ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 32, Number 4, Winter 2002

THE AFFINITY OF SET THEORY AND ABELIAN GROUP THEORY

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ABSTRACT. This paper reviews the uses of set theory to solve some long-standing problems in a number of different areas of abelian group theory. In some cases the solution is an independence result (from ZFC, the ordinary axioms of set theory); in other cases the result is a theorem of ZFC proved by combinatorial methods. In the interests of breadth, and to keep within the prescribed bounds of space, some depth and detail have been sacrificed and the emphasis is on key developments in the early history of each area. In general, except for Butler groups, a cut-off date of about 1990 has been observed, except for brief references to selected later developments. Also, for reasons of space, the bibliography is not complete.

1. Slenderness and reflexivity. One of the first cases where set theory beyond ZFC had an impact on abelian group theory was the appearance of measurable cardinals in the study of slender groups. As is well known, the theory of slenderness originated with Łoś and first appeared in Fuchs' 1958 book [**37**]. While the definition of slender involves homomorphisms from $\mathbf{Z}^\omega$ into the slender group L, Theorem 47.2 of [**37**] states a property that holds for homomorphisms from $\mathbf{Z}^\kappa$ into L provided that κ is a cardinal less than the first measurable cardinal, if there is one. (This is anticipated in [**20**].) Fuchs recalls that Łoś, when he outlined the proof,

> "had a clear idea that the cardinality restriction was unavoidable. The proof in the book follows closely his outline. I was surprised, because I had never seen anything like that before and was hoping that the restriction could be removed, but as we worked on the details of the proof, it became clear to me that it was impossible to get rid of it."

The notion of a measurable cardinal was defined by Ulam in a 1930 paper [**91**]. Ulam and Tarski proved that a measurable cardinal λ

Partially supported by NSF DMS 98-03126.
Received by the editors on July 16, 2001, and in revised form on September 18, 2001.

is strongly inaccessible; a footnote in [**37**] incorrectly states that the least measurable cardinal does not exceed the first inaccessible; in fact, as Kanamori [**57**, p. 27] says, “whether the least measurable cardinal is strictly larger than the least inaccessible cardinal became a focal question, and was settled only thirty years later” (by Hanf and Tarski in the affirmative).

A consequence of Łoś’s theorem is that, for cardinals κ less than the first measurable cardinal, $\mathbf{Z}^\kappa$ is reflexive, that is, naturally isomorphic to its double dual; the key is that $\mathrm{Hom}\,(\mathbf{Z}^\kappa, \mathbf{Z})$ is naturally isomorphic to $\oplus_\kappa \mathrm{Hom}\,(\mathbf{Z}, \mathbf{Z}) \cong \mathbf{Z}^{(\kappa)}$. (In 1954, Zeeman [**92**] proved this for κ less than the first strongly inaccessible cardinal, generalizing Specker who proved it for $\kappa = \aleph_0$ and independently Ehrenfeucht and Łoś [**20**] announced it for κ less than the first measurable cardinal.) In the early 1980’s Eda [**17, 19**] generalized the result so that it applies to all cardinals. The crucial idea is to use ω_1-complete ultrafilters, i.e., those closed under countable intersections. (A cardinal κ is $\geq$ the first measurable cardinal if and only if there is an ω_1-complete ultrafilter on κ which is not principal.) In particular, for arbitrary κ, $\mathrm{Hom}\,(\mathbf{Z}^\kappa, \mathbf{Z})$ is naturally isomorphic to $\oplus_D \mathrm{Hom}\,(\mathbf{Z}^\kappa/D, \mathbf{Z})$ where the direct sum is over all ω_1-complete ultrafilters, D, and $\mathbf{Z}^\kappa/D$ denotes the ultraproduct with respect to D. It follows that the double dual of $\mathbf{Z}^\kappa$ is free for all cardinals κ, but $\mathbf{Z}^\kappa$ is reflexive only if κ is below the first measurable cardinal. (See [**31**] for an exposition.) It remains an open question whether there is a reflexive group of measurable cardinality.

