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THE KRULL-SCHMIDT PROPERTY FOR IDEALS AND MODULES OVER INTEGRAL DOMAINS

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Dedicated to Jim Reid

ABSTRACT. We examine when versions of the Krull-Schmidt property hold for (1) direct sums of ideals of integral domains, (2) direct sums of indecomposable submodules of finitely generated free modules, and (3) direct sums of rank one torsion-free modules. Our main results are formulated for modules over h-local integral domains without recourse to finite generation for the modules. This leads to some new results for Krull-Schmidt properties of modules over Noetherian and Prüfer domains.

1. Introduction. Let R be a commutative integral domain and $\mathcal{C}$ a class of R-modules. The *Krull-Schmidt property* holds for $\mathcal{C}$ if, whenever

$$G_1 \oplus G_2 \oplus \cdots \oplus G_n \cong H_1 \oplus H_2 \oplus \cdots \oplus H_m$$

for $G_i, H_j \in \mathcal{C}$, then $n = m$ and, after reindexing, $G_i \cong H_i$ for all $i \leq n$. If, instead of $G_i \cong H_i$, we require only that $k > 0$ exists such that $G_i^{(k)} \cong H_i^{(k)}$ for all i, then we say the *weak Krull-Schmidt property* holds for $\mathcal{C}$. (We write $G^{(k)}$ for a direct sum of k copies of a module G.)

In this article we examine Krull-Schmidt properties for certain classes of indecomposable torsion-free modules over commutative integral domains. By a *torsionless* module over a domain R, we mean a submodule of a finitely generated free R-module. An integral domain R has the *torsion-free Krull-Schmidt property, TFKS*, if the class of indecomposable torsionless R-modules has the Krull-Schmidt property; R has *weak TFKS* if this class has the weak Krull-Schmidt property.

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We also study a weaker Krull-Schmidt property, one that asserts uniqueness of decomposition only for ideals. We say a domain R has *unique decompositions into ideals, UDI*, if the class of ideals of R has the Krull-Schmidt property. Similarly, R has *weak UDI* if the class of ideals of R has the weak Krull-Schmidt property. Thus UDI, TFKS and their weak forms offer Krull-Schmidt properties for the category of torsionless modules. Moving outside of this category, we also examine Krull-Schmidt for rank one modules, those torsion-free modules that are isomorphic to submodules of the quotient field. Notice that if $X_1 \oplus \cdots \oplus X_n \cong Y_1 \oplus \cdots \oplus Y_m$ for rank one modules X_i and Y_j, then it is easy to see $n = m$. For this reason, we need only consider the case $n = m$ when treating UDI, weak UDI and Krull-Schmidt for rank one modules.

Levy and Odenthal have completely described the (not necessarily commutative) one-dimensional orders over Noetherian domains that possess TFKS (as well as those for which Krull-Schmidt holds for finitely generated modules) [**8**]. In [**6**], we classified Noetherian domains with UDI. Where the present study differs from these articles is that we seek an approach to Krull-Schmidt that proceeds from not-necessarily Noetherian assumptions. We work over h-local integral domains (defined below). This is somewhat natural in the context of Krull-Schmidt properties: the weakest Krull-Schmidt property that we study (weak UDI) implies h-locality for Noetherian domains.

Notation and terminology. If R is an integral domain, we write $\overline{R}$ for the integral closure of R in its quotient field, Q. A domain R is *h-local* if (i) every nonzero element of R is contained in only finitely many maximal ideals of R, and (ii) each nonzero prime ideal of R is contained in a unique maximal ideal of R; equivalently, for each maximal ideal M of R, $R_{[M]}R_M = Q$, where $R_{[M]} = \cap\{R_N \mid N$ is a maximal ideal different from $M\}$ [**10**, Theorem 22]. In particular, if R is h-local, $R_N R_M = Q$ for all distinct maximal ideals M and N of R. If X and Y are R-submodules of Q, then $[Y : X]$ denotes the R-module, $\{q \in Q : qX \subseteq Y\}$. We abbreviate $[X : X]$ by $E(X)$; the notation is motivated by the observation that $E(X) \cong \mathrm{End}_R(X)$ if $X \neq 0$.

Recall that two R-modules G and H are *locally isomorphic* if $G_M \cong H_M$ for all maximal ideals M of R, and G and H are *power isomorphic* if $n > 0$ exists such that $G^{(n)} \cong H^{(n)}$. If G is locally isomorphic to H, we write $G \cong_l H$. If G is power isomorphic to H, we write $G \cong_\wp H$. The *divisible hull* of a torsion-free R-module G is $QG := Q \otimes_R G$. We identify G with its image in QG and, if M is a maximal ideal of R, we view G_M as contained in QG. A torsionless module is *completely decomposable* if it is isomorphic to a direct sum of ideals.

In the proof of Theorem 3.4 (and only in this proof), we use the following terminology from the theory of torsion-free abelian groups. Two rank one modules are *quasi-isomorphic* if each is isomorphic to a submodule of the other. A *type* τ is the quasi-isomorphism class of a rank one module. The collection of types is partially ordered by the relation $\tau_1 \leq \tau_2$ whenever U_1 is isomorphic to a submodule of U_2 (where τ_i is the type of the R-module U_i). If A is a torsion-free module and $a \in A$, then the *type* of a is the quasi-isomorphism class of the pure submodule of A generated by a, i.e., the submodule $\{b \in A : rb = sa$ for some $r, s \in R\}$. Given a type τ, define $A(\tau) = \{a \in A :$ type of $a \geq \tau\}$.

Finally, we occasionally use the notion of the kth exterior power $\wedge^k G$ of a module G. In particular, if G is a torsion-free R-module and $G = X_1 \oplus X_2 \oplus \cdots \oplus X_n$ for some rank one submodules X_i of G, then $\wedge^n G \cong X_1 \otimes_R X_2 \otimes_R \cdots \otimes_R X_n$. This property, as well as more background on the exterior power construction, can be found in [**1**, Chapter II, Sections 3-5].

2. Krull-Schmidt and Pic$\,(R)$**.** We use the notion of the Picard group of an integral domain to distinguish between weak UDI and UDI. Recall that if R is an integral domain, the *Picard group* of R is the abelian group consisting of the invertible fractional ideals of R modulo the principal fractional ideals of R. In this section we show that the Picard group of a weak UDI domain R measures how close R is to having UDI.

For the purpose of proving some technical lemmas in this section, we introduce the following notion. If R is a domain and S is an overring of R, that is, a ring S that contains R and is contained in the quotient field of R, then (R, S) is a *weak UDI pair* if every overring T of R such that $R \subseteq T \subseteq S$ has weak UDI. Similarly, (R, S) is a *weak TFKS pair* if every overring of R contained in S has weak TFKS. It is easy to see that if R has weak UDI (weak TFKS) and S is a fractional overring of R, then (R, S) is a weak UDI pair, respectively, weak TFKS pair. We make use of this fact without further comment.

A maximal ideal M of an integral domain R is *complemented* if every ideal of R not contained in M is invertible. We collect some simple observations about complemented maximal ideals in the next lemma.

Lemma 2.1. *Let R be an h-local domain with complemented maximal ideal M.*

(i) *If P and Q are comaximal prime ideals, then P or Q is a maximal ideal.*

(ii) *For all maximal ideals $N \neq M$ of R, R_N is a* DVR.

(iii) *If S is a fractional overring of R and $N \neq M$ is a maximal ideal of R, then SN is a maximal ideal of S.*

(iv) $\mathrm{Pic}\,(R) = 0$ *if and only if every maximal ideal of R distinct from M is principal.*

Proof. For the proof of (i), suppose P and Q are comaximal. Then, without loss of generality, we may assume P is an invertible prime ideal. If P is not maximal, then it is contained in an invertible maximal ideal N of R, which is impossible. Statement (ii) follows from (i) and the fact that R is h-local. Indeed, if $N \neq M$ is a maximal ideal of R, then since each nonzero prime ideal of R is contained in a unique maximal ideal of R, (i) implies R_N is one-dimensional. By assumption, N is an invertible ideal of R, so it follows that R_N is a DVR. Statement (iii) is a consequence of (ii). For if N is a maximal ideal of R such that $N \neq M$, R_N is a DVR, so since S_N is a fractional overring of R_N, $R_N = S_N$. Then $SN_N = N_N$ and $S_N/SN_N = R_N/N_N$ so SN_N is a maximal ideal of S_N. Furthermore, local verification shows that $SN = SN_N \cap S$ so SN is a maximal ideal of S. Finally, to prove (iv), suppose every maximal ideal of R distinct from M is principal. Let I be an invertible ideal of R. Then $II^{-1} = R$ implies that $xI \not\subseteq M$ for some $0 \neq x \in I^{-1}$. But $xI \cong I$, so we may assume without loss of generality that $I \not\subseteq M$. Since R is h-local, I is contained in at most finitely many maximal ideals, say $N_1, \ldots , N_k$. For each $i \leq k$, R_{N_i} is a DVR, so $IR_{N_i} = N_i^{j(i)} R_{N_i}$ for some $j(i) > 0$. Local verification shows that $I = N_1^{j(1)} \cdots N_k^{j(k)}$, so I is principal. Conversely, if $\mathrm{Pic}\,(R) = 0$, then clearly every maximal ideal of R distinct from M is principal since M is complemented. $\square$

Lemma 2.2. *Let S be an overring of an integral domain R. If (R,S) is a weak* UDI *pair and I and J are comaximal ideals of S, then $S = R+I$ or $S = R+J$.*

Proof. If either $I = S$ or $J = S$, the claim is clear, so suppose S is not quasilocal and neither $I = S$ nor $J = S$. Observe that $1 \in I+J \subseteq [R+I : S] + [R+J : S]$. Thus $a \in [R+I : S]$ and $b \in [R+J : S]$ exist such that $1 = a+b$. Define a homomorphism $\phi : (R+I) \oplus (R+J) \rightarrow S$ by $\phi(x,y) = x+y$ for all $x \in R+I$ and $y \in R+J$. Since $I+J = S$, ϕ is surjective. Define a homomorphism $\gamma : S \rightarrow (R+I) \oplus (R+J)$ by $\gamma(s) = (as, bs)$ for $s \in S$, and observe that γ is a splitting map for ϕ since, if $s \in S$, then $\phi(\gamma(s)) = \phi(as,bs) = as+bs = s$. Thus S is isomorphic to a summand of $(R+I)\oplus(R+J)$ and, since $T := (R+I)\cap(R+J)$ has weak UDI and S is a fractional ideal of T, $S^{(n)} \cong (R+I)^{(n)}$ or $S^{(n)} \cong (R+J)^{(n)}$ for some $n > 0$. Taking the nth exterior power of each side with respect to $R+I$ and $R+J$, respectively, yields $S \cong R+I$ or $S \cong R+J$ since $R+I$ and $R+J$ are subrings of S. Finally, isomorphism can be replaced by equality, again since $R+I$ and $R+J$ are rings. $\square$

Lemma 2.3. *Let R be an integral domain. If S is an overring of R such that (R,S) is a weak* UDI *pair, then S is quasilocal or $S = R+N$ for some maximal ideal N of S. Furthermore, for each maximal ideal M of R, there are at most three maximal ideals of S lying over M, and if (R,S) is a weak* TFKS *pair, then there are at most two maximal ideals of S lying over M.*

Proof. If S has at least two distinct maximal ideals N and N', then, by Lemma 2.2, $S = R+N$ or $S = R+N'$. Assume that there are four distinct maximal ideals N_1, N_2, N_3, N_4 of S lying over a maximal ideal M of R. Define $I = N_1N_2$ and $J = N_3N_4$ and note that I and J are comaximal ideals of S. Without loss of generality, we may assume by Lemma 2.2 that $S = R+I$. Now $n \in N_1$ exists such that $n \notin I$, yet $n = r+i$ for some $r \in R$ and $i \in I$. Thus $n-i \in R$. Since $n-i \in N_1$, it must be that $n - i \in M$. Thus $n \in I + M = I$, a contradiction that implies there cannot be more than three maximal ideals of S lying over M.

Now suppose (R, S) is a weak TFKS pair and that S has three distinct maximal ideals, N_1, N_2 and N_3. Define $I_1 = N_2N_3$, $I_2 = N_1N_3$ and $I_3 = N_1N_2$. Then $S = I_1+I_2+I_3 \subseteq [R+I_1 : S]+[R+I_2 : S]+[R+I_3 : S]$ so, for each $i = 1, 2, 3$, $u_i \in [R+I_i : S]$ exists such that $1 = u_1+u_2+u_3$. The map $\sigma : (R + I_1) \oplus (R + I_2) \oplus (R + I_3) \to S$ given by $\sigma(a, b, c) = a+b+c$ is split by the map $\delta : S \to (R+I_1)\oplus(R+I_2)\oplus(R+I_3)$ defined by $\delta(s) = (su_1, su_2, su_3)$ for all $s \in S$. Set $T = (R+I_1)\cap(R+I_2)\cap(R+I_3)$. Then, since (R, T) is a weak TFKS pair, $\mathrm{Ker}\,(\delta)$ is a torsionless T-module, and the $R + I_j$ are fractional ideals of T, it follows that $S \cong_{\wp} R + I_i$ for some $i = 1, 2, 3$. As above, this implies $S = R + I_i$. However, by an argument similar to the one above, this leads to a contradiction that implies S has at most two maximal ideals lying over M. $\square$

Lemma 2.4. *If R has weak* UDI*, then R has a complemented maximal ideal M and the Picard group of R is torsion. If R has* UDI*, then* $\mathrm{Pic}\,(R) = 0$.

Proof. First observe that, if I is an invertible ideal of R, then since I is a summand of a free R-module and R has weak UDI, $I^{(n)} \cong R^{(n)}$ for some $n > 0$. Thus, taking the nth exterior power of both sides yields $I^n \cong R$, and it follows that $\mathrm{Pic}\,(R)$ is torsion. If R has UDI, then, since I is a summand of a free R-module, I is principal. It remains to show that R has a complemented maximal ideal. If R is Dedekind, then every maximal ideal of R is complemented, so suppose R is not Dedekind and assume R is not quasilocal (for otherwise the claim is clear). If I and J are comaximal ideals of R, then since $I + J = R$, it follows that $I \oplus J \cong R \oplus (I \cap J)$. Hence, there exists $n > 0$ such that either $I^{(n)}$ or $J^{(n)}$ is free, so either I or J is projective, i.e. invertible. Let A be the sum of all noninvertible ideals of R (by assumption there is at least one noninvertible ideal). If $A = R$, then it follows that R is the sum of finitely many noninvertible ideals, the sum of any two of which is noninvertible. Hence if $A = R$, R is the sum of two noninvertible ideals, but, as noted above, this is impossible unless R is a Dedekind domain. It follows that a maximal ideal M containing A exists. If B

is an ideal of R not contained in M, then B is not contained in A and must be invertible, proving that M is complemented. □

Lemma 2.5. *Let R be an h-local domain with complemented maximal ideal M, and suppose S is an overring of R and (R_M, S_M) is a weak* UDI *pair. If B is an invertible fractional ideal of S, then $B = SA$ for some invertible fractional ideal A of R.*

Proof. We claim first that S_M has at most four maximal ideals. If S_M is quasilocal, the claim is clear, so suppose S_M has more than one maximal ideal. If N is a maximal ideal of S_M such that $S_M = R_M + N$, then $S_M/N \cong R_M/(R_M \cap N)$ so N lies over the maximal ideal M of R. By Lemma 2.3, there are at most three maximal ideals of S_M lying over M. If L is a maximal ideal of S_M such that $S_M \neq R_M + L$, then, by Lemma 2.3, $S_M = R_M + N$ for every maximal ideal N of S_M distinct from L. Thus, all the maximal ideals of S_M except possibly one contract to maximal ideals of R_M. In particular, S_M has at most four maximal ideals. Since B is an invertible fractional ideal of S, it follows that $BS_M = S_M a$ for some $a \in B$.

Now let $\{M_\alpha\}$ denote the set of all maximal ideals of R such that $M_\alpha \neq M$ and $S_{M_\alpha} \neq Q$ where Q is the quotient field of R. By Lemma 2.1(ii), R_{M_α} is a DVR, so $S_{M_\alpha} = R_{M_\alpha}$. Then, for all α, $BR_{M_\alpha} = R_{M_a} a_\alpha$ for some $a_\alpha \in B$. Define $A = R_M a \cap (\cap_\alpha R_{M_\alpha} a_\alpha) \cap (\cap_N R_N)$ where N ranges over the maximal ideals of R not in $\{M_\alpha\} \cup \{M\}$. Then, since R is h-local, if N is a maximal ideal of R distinct from a particular M_α, $R_{M_\alpha} R_N = Q$ [**10**, Theorem 22]. Also, since R is h-local, localizations commute with infinite intersections [**4**, Lemma IV. 3.10], so $AR_M = R_M a$, $AR_{M_\alpha} = R_{M_\alpha} a_\alpha$ for all α and $AR_N = R_N$ for all maximal ideals N of R not in $\{M_\alpha\} \cup \{M\}$. Furthermore, $SA = B$ and A is a locally free R-submodule of Q. In fact, since B is a finitely generated S-submodule of Q and every nonzero element of R is contained in at most finitely many maximal ideals of R, it follows that $BR_{M_\alpha} = SR_{M_\alpha} = R_{M_\alpha}$ for all but finitely many α. This implies that A is a fractional ideal of R. Since R is h-local, A is a finitely generated fractional ideal [**10**, Theorem 26]. Hence A is an invertible fractional ideal of R. □

Lemma 2.6. *Let R be an h-local domain, and let X and Y be rank one R-modules such that $(R, E(X))$ is a weak* UDI *pair. Then $X \cong_{\wp} Y$ if and only if $XA = Y$ for some invertible fractional ideal A of R.*

Proof. Suppose $X^{(n)} \cong Y^{(n)}$. Then the canonical homomorphism, $X \otimes_R \mathrm{Hom}_R(X, Y) \to Y$ is surjective, and it follows that $X[Y : X] = Y$. The existence of a splitting map for the induced surjection $X^{(n)} \to Y$ shows that $1 \in [Y : X][X : Y] \subseteq E(X)$. In particular, $[Y : X][X : Y] = E(X)$, and it follows that $[Y : X]$ is an invertible fractional ideal of $E(X)$. (Indeed, if $q \in [X : Y] \cap [Y : X]$, then $q[Y : X] \subseteq E(X)$ and $q[X : Y] \subseteq E(X)$.) Set $B := [Y : X]$ and $S := E(X)$. By Lemma 2.5, a fractional invertible ideal A of R exists such that $SA = B$. Thus $XA = XSA = XB = Y$. Conversely, suppose $XA = Y$ for some invertible fractional ideal of R. By Lemma 2.4, $\mathrm{Pic}\,(R)$ is torsion, so $A^n \cong R$ for some $n > 0$. It follows that $A^{(n)} \cong R^{(n)}$ (see [**7**], for example). Since A and R are flat R-modules, it follows that $Y^{(n)} \cong (XA)^{(n)} \cong (X \otimes_R A)^{(n)} \cong X \otimes_R A^{(n)} \cong X \otimes_R R^{(n)} \cong (X \otimes_R R)^{(n)} \cong X^{(n)}$, and the claim is proved. □

Theorem 2.7. *An h-local integral domain R has* UDI *if and only if R has weak* UDI *and* $\mathrm{Pic}\,(R) = 0$.

Proof. Suppose R has weak UDI, $\mathrm{Pic}\,(R) = 0$ and $I_1 \oplus \cdots \oplus I_n \cong J_1 \oplus \cdots \oplus J_n$ for some ideals $I_1, \ldots, I_n, J_1, \ldots, J_n$ of R. After reindexing we may assume that, for each $j \leq n$, $I_j \cong_{\wp} J_j$. Since $E(I_j)$ is a fractional ideal of R for each $j \leq n$, it follows that $(R, E(I_j))$ is a weak UDI pair. By Lemma 2.6, $I_j = AJ_j$ for some invertible fractional ideal A of R. By assumption A is a principal ideal of R, so $I_j \cong J_j$. The converse is clear from Lemma 2.4. □

It follows that if R is a domain with only finitely many maximal ideals, then R has weak UDI if and only if R has UDI.

3. Main reductions. In this section we prove reduction theorems for our various Krull-Schmidt properties. The first two lemmas, proved elsewhere, play a role similar to that of the "package deal" theorems of

Levy and Odenthal [**9**].

Lemma 3.1 [**7**, Proposition 2.8, Theorems 2.11 and 2.13]. *Let R be an h-local integral domain and G and H be torsionless R-modules.*

(i) *If* $\mathrm{Pic}\,(R) = 0$*, then* $G \cong_l H$ *if and only if* $G \oplus R \cong H \oplus R$*.*

(ii) *If* $\mathrm{Pic}\,(R) = 0$*,* $G \cong_l H$ *and* G *has a summand isomorphic to an ideal of* R*, then* $G \cong H$*.*

(iii) *If* $\mathrm{Pic}\,(R)$ *is torsion and* $G \cong_l H$*, then* $G \cong_{\wp} H$*.*

Lemma 3.2. *Let R be an h-local domain with complemented maximal ideal. If $G \cong_l H$, then G is indecomposable if and only if H is indecomposable.*

Proof. The lemma is proved in [**7**, Corollary 3.2] under the more general hypothesis that R is an h-local domain and R_N is a valuation domain for almost all maximal ideals N of R. That this hypothesis is indeed more general is a consequence of Lemma 2.1(ii). □

Lemma 3.3. *Let R be an h-local domain with complemented maximal ideal M. If $G := G_1 \oplus \cdots \oplus G_n$ and $H := H_1 \oplus \cdots \oplus H_m$ are direct sums of torsionless R_M-modules such that $G \cong H$, then torsionless R-modules $G' := G'_1 \oplus \cdots \oplus G'_n$ and $H' := H'_1 \oplus \cdots \oplus H'_m$ exist such that $G' \cong_l H'$ and, for all $i \leq n$, $j \leq m$, $G_i = (G'_i)_M$ and $H_j = (H'_j)_M$.*

Proof. We may assume that, for each $i \leq n$ and $j \leq m$, free R-modules $E_i \subseteq QG_i$ and $F_j \subseteq QH_j$ exist such that $G_i \subseteq (E_i)_M$ and $H_j \subseteq (F_j)_M$. For each $i \leq n$ and $j \leq m$, let G'_i and H'_j be torsionless R-modules defined by $G'_i = G_i \cap E_i$ and $H'_j = H_j \cap F_i$. Then, for all i, j, $G_i = (G'_i)_M$ and $H_j = (H'_j)_M$. By Lemma 2.1(ii), R_N is a DVR for each maximal ideal $N \neq M$. Thus, the torsionless R_N-modules $(G')_N$ and $(H')_N$ are free of the same rank for all maximal ideals $N \neq M$. It follows that $G' \cong_l H'$. □

Theorem 3.4. *Let R be an h-local domain. The following statements hold for R.*

(i) *R has weak* UDI *if and only if* Pic$\,(R)$ *is torsion and R has a complemented maximal ideal M such that R_M has* UDI.

(ii) *R has* UDI *if and only if* Pic$\,(R) = 0$ *and R has a complemented maximal ideal M such that R_M has* UDI.

(iii) *R has weak* TFKS *if and only if* Pic$\,(R)$ *is torsion and R has a complemented maximal ideal M such that R_M has weak* TFKS.

(iv) *R has* TFKS *if and only if locally isomorphic torsionless modules are isomorphic and R has a complemented maximal ideal M such that R_M has* TFKS.

(v) *R has the Krull-Schmidt property for rank one modules if and only if* Pic$\,(R) = 0$ *and R has a complemented maximal ideal M such that R_M has the Krull-Schmidt property for rank one modules.*

Proof. (i) Suppose first that R has weak UDI. By Lemma 2.4, Pic$\,(R)$ is torsion and a complemented maximal ideal M exists. If G and H are completely decomposable torsionless R_M-modules such that $G \cong H$, then, by Lemma 3.3, completely decomposable torsionless R-modules G' and H' exist such that $G = G'_M$, $H = H'_M$ and $G' \cong_l H'$. Thus, by Lemma 3.1, $G' \cong_\wp H'$ and, since R has weak UDI, it follows that R_M has weak UDI. By Theorem 2.7 and the remark that follows it, R_M has UDI.

To prove the converse, assume that M is a complemented maximal ideal of R, R_M has UDI and the Picard group of R is torsion. Suppose $G := I_1 \oplus \cdots \oplus I_n$, $H := J_1 \oplus \cdots \oplus J_n$ are direct sums of ideals of R such that $G \cong H$. Since R_M has weak UDI we have, after reindexing, $(I_j)_M \cong_\wp (J_j)_M$ for each j. For each maximal ideal $N \neq M$, R_N is a DVR so $I_j^{(k)} \cong_l J_j^{(k)}$ for some $k > 0$. Thus, by Lemma 3.1, $I_j \cong_\wp J_j$ for all $j \leq n$.

(ii) If R has UDI, then R has trivial Picard group and complemented maximal ideal M (Lemma 2.4) and, by (i), R_M has UDI. The converse follows from (i) and Theorem 2.7.

(iii) Suppose R has complemented maximal ideal M, R_M has weak TFKS and Pic$\,(R)$ is torsion. By Lemma 3.2, if G is an indecomposable torsionless R-module, then G_M is indecomposable. Let $G_1, \ldots, G_n$, $H_1, \ldots, H_m$ be indecomposable torsionless R-modules such that $G_1 \oplus \cdots \oplus G_n \cong H_1 \oplus \cdots \oplus H_m$. Passing to R_M, each $(G_i)_M$ and $(H_i)_M$

remains indecomposable. Thus the assumption that R_M has weak TFKS implies that $n = m$ and, after reindexing, $(G_i)_M \cong_\wp (H_i)_M$ for all $i \leq n$. Since R is h-local, R_N is a DVR for each maximal ideal $N \neq M$. Thus, for each maximal ideal $N \neq M$, $(G_i)_N$ and $(H_i)_N$ are free R_N-modules of the same rank. Thus $G^{(k)} \cong_l H^{(k)}$ for some $k > 0$ and, by Lemma 3.1, $G_i \cong_\wp H_i$ for all $i \leq n$.

Now suppose R has weak TFKS. By Lemma 2.4 it suffices to show that R_M has weak TFKS where M is a complemented maximal ideal of R. Suppose $G_1 \oplus \cdots \oplus G_n \cong H_1 \oplus \cdots \oplus H_m$ for indecomposable torsionless R_M-modules G_i and H_j. By Lemma 3.3, indecomposable torsionless R-modules G_i' and H_j' exist such that $G_i = (G_i')_M$ and $H_j = (H_j')_M$ for each $i \leq n$ and $j \leq m$. Moreover, if $G' := G_1' \oplus \cdots \oplus G_n'$ and $H' := H_1' \oplus \cdots \oplus H_m'$, then $G' \cong_l H'$. Thus, by Lemma 3.1, $G' \cong_\wp H'$ and, since R has weak TFKS, we have $n = m$ and, after reindexing, $G_i' \cong_\wp H_i'$ for all i. Thus $G_i \cong_\wp H_i$ for all i, proving R_M has weak TFKS.

(iv) Suppose R has TFKS and G and H are locally isomorphic torsionless R-modules. Then $\operatorname{Pic}(R) = 0$ by (ii) and, by Lemma 3.1(i), $G \oplus R \cong H \oplus R$. Since R has TFKS, $G \cong H$ and we conclude that locally isomorphic torsionless R-modules are isomorphic. By (ii), R has a complemented maximal ideal M. The proof that R_M has TFKS is similar to the proof that R_M has weak TFKS in (iii). Suppose $G_1 \oplus \cdots \oplus G_n \cong H_1 \oplus \cdots \oplus H_m$ for indecomposable torsionless R_M-modules G_i and H_j. Let G', H', G_i' and H_j' be as in Lemma 3.3. Then $G' \cong_l H'$ so, by Lemma 3.1, $G' \oplus R \cong H' \oplus R$. Since R has TFKS, we have $n = m$ and, after reindexing, $G_i' \cong H_i'$ for all $i \leq n$. Thus, $G_i \cong H_i$ for all i and R_M has TFKS.

Conversely, note that if locally isomorphic torsionless R-modules are isomorphic then since every invertible ideal of R is locally isomorphic to R, we have $\operatorname{Pic}(R) = 0$. Thus, by (iii), R has weak TFKS and it suffices to check that if $G \cong_\wp H$ for torsionless indecomposable R-modules G and H, then $G \cong H$. By Lemma 3.2, G_M and H_M are indecomposable R_M-modules. Since R_M has TFKS, $G_M \cong H_M$. Moreover, R_N is a DVR for all maximal ideals $N \neq M$ of R so $G \cong_l H$. By assumption, $G \cong H$.

(v) If R has the Krull-Schmidt property for rank one modules, then, by Lemma 2.4, $\operatorname{Pic}(R) = 0$ and R has a complemented maximal ideal M. Since rank one R_M-modules are rank one R-modules, R_M has the Krull-Schmidt property for rank one modules. Conversely, suppose M is a complemented maximal ideal of R, R_M has the Krull-Schmidt property for rank one modules and $\mathrm{Pic}\,(R) = 0$. As in [**6**, Theorem 4.3], our argument is modeled on a classical proof of a theorem of Baer for abelian groups. Recall the terminology and notation of quasi-isomorphism and types from the introduction. Suppose $G := X_1 \oplus \cdots \oplus X_n$ and $H := Y_1 \oplus \cdots \oplus Y_n$ are direct sums of rank one R-modules X_i and Y_j such that $G \cong H$. Since types are preserved under isomorphism, $G(\tau) \cong H(\tau)$ and $G/G(\tau) \cong H/H(\tau)$ for all types τ. Select a type τ that is maximal with respect to the types of the X_i and Y_i. Then $G(\tau) \cong H(\tau)$ and, without loss of generality, we may assume $X_1 \oplus \cdots \oplus X_k \cong Y_1 \oplus \cdots \oplus Y_k$, where $k \leq n$ and each X_i and Y_i has type τ. After reindexing, we may assume $(X_i)_M \cong (Y_i)_M$ for all $i \leq k$. Moreover, since the X_i and Y_i have the same type and R_N is a DVR for all $N \neq M$, it follows that $(X_i)_N \cong (Y_i)_N$ for all $N \neq M$. Thus $X_i \cong_l Y_i$ for all $i \leq k$ and X_i and Y_i are quasi-isomorphic rank one modules. In particular, $[X_i : Y_i][Y_i : X_i] = E(Y_i)$, since localizations commute with brackets of quasi-isomorphic rank one modules over h-local domains [**4**, Lemma IV.3.10]. Set $B = [X_i : Y_i]$ and $S = E(Y_i)$. Then B is an invertible fractional ideal of S, and (R_M, S_M) is a weak UDI pair, so by Lemma 2.5, $B = SA$ for some invertible fractional ideal A of R. Since $\mathrm{Pic}\,(R) = 0$, B is a principal fractional ideal of S, and it follows that $X_i \cong Y_i$ for all $i \leq k$. Since $G/G(\tau) \cong H/H(\tau)$, an inductive argument completes the proof that R has the Krull-Schmidt property for rank one modules. □

It would be interesting to know whether the requirement in (iv) that R_M has TFKS can be amended to require only that R_M has weak TFKS. (Compare Theorem 2.7.) In the Prüfer and Noetherian cases, the answer is affirmative.

Corollary 3.5. *Let R be a domain that is Noetherian or Prüfer. Then R has* TFKS *if and only if R has weak* TFKS *and locally isomorphic torsionless modules are isomorphic.*

Proof. Apply Lemma 3.1 and Theorem 3.4 and use the fact that over quasilocal Prüfer or Noetherian domains, power isomorphism of torsionless modules implies isomorphism [**3**, Theorem 2.11] and [**4**, Proposition I.10.7 and Theorem XV.7.4]. □

Theorem 2.7 suggests that the requirement in Corollary 3.5 that locally isomorphic torsionless modules are isomorphic might be replaced by the weaker requirement that $\mathrm{Pic}\,(R) = 0$. We do not know if this is the case.

4. Noetherian case. The characterization of one-dimensional Noetherian domains satisfying TFKS is contained in [**8**]. In this section we give our own version of Levy and Odenthal's description of TFKS, which depends upon the splitting of the singular maximal ideal in the integral closure of R.

A characterization of Noetherian domains with UDI was given in [**6**]; however, the one-dimensional version is more easily stated and is all we require here.

Theorem 4.1 [**6**, Theorem 3.2]. *Let R be a one-dimensional Noetherian domain. Then R has* UDI *if and only if R has a maximal ideal M such that every other maximal ideal is principal and, if M splits in $\overline{R}$, then M satisfies one of the following conditions.*

(i) $M\overline{R} = P_1 P_2^{e_2}$ *such that* $e_2 \geq 1$, $\overline{R}/P_1 \cong R/M$ *where* P_1, P_2 *are distinct maximal ideals of* $\overline{R}$.

(ii) $M\overline{R} = P_1 P_2 P_3$ *where* P_1, P_2, P_3 *are distinct maximal ideals of* $\overline{R}, \overline{R}/P_j \cong R/M$ *for all* $j = 1, 2, 3$ *and* $\overline{R}$ *is finitely generated over* R.

Theorem 4.2 (After Levy and Odenthal). *Let R be a one-dimensional Noetherian domain. Then R satisfies* TFKS *if and only if locally isomorphic torsionless modules of R are isomorphic and R has a complemented maximal ideal M such that if M splits in $\overline{R}$, then $\overline{R}$ is finitely generated over R and has exactly two maximal ideals P_1, P_2 lying over M, $M\overline{R} = P_1 P_2$ and $\overline{R}/P_1 \cong \overline{R}/P_2 \cong R/M$.*

Proof. We first interpret a commutative version of one of the main theorems of [**8**], which we designate by (LO): A Noetherian one-dimensional domain R has TFKS if and only if locally isomorphic torsionless modules are isomorphic and R has a complemented maximal ideal M such that either

(a) $\overline{R}_M$ *is local or*

(b) $\overline{R}$ *is finitely generated over* R *and* $\overline{R}$ *has exactly two maximal ideals* P_1, P_2 *over* M *such that, for some ring* Ψ*, there are ring epimorphisms* $f : \overline{R} \to \Psi$ *and* $g : \overline{R} \to \Psi$ *such that* $\operatorname{Ker} f = P_1^h$ *and* $\operatorname{Ker} g = P_2^h$ *for some* $h \geq 1$ *and* $R = \{x \in \overline{R} \mid f(x) = g(x)\}$.

Assume that R satisfies TFKS. Then, by Theorem 3.4, locally isomorphic torsionless modules of R are isomorphic, R has a complemented maximal M and R_M satisfies TFKS. If R is a PID, there is nothing to show so suppose $\overline{R}_M \neq R_M$. By (LO) and Theorem 4.1, it is enough to consider the case where $\overline{R}$ is a finitely generated R-module, $M\overline{R} = P_1P_2^{e_2}$ for distinct maximal ideals P_1, P_2 of $\overline{R}$ and $\overline{R}/P_1 \cong R/M$. By (LO), $\overline{R}/P_1^h \cong \overline{R}/P_2^h$ as rings for some $h \geq 1$. Observe that $\overline{R}/P_1^h$ has vector space dimension h over R/M, while $\overline{R}/P_2^h$ has dimension $h \cdot d$ where d is the dimension of $\overline{R}/P_2$ over R/M. Therefore, $\overline{R}/P_2 \cong R/M$ also. Write $h = me_2 + l$ with $0 \leq l < e_2$. When $l = 0$, from the fact that $M^m \subseteq P_2^{e_2 m}$, it follows that $f(M^m)$ must be zero. This implies $M^m \subseteq P_1^h$. But $M\overline{R}_{P_1} = P_1\overline{R}_{P_1}$ and so $h \leq m$. This can only happen when $e_1 = 1$, as claimed. Finally, if $l > 0$, then in this case, $M^{m+1} \subseteq P_2^h$, implying that $M^{m+1} \subseteq P_1^h$ as before. Then $m + 1 \geq h$, implying that $m(1 - e_2) \geq l - 1$. Again we must have $e_2 = 1$.

To establish the converse, note that, by comparing (LO) with Theorem 4.1, we need only consider the possibility that M splits in $\overline{R}$. For each $j = 1, 2$, since $\overline{R}/P_j \cong R/M$, $\overline{R} = R + P_j$. Moreover, $P_j^k/P_j^{k+1} \cong R/M$ for all $k > 0$, so $P_j^k = M^k + P_j^{k+1}$. It follows that $\overline{R} = R + P_j^k$ for all $k > 0$. Let h be the smallest positive integer such that $M^h\overline{R} \subseteq R$ and define $\iota : R/M^h \to \overline{R}/M^h\overline{R} = \overline{R}/P_1^h \oplus \overline{R}/P_2^h$. For $\pi_j : \overline{R}/M^h\overline{R} \to \overline{R}/P_j^h$ equal to the coordinate projection, $\delta_j = \pi_j \iota$ is an isomorphism because $R + P_j^h = \overline{R}$ and a dimension argument for $j = 1, 2$. Let $\beta_j : \overline{R} \to \overline{R}/P_j^h$ be the natural map and $f_j = \delta_j^{-1}\beta_j$. Then $R = \{x \in \overline{R} \mid f_1(x) = f_2(x)\}$ and so (b) of (LO) holds. Therefore, R has TFKS. $\square$

Corollary 4.3. *Let R be a one-dimensional Noetherian domain. Then weak* TFKS *holds for R if and only if* $\mathrm{Pic}\,(R)$ *is torsion and R has a complemented maximal ideal M such that if M splits in $\overline{R}$, then $\overline{R}$ is finitely generated over R and M splits as in Theorem* 4.2.

Proof. Apply Theorems 3.4, Corollary 3.5 and Theorem 4.2. □

It is shown in [**6**, Theorem 4.3] that the Krull-Schmidt property holds for rank one modules of a one-dimensional UDI Noetherian domain. In Theorem 4.5, we establish the converse.

Lemma 4.4. *If R is an integral domain and every overring of R has weak* UDI*, the set of nonmaximal prime ideals of R is linearly ordered.*

Proof. Suppose P and Q are incomparable nonmaximal prime ideals of R. Examination of the proof of Lemma 2.1(i) shows the hypothesis of h-locality is not needed, so it follows that P and Q are contained in a common maximal ideal M of R. Thus $P+Q$ is a proper ideal of R. By assumption, $P+Q \neq P \cup Q$, so let $r \in (P+Q) \setminus (P \cup Q)$. Define $S := R[r^{-1}]$. Then S is an overring of R contained in both R_P and R_Q. Also $SP \neq S$ and $SQ \neq S$, but $S(P+Q) = S$. This implies SP and SQ are comaximal ideals of S. By Lemma 2.2, we may assume without loss of generality that $S = R + SP$. Thus $1 \in Rr + SP \subseteq Q + SP$ so $1 = q + pr^{-k}$ for some $k \geq 0$. Hence $r^k(1-q) \in P$ and, since $r \notin P$, $1 - q \in P$. But this implies $1 \in Q + P$, a contradiction. □

Theorem 4.5. *A Noetherian domain R is a one-dimensional* UDI *domain if and only if the Krull-Schmidt property holds for rank one modules of R.*

Proof. The Krull principal ideal theorem implies that the set of prime ideals of a Noetherian domain is linearly ordered only if the domain has Krull dimension one. Hence, by Lemma 4.4, the Krull-Schmidt property for rank one modules implies R is one-dimensional. The converse is established in [**6**, Theorem 4.3]. □

Example 4.6. *Overrings of* UDI *domains need not have weak* UDI. Simply choose R to be an integrally closed local domain of Krull dimension greater than one. Then R has UDI since all its ideals have local endomorphism rings [**4**, Theorem I.9.8], but by Lemma 4.4, R has an overring without UDI since the prime ideals of R are not linearly ordered.

In [**13**], a geometric interpretation of our Krull-Schmidt properties is given. A number of examples of UDI orders in algebraic number fields is given in [**6**], many of which occur in quadratic number fields. Evidently, if R is an order in a quadratic number field, R has (weak) UDI if and only if R has (weak) TFKS. This is because every ideal of R is 2-generated and, hence, torsionless R-modules are completely decomposable [**14**]. Thus, using [**6**], Examples 4.6 and 4.7, one can list a number of examples of TFKS domains.

5. Non-Noetherian case. In this section we briefly treat the non-Noetherian case of UDI and TFKS, with special emphasis on the Prüfer case. As noted in the introduction, all quasilocal Prüfer domains (= valuation domains) have UDI. Using Theorem 3.4, it is then easy to describe the h-local Prüfer domains with UDI. Since the rank one modules of a valuation domain are either divisible or isomorphic to ideals, the equivalence of UDI and Krull-Schmidt for rank one modules is also immediate.

Proposition 5.1. *The following statements hold for R, an h-local Prüfer domain.*

(i) *R has* UDI *if and only if R has the Krull-Schmidt property for rank one modules*; *if and only if R is a Bézout domain with complemented maximal ideal.*

(ii) *R has weak* UDI *if and only if* $\mathrm{Pic}\,(R)$ *is torsion and R has a complemented maximal ideal.*

The Prüfer cases of TFKS and weak TFKS are not as transparent. This is because, to our knowledge, the quasilocal case remains undescribed.

Combining the Prüfer description of UDI with results on decompositions of torsion-free modules, we obtain characterizations of some strong forms of the Krull-Schmidt property. By way of application, we are interested in some of the following variations of the Krull-Schmidt property. In order to characterize these properties, we recall several related notions. A Prüfer domain R satisfies (##) if every prime ideal of R is the radical of a finitely generated ideal of R (see [**5**, Theorem 3]). In particular, if every nonzero ideal of a Prüfer domain R is contained in at most finitely many maximal ideals of R, then R satisfies (##), but the converse is not true ([**5**, Theorem 5]). A Prüfer domain R is h-local if and only if R satisfies (##) and each nonzero prime ideal of R is contained in a unique maximal ideal of R [**11**, Proposition 3.4].

Recall that a D-ring is an integral domain R for which every torsion-free finite rank R-module decomposes into a direct sum of rank one R-modules. There is an extensive theory of D-rings due to Matlis (see [**10**]).

We need also the notion of an almost maximal ring, that is, a ring for which R/I is a linearly compact R-module for all nonzero ideals I. A Prüfer domain R is almost maximal if and only if Q/R is an injective R-module and R is h-local (see [**2**, Proof of (2) $\Leftrightarrow$ (5) in Theorem 4.8]).

The main thrust of the next proposition is that we do not have to assume R is an h-local domain.

Proposition 5.2. *Let R be an integral domain.*

(i) *R is an h-local Bézout domain with complemented maximal ideal if and only if, for each R-module $G := I_1 \oplus \cdots \oplus I_n$, that is a direct sum of ideals of R, every pure submodule of G is a summand of G that is isomorphic to a direct sum of the I_js.*

(ii) *R is an almost maximal Bézout domain with complemented maximal ideal if and only if, for each torsionless R-module G, $G \cong I_1 \oplus \cdots \oplus I_n$ for some ideals I_j of R, and every pure submodule of G is a summand of G that is isomorphic to a direct sum of the I_js.*

(iii) *R is an almost maximal Bézout domain with complemented maximal ideal if and only if R is a Prüfer (##) domain such that every torsionless R-module decomposes uniquely, up to isomorphism, into a direct sum of rank one modules.*

(iv) *R is a quasilocal D-ring if and only if every torsion-free finite* *rank R-module decomposes uniquely, up to isomorphism, into a direct sum of rank one R-modules.*

Proof. (i) An integral domain R is an h-local Prüfer domain if and only if pure submodules of completely decomposable torsionless R-modules are summands ([**12**, Theorem 3.2]). This implies h-local Prüfer domains have the property that pure submodules of completely decomposable torsionless modules are completely decomposable. Thus, the asserted property holds if and only if R is an h-local Prüfer domain with UDI.

(ii) By (i), the stated decomposition property implies that R is an h-local Prüfer domain with UDI. A Prüfer domain R is almost maximal if and only if R is h-local and every torsionless R module is completely decomposable [**4**]. Thus the given decomposition property and (i) imply R is an almost maximal Bézout domain with complemented maximal ideal. The converse is clear from (i) and the cited result.

(iii) Assume torsionless R-modules decompose uniquely into a direct sum of modules and that R is a Prüfer (##) domain. We show first that R is a locally almost maximal Prüfer domain. Let M be a maximal ideal of R and G a torsionless R_M-module. Then there is a free R-module F such that $G \subseteq F_M$. Define $G' := G \cap F$; then G' is a torsionless R-module and $G'_M = G$. By assumption G', hence G, is completely decomposable. If every torsionless R_M-module is completely decomposable, then R_M is an almost maximal valuation domain [**4**, Theorem XV.2.3]. Thus each localization of R at a maximal ideal is an almost maximal valuation domain. We show R is h-local. Since R satisfies (##), it is enough to check that each nonzero prime ideal of R is contained in a unique maximal ideal of R. Suppose P is a prime ideal of R contained in at least two maximal ideals N_1 and N_2 of R. Set $S := R \setminus (N_1 \cup N_2)$, $T := R_S$, $A := M_S$, $B := N_S$ and $L := P_S$. Then $LT_L = L$ and T/L has quotient field T_L/L. Since R_{N_1} and R_{N_2} are almost maximal valuation domains, T_A/L and T_B/L are independent maximal valuations with common quotient field T_L/L. As such, each must have a divisible value group [**15**, Theorem A]. But since R has UDI, R has a complemented maximal ideal and at least one of A and B is principal. In particular, the value group of T_A/L or T_B/L must have a copy of $\mathbf{Z}$ as a summand. This contradiction implies each nonzero prime ideal of R is contained in a unique maximal ideal of

R. Consequently, since R satisfies (##), R is an h-local locally almost maximal domain; hence, R is almost maximal [**4**, Theorem IV.3.9]. Since R has UDI, R must be a Bézout domain. This proves the claim. The converse follows from (ii).

(iv) Suppose R is a quasilocal D-ring. Then the integral closure of R is a valuation domain [**10**, Theorem 72]. Thus the integral closure of every overring of R is a valuation ring, hence quasilocal, and this forces every overring of R to be quasilocal. In particular, rank one modules have quasilocal endomorphism rings so the Krull-Schmidt property holds for rank one modules. Since R is a D-ring, (d) follows. Conversely, suppose torsion-free finite rank R-modules decompose uniquely into direct sums of rank one R-modules. Then $\overline{R}$ is a D-ring with UDI. An integrally closed D-ring is the intersection of at most 2 maximal valuation domains [**10**]. However, $\overline{R}$ has a complemented maximal ideal, so if $\overline{R}$ has two maximal ideals M and N, one of these ideals, say N, is principal. In particular, the maximal valuation domain $\overline{R}_N$ has a nondivisible value group. As in the proof of (iii), this is in contradiction to the fact that two independent maximal valuation domains having the same quotient field each have divisible value group. Thus $\overline{R}$ is quasilocal; hence, R is quasilocal and a D-ring. $\square$

Using the following proposition, one can construct examples of non-Noetherian UDI domains for which h-locality fails in a strong way.

Proposition 5.3. *Let R be an integral domain with a prime ideal P such that $PR_P = P$. Then the Krull-Schmidt property holds for rank one modules of R if R_P is a* DVR *and the Krull-Schmidt property holds for rank one modules of R/P.*

Proof. For each ideal I of R, there is an exact sequence,

$$\mathrm{Hom}_R(I,P) \longrightarrow \mathrm{Hom}_R(I,R_P) \longrightarrow \mathrm{Hom}_R(I,R_P/P) \longrightarrow \mathrm{Ext}_R(I,P).$$

Now P is a principal ideal of R_P, so $\mathrm{Ext}_R(I,P) \cong \mathrm{Ext}_{R_P}(IR_P,P) = 0$. Also we have $\mathrm{Hom}_R(I,R_P/P) \cong \mathrm{Hom}_R(I/IP,R_P/P)$, hence $\mathrm{Hom}_R(I,P) \to \mathrm{Hom}_R(I,R_P)$ is surjective if and only if $I = IP$. Note that, if $I = IP$, then clearly $\mathrm{Hom}_R(I/IP,R_P/P) = 0$. On the other hand, if $\mathrm{Hom}_R(I/IP, R_P/P) = 0$, then, since R_P/P is the quotient field of R/P and I/IP is a torsion-free R/P-module, it must be the case that $I = IP$.

Tensoring both sides of $I_1 \oplus \cdots \oplus I_n \cong J_1 \oplus \cdots \oplus J_n$ with R/P yields $I_1/PI_1 \oplus \cdots \oplus I_n/PI_n \cong J_1/PJ_1 \oplus \cdots \oplus J_n/PJ_n$. After reindexing, we may assume that $m, m' \leq n$ exist such that $I_1/PI_1 \oplus \cdots \oplus I_m/PI_m \cong J_1/PJ_1 \oplus \cdots \oplus J_{m'}/PJ_{m'}$ and no I_k/PI_k or J_l/PJ_l is trivial for $k \leq m$, $l \leq m'$. The preceding argument shows that, for all $k \leq m$, $\mathrm{Hom}_R(I_k, P) \to \mathrm{Hom}_R(I_k, R_P)$ is not surjective, so I_k is isomorphic to an R-submodule of R_P that is not contained in P. Similarly, for all $l \leq m'$, J_l is isomorphic to an R-submodule of R_P that is not contained in P. Thus we assume that $I_k, J_l \subseteq R_P$ but $I_k, J_l \not\subseteq P$ for all $k \leq m$, $l \leq m'$. In particular, $I_k R_P = J_l R_P = R_P$ implies $I_k P = J_l P = P$ for all $k \leq m$, $l \leq m'$. Since I_k/P, $J_l/P \subseteq R_P/P$, each I_k/P, J_l/P is a rank one R/P-module. Thus $m = m'$ and, after reindexing, we may conclude that $I_k/P \cong J_k/P$ for each $k \leq m$. It follows that $a, b \in R$ exists with $b \notin P$ such that $aI_k + P = bJ_k$. If $a \in P$, then $aI_k \subseteq PI_k = P$ and $P = bJ_k$. However, this implies that $R_P J_k = J_k$, hence $PJ_k \neq P$ since R_P is a DVR. This contradiction forces $a \notin P$. Thus, since $P = PI_k \subseteq I_k$ and $a^{-1}P = P$, we have $P \subseteq aI_k$ proving that, for all $k \leq m$, $I_k \cong J_k$. If $k > m$, then $I_k P = I_k$ and $J_k P = J_k$. Since R_P is a DVR, I_k and J_k are principal ideals of R_P, hence $I_k \cong J_k$. It follows that R has UDI.

Finally, observe that if X is a proper submodule of Q, the quotient field of R, then $X_P \neq Q$ since $X_P = Q$ would force $Q = XP \subseteq X$. Thus X is a fractional ideal of the DVR R_P and, since R_P is a fractional ideal of R, X is a fractional ideal of R. It follows that every proper rank one R-module is a fractional ideal of R. This proves the claim. □

Example 5.4. *There exist domains that are not h-local but that satisfy the Krull-Schmidt property for rank one modules.* Let R be any domain that has the Krull-Schmidt property for rank one modules. Denote its quotient field by Q and define $S := R + XQ[X]_{(X)}$. Then by Proposition 5.3, S has UDI. If R has infinitely many maximal ideals (e.g. $R = \mathbf{Z}$), then $XQ[X]_{(X)}$ is a prime ideal of S that is contained in infinitely many maximal ideals of S; hence, S is not h-local.

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