcal{C}$ is a Baer-Kaplansky category.*

Proof. Let M and N be two modules in $\mathcal{C}$ and let $\Phi : \mathrm{End}\,(M) \rightarrow \mathrm{End}\,(N)$ be an isomorphism. Let $M = R \bigoplus_i M_i$ be a decomposition into indecomposable cyclic submodules, let $G = \mathrm{End}\,(M)$ and $H = \mathrm{End}\,(N)$. Let $e_R, e_i \in G$ be the canonic projections onto R and M_i, respectively, and let $N_R = N(e_R\Phi)$ and $N_i = N(e_i\Phi)$. Now $\mathrm{End}\,(R) \approx R$ so $\mathrm{End}\,(N_R) \approx R$ and therefore N_R is a faithful R-module. Since N_R is a summand of N it is finitely generated so by Lemma 11 of [**12**] it has a summand isomorphic to R. But N_R is indecomposable (since $e_R\Phi$ is a primitive idempotent) so $N_R \cong R$.

Now G contains a subring $\mathrm{End}\,(R) \approx R$ and a subgroup $\mathrm{Hom}\,(R, M_i)$ which is a left R-module and Φ maps these isomorphically onto $\mathrm{End}\,(N_R)$ and $\mathrm{Hom}\,(N_R, N_i)$, respectively. Hence $\mathrm{Hom}\,(R, M_i)$ and $\mathrm{Hom}\,(N_R, N_i)$ are isomorphic R-modules. But $\mathrm{Hom}\,(N_R, N_i)$ is isomorphic to $\mathrm{Hom}\,(R, N_i) \cong N_i$ and $\mathrm{Hom}\,(R, M_i)$ is isomorphic to M_i so M_i is isomorphic to N_i. By [**5**, Proposition 1], M is isomorphic to N and so $\mathcal{C}$ is a B-K category. □

Corollary 5. *If R is an indecomposable* FGC *ring, then the category of finitely generated faithful R-modules is a Baer-Kaplansky category.*

Proof. By [**12**, Lemma 11] every faithful R-module has a summand isomorphic to R. □

Proposition 6. *Let R be an almost maximal valuation ring and $\mathcal{C}$ a category of* (*possibly infinite*) *direct sums of cyclic R-modules. If each object in $\mathcal{C}$ has a summand isomorphic to R, then $\mathcal{C}$ is a* B-K *category.*

Proof. The proof is as for Theorem 4 after observing that an indecomposable summand of N with local endomorphism ring is isomorphic to one of its cyclic summands (by Warfield's result [**13**]). □

For abelian groups, Theorem 4 and Proposition 6 are the well-known Baer-Kaplansky results. For if A is an abelian group which is torsion, then it is a direct sum of cyclic primary groups $\mathbf{Z}/p_i^{n_i}\mathbf{Z}$, for primes p_i. The sum of one copy of the primary cyclics of highest exponents is in fact a ring (under the usual multiplication) and the group A is a module over this ring. If A is not torsion but is finitely generated, then it contains a copy of **Z** as a summand. In both cases the hypotheses of Theorem 4 and Proposition 6 are satisfied. Hence in this sense FGC rings are true generalizations of the ring of integers **Z**.

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Department of Mathematics, Macquarie University, NSW 2109, Australia
E-mail address: ivanov@maths.mq.edu.au

Department of Mathematics, University of Exeter, Exeter, United Kingdom
E-mail address: p.vamos@ex.ac.uk