One can define a notion of M-slender by replacing $\mathbf{Z}^\omega$ in the definition of slender with a suitable subgroup M of $\mathbf{Z}^\omega$. Göbel and Wald [**44**] showed that Martin’s axiom implies that there are the maximum possible number of different notions of M-slender; on the other hand, Blass and Laflamme [**4**] showed that it is consistent with ZFC that there are exactly the minimal possible number, four, of different notions. If the continuum hypothesis fails, it becomes an interesting question to ask for the minimal size of a subgroup M of $\mathbf{Z}^\omega$ such that $\mathbf{Z}$ is M-slender; Eda [**18**] and Blass [**3**] considered the question and Blass, in particular, related it to several other, well-studied, so-called cardinal invariants of the real numbers.

In 1985 I visited Alan Mekler at Simon Fraser University and gave a course on reflexive groups, with the aim of understanding and completing a construction by Shelah of strongly nonreflexive groups. This led to [**33**] and to the idea of writing a book [**31**] with Alan Mekler on the broader subject of set-theoretic methods in module theory. It contains a chapter of new results on dual groups by Mekler and Shelah, as well as an exposition of results by the "Japanese school." (The new edition [**32**] contains later developments on dual groups, in particular by Mekler's student Greg Schlitt.)

2. Ext. Whitehead's problem asks whether $\mathrm{Ext}\,(A, \mathbf{Z}) = 0$ implies that A is free. In the 1950s, Stein and Ehrenfeucht showed that the answer is affirmative for countable A. For uncountable A prior to 1973, Nunke [**73**] says: "Many people: J. Rotman, myself, S. Chase, P. Griffith have studied this problem obtaining meager results." In hindsight, this was because the axioms of ZFC would not allow much more to be proved. In one case, Chase [**5**] went beyond ZFC and invoked a weak form of the continuum hypothesis (CH) to prove that, in modern terminology, every Whitehead group is strongly $\aleph_1$-free.

Saharon Shelah relates that in 1973 he had the habit of looking every week at the new books displayed in the Hebrew University library.

> "One day [in July 1973] I have come and see the second volume of László; its colour was attractive green. I take it and ask myself isn't everything known on [abelian groups]... I start to read each linearly; after reading about two thirds of the first volume I move to the second volume and read the first third. I mark the problems (I think six) which attract me—combination of being stressed by László, seem to me I have a chance, and how nice the problem look."

By September 4, 1973, Shelah had submitted to the Israel Journal a paper [**77**] proving that the solution to Whitehead's problem is independent of ZFC and answering some other open problems as well.

> "I have thought the most important is to build indecomposable abelian groups in every cardinality. I thought the independence of Whiteheads' problem will be looked on suspiciously. As you know abelian group theorists thought differently".

The paper proved that two different set-theoretic hypotheses (Martin's axiom $+\neg$CH and $V = L$) each consistent with ZFC, implied, respectively, negative and affirmative answers to Whitehead's problem for groups of cardinality $\aleph_1$. The affirmative result used the fact that, under $V = L$, Jensen's "diamond" prediction principles, $\diamond(S)$, held at *every* stationary subset S of $\aleph_1$. These principles have now become familiar and useful instruments in the abelian group theorist's toolbox, but at the time they were unfamiliar to most algebraists and the proof was not easy to read. Consequently, some false claims circulated that the affirmative result could be derived from the continuum hypothesis (CH) alone.

In the exposition [**22**] of Shelah's independence result, I used strongly $\aleph_1$-free groups in the proof that MA $+\neg$CH implies there are Whitehead groups which are not free. However, Shelah's original paper had used a larger class of groups (those not satisfying "Possibility I"), previously unknown to group theorists, which I later called the Shelah groups. Shelah's insight was demonstrated by the fact that it later became clear that this larger class of groups consisted of precisely the groups of cardinality $\aleph_1$ which are Whitehead in a model of MA $+\neg$CH; by constructing a Shelah group which is not strongly $\aleph_1$-free, Shelah showed that Chase's result requires the weak CH, cf. [**82**].

Among early workers who made use of Shelah's method were Mekler [**66**], Huber [**53, 54, 55**], Hiller [**52**], Sageev [**75, 76**] and myself [**23**]. The first four focused on the structure of Ext. Hiller (a graduate student in topology at MIT) noticed the application to a topological question posed by Kan and Whitehead [**56**]. Huber and Hiller, in a joint paper with Shelah [**51**], provided a definitive account of the torsion-free rank of Ext $(A, \mathbf{Z})$ assuming $V = L$. Huber recalls how Shelah improved the paper:

> "Saharon himself came to Zurich for a talk on August 22, 1977. After the talk I introduced myself to him. In no time he realized how to adapt my argument to obtain a much better result. We walked to downtown Zurich and had a Swiss cheese fondue in the 'Dézaley.' While steering the bread in the cheese Saharon was sketching the proof of the new result on a copy of the preprint. I kept that preprint for a long time. Besides Saharon's hand-writing there were some spots of melted cheese from the fondue!"

In particular, $V = L$ implies Ext $(A, \mathbf{Z})$ cannot equal $\mathbf{Q}$. Later, Shelah [**85**] proved, by a difficult direct forcing argument, that it is consistent with GCH that Ext $(A, \mathbf{Z}) = \mathbf{Q}$.

By 1976 Shelah had succeeded in proving that Whitehead's problem is independent of CH. (For more on this early history, see [**27**].) In the course of this investigation, he and Devlin discovered the weak diamond principle, which is equivalent to weak CH (and can be used to prove Chase's result). He also introduced the uniformization principles, which are consistent with GCH and have become another important tool; they have been used in algebra by Shelah, Mekler, Trlifaj and myself, among others. They can also be used to give a purely combinatorial equivalent to the Whitehead problem. (See [**32**, Chapter 13] for some of these uses.)

The problem of whether Baer groups (groups A such that $\mathrm{Ext}\,(A,T) = 0$ for all torsion groups T) are free was finally settled, in the affirmative, in ZFC, by Griffith in 1969. Kaplansky, in 1962, began the study of Baer modules over arbitrary domains; for nonhereditary domains, Griffith's method does not work, but Fuchs realized that Shelah's argument under $V = L$ could be carried out in ZFC when the second factor in Ext ranged over the class of torsion groups, rather than a single group. The result was a proof in ZFC that Baer modules over valuation domains are free [**28**] and a reduction of the problem for modules over arbitrary domains to the countably-generated case [**29**].

3. Almost free groups. It is a consequence of Reinhold Baer's fundamental work on torsion-free groups that there are $\aleph_1$-free groups, e.g., $\mathbf{Z}^\omega$, which are not free. Problem 10 in Fuchs' 1970 book [**38**] asked for which cardinals κ are there κ-free groups which are not κ^+-free. In 1972, Griffith [**46**] showed that, for all $n \in \omega$, there are $\aleph_n$-free groups which are not free, but didn't determine their cardinality. In 1974, Hill [**49**] constructed $\aleph_n$-free groups of cardinality $\aleph_n$ which are not free; the proof was an ingenious inductive construction of a "smooth chain" $\{G_\alpha : \alpha < \omega_n\}$ of free groups of cardinality $\aleph_{n-1}$, with requirements on the quotients G_β/G_α, $\alpha < \beta$. (Some of the ideas go back to Hill's 1969 construction [**47**] of a "Fuchs-5 group" of cardinality $\aleph_1$.)

We will say that *there is an almost free group* of cardinality κ if there is a κ-free group of cardinality κ which is not free. By 1973, several people, including Alan Mekler, David Kueker and John Gregory, had noticed that there is no almost free group of a weakly compact cardinality. In 1973, John Gregory announced in an abstract [**45**] that $V = L$ implies that an almost free group of cardinality κ exists, even one which is strongly κ-free, whenever κ is regular but not weakly compact. In trying to understand this, I discovered a ZFC result [**21**]: that the existence of an almost free group of cardinality κ, for a regular κ, implied the existence of an almost free group of cardinality κ^+. A key to this (and Gregory's result) is the role of stationary sets; in order for the union of the smooth chain $\{G_\alpha : \alpha < \kappa^+\}$ of free groups of cardinality κ to be nonfree, it is sufficient that the set of "bad places," i.e., the set of α such that G_β/G_α is nonfree for some $\beta > \alpha$, be stationary. I described how to turn this set into an invariant of an almost free group, the so-called Γ-invariant [**24**, p. 259].

One of the most fortunate events of my mathematical career was that Alan Mekler came from Toronto to become a graduate student at Stanford in 1970, the same year I arrived there as an assistant professor. It wasn't too long before he came to me with some interesting new results about almost free groups, and from then on I was learning as much from him as the other way around. His Ph.D. work on almost free groups (commutative and noncommutative) was published in 1980 [**67**], though the work was done much earlier. Shelah has said that he enjoyed working with Alan because of the quickness of his mind and the breadth of his knowledge and interests (covering many areas of logic and algebra). Manfred Dugas says

> "We are forever grateful for the great job Alan did lecturing to us and our students [at Essen] on set theory, which did enable us to make a contribution to the subject."

Alan was exuberantly brilliant, and it was a great loss, both to mathematics and to many of us personally, when he died at age 44 in 1992.

In 1974 Shelah proved his "Singular Compactness Theorem," a theorem of ZFC which (as one special case) says that there are no almost free groups of singular cardinality [**78**]. Shelah was aware of Hill's proofs of the latter fact for singular cardinals of cofinality ω [**48**] and cofinality ω_1 [**50**]. An important motivation of seeking this result was that it completed the proof from $V = L$ that every Whitehead group, of arbitrary cardinality, is free. (See [**27**] for more on the history of "compactness" results.)

Shelah formulated the singular compactness theorem in an axiomatic setting, so that it applies to an abstract notion of "almost free"; it has been applied, for example, in settings where "free" means being the union of a certain kind of chain of subgroups, cf. [**1**] or [**29**]. Another application that Shelah had in mind from the beginning was to transversal theory, a branch of combinatorial theory. There was, by 1974, a series of results in that area parallel to those about almost free groups (for example, an analog of Gregory's result). He conjectured, as early as 1975, that this was not an accident and by 1981 was able to prove it; this theorem (of ZFC) is easy to state but quite difficult to prove. It says that, for any cardinal λ, there is an almost free abelian group of cardinality λ if and only if the following combinatorial property, denoted NPT(λ), holds:

> ...there is a family of size λ of countable sets which does not have a transversal (a one-to-one choice function) but such that every subfamily of size $< \lambda$ does have a transversal.

The proof was published in 1985 [**88**] with thanks to Alan Mekler for "industriously refereeing the paper," and writing an appendix with an alternate version of the proof. The most complete exposition of the proof is published in [**31**, Chapter 7]. (We had assistance from Menachem Magidor; Mekler was able to understand his explanations, which at first were beyond me.) It remains open whether the existence of a λ-free nonfree *noncommutative* group of cardinality λ implies $NPT(\lambda)$.

The most comprehensive answer, so far, to the question of which cardinals λ satisfy NPT(λ) is found in a large paper by Magidor and Shelah [**67**]; it was finally published in 1994, but existed in various states for many years before that while it went through many improvements, including the elimination of some hypotheses on cardinal arithmetic by means of Shelah's powerful "pcf theory." In particular, it is proved (in ZFC) that NPT(λ) holds for every regular λ less than $\aleph_{\omega^2}$ (a considerable strengthening of the Hill result); moreover, it is consistent with GCH (assuming the consistency of the existence of certain large cardinal) that NPT($\aleph_{\omega^2+1}$) fails. But it remains open whether it is consistent that there are only countable many λ satisfying NPT(λ).

In 1980–81 I worked on the structure and classification of $\aleph_1$-separable groups of cardinality $\aleph_1$ under various set-theoretic hypotheses [**26**]. When I communicated the results to Mekler, he not only saw how to simplify some of the proofs but found the deeper reason for the positive results that followed from the Proper Forcing Axiom: a structural property he called "being in standard form" [**69**], which he presented at the first Honolulu conference on abelian groups.

4. Indecomposable groups. Problem 21 of Fuchs' 1958 treatise [**37**] asked whether indecomposable groups of arbitrarily large cardinality exist. By 1958, the barrier of the continuum had been surmounted and indecomposable groups of cardinality $2^{2^{\aleph_0}}$ had been constructed by Fuchs, Hulanicki and Sasiada. By the time that Fuchs' 1973 book [**39**] appeared, the work of Fuchs [**36**] and Corner [**6**] had led to a proof of the existence of indecomposables of all cardinalities less than the first strongly inaccessible cardinal. A construction that worked for all cardinals was still elusive, although Fuchs [**40**] was able to go beyond the first measurable cardinal.

In the same paper [**77**] in which he showed that the Whitehead problem was undecidable, Shelah proved (in ZFC) that, for every cardinal λ, there is a rigid system $\{G_i : i < 2^\lambda\}$ of abelian groups of cardinality λ; that is, for every i, $\mathrm{End}\,(G_i) \cong \mathbf{Z}$ and for $i \neq j$, $\mathrm{Hom}\,(G_i, G_j) = 0$. In particular, each G_i is indecomposable. Shelah followed up in [**79**] with a related result for p-groups (although just for arbitrarily large λ-specifically for strong limit cardinals λ, that is, $2^\kappa < \lambda$ whenever $\kappa < \lambda$). Dugas and Göbel [**11**] corrected an error in the latter and generalized the result to realize other rings as endomorphism rings.

Shelah's proof was based on a general combinatorial method which he tried to advertise in his 1974 Vancouver ICM lecture (and which was published in [**81**]), but "[a]s the suggestion has not been followed up, ... we develop from it 'black boxes' which hopefully can be used by algebraists." [**86**, p. 240]

This is carried out more explicitly in the second half of the paper [**87**], published in the proceedings of the 1984 Udine conference; there he separated out the combinatorics from the algebra and extended it to a wider class of cardinals than just strong limit cardinals. The two papers began life as one handwritten paper during the 1980–81 Jerusalem Model Theory Year. Corner and Göbel extracted another version of the combinatorics from their reading of a preliminary version of the paper. It was they who invented the term "Black Box" (as described by Göbel)

> "... on a nice summer evening at home in our backyard in 1982 sitting together with Tony Corner: We had just (essentially) finished the joint paper in the Proc. LMS with an appendix on the ____, and so we searched for a name for the appendix and immediately liked the name Black Box because it somehow describe what it is good for. ... I mentioned the [name] Black Box to Saharon who immediately agreed and used that name."

It was through the Corner-Göbel paper [**7**], which contained a wide range of applications, that the term "Black Box" and the usefulness of the method became known to algebraists. Since then a whole industry of applications of the Black Box, to realize endomorphism rings and give negative solutions to the Kaplansky test problems, among other uses, has developed. One can consult, for example [**31**, Chapter 13], [**32**, Chapter 14] or [**42**] for more information.

An additional element is added if one asks for indecomposable groups which are almost free, or better. Mekler proved, in his thesis, that MA $+\neg$CH implies that strongly $\aleph_1$-free groups of cardinality $\aleph_1$ are $\aleph_1$-separable. Assuming $V = L$, we constructed strongly κ-free groups of cardinality κ which are indecomposable [**30**]. Dugas [**10**] generalized this to obtain rigid groups under the same hypothesis. Shelah [**84**] used weak diamond, instead of diamond. Dugas and Göbel [**12**] proved the most general result in this line by realizing suitable rings as endomorphism rings of strongly κ-free groups, assuming $V = L$.

5. ***P*-groups.** In the early 1950s, Fuchs and Kulikov independently gave necessary and sufficient conditions, involving cardinal arithmetic, for a sequence of separable p-groups indexed by an ordinal to be the Ulm sequence of a p-group, see [**37**, Section 38].

In 1974 Warfield posed a question about uniquely ω-elongating p-groups, which he said "was raised long ago by Peter Crawley." Charles Megibben [**64**] employed the set-theoretic methods pioneered by Shelah to prove that Crawley's problem for groups of cardinality $\aleph_1$ is undecidable in ZFC (by assuming $V = L$ on the one hand and MA $+\neg$CH on the other). Mekler [**69**] built on this and used proper forcing to show that the problem is undecidable in ZFC + GCH. Mekler and Shelah [**70, 71**] completed the story of what happens for arbitrary cardinality when $V = L$. Megibben also studied ω_1-separable p-groups in [**65**] and proved, among other things, that it is undecidable in ZFC whether the socle determines if the p-group is ω_1-separable.

In a series of papers in the 1960s, Nunke studied the structure of torsion products and, in particular, considered the question of when $\mathrm{Tor}\,(A, B)$ is a direct sum of countable groups (dsc). He gave a complete answer when A and B are p-groups of different lengths. The remaining cases can be reduced to those where A and B are C_λ-groups of length λ for some $\lambda \leq \omega_1$. Patrick Keef proved [**58**] that there are C_{ω_1}-groups A and B of length ω_1 such that $\mathrm{Tor}\,(A, B)$ is not a disc if and only if Kurepa's hypothesis (KH) fails. KH is the combinatorial hypothesis that there is a set Y of subsets of ω_1 of size $\aleph_2$ such that, for all $\alpha < \omega_1$, $\{y \cap \alpha : y \in Y\}$ is countable. It is known that KH is independent of GCH (and implied by $V = L$). A simplified proof of Keef's result, using valuated vector spaces, was given by Cutler and Dimitric [**8**]. Keef has given other applications of set theory to the study of torsion products, for example in [**59**] and [**60**]. See [**61**] for a survey and bibliography.

Ulm's theorem implies that countable p-groups are determined by their socles. Dugas and Vergohsen [**16**] proved that $V = L$ implies that the only separable p-groups of cardinality $\aleph_1$ which are determined by their socles are the Σ-cyclic and torsion-complete ones. Shelah [**89**] derived the same result from GCH and Mekler and Shelah [**72**] proved that the result is independent of ZFC.

6. Butler groups. A finite rank group is called a Butler group if it is a pure subgroup of a completely decomposable group; these groups are named for M.C.R. Butler, who studied them in a 1965 paper. In 1983 Bican and Salce [**2**] gave two different generalizations of the notion to infinite rank torsion-free groups:

- A is B_1 if $\mathrm{Bext}^1(A, T) = 0$ for all torsion T;
- A is B_2 if A is the union of a smooth chain $\{A_\alpha : \alpha < \sigma\}$ such that $A_{\alpha+1} = A_\alpha + G_\alpha$ for all α, where G_α is a finite rank Butler group.

They showed that the two notions coincide for all countable groups and that B_2 always implies B_1, for arbitrary cardinality. The natural question was then: does B_1 always imply B_2? This was answered in the affirmative for groups of cardinality $\leq \aleph_\omega$, assuming CH, by Dugas-Hill-Rangaswamy [**14**]. Then Fuchs and Magidor, in a collaboration that began at the Curaçao conference in 1991, gave an affirmative answer for all cardinalities under the hypothesis $V = L$ [**41**]; the proof used Jensen's "square" principles. (In [**35**], Foreman and Magidor identify a "very weak square" principle, which is sufficient for the result and consistent with the existence of very large cardinals.) Only recently have Shelah and Strüngmann [**90**] constructed a model of ZFC in which CH fails ($2^{\aleph_0} = \aleph_4$, for example) and there is a B_1-group which is not B_2.

Another key question in the subject has been whether $\mathrm{Bext}^2(A, T) = 0$ for all torsion-free A and torsion T. Dugas and Thomé [**15**] proved that an affirmative answer for all groups A of cardinality $\aleph_2$ is equivalent to CH. Rangaswamy [**74**] showed that, under CH, an affirmative answer implies that every B_1-group is B_2. In their paper, Fuchs and Magidor proved that $V = L$ implies an affirmative answer. However, Magidor and Shelah [**63**] have proved (with the necessary assumption of the consistency of some large cardinals) that it is consistent with GCH that the answer is negative. It remains open whether it is consistent with CH that there is a B_1-group which is not B_2.

Acknowledgments. I would like to thank Manfred Dugas, László Fuchs, Rüdiger Göbel, Martin Huber and Saharon Shelah for their help; their reminiscences herein are taken from emails to me, with their permission.

REFERENCES

1. T. Becker, L. Fuchs and S. Shelah, *Whitehead modules over domains*, Forum Math. **1** (1989), 53–68.

2. L. Bican and L. Salce, *Butler groups of infinite rank*, Abelian Group Theory, Lecture Notes in Math., vol. 1006, Springer, New York, 1983, pp. 171–189.

3. A. Blass, *Cardinal characteristics and the product of countably many infinite cyclic groups*, J. Algebra **169** (1994), 512–540.

4. A. Blass and C. Laflamme, *Consistency results about filters and the number of inequivalent growth types*, J. Symbolic Logic **54** (1989), 50–56.

5. S.U. Chase, *On group extensions and a problem of J.H.C. Whitehead*, in *Topics in abelian groups*, Scott, Foresman and Co., 1963, pp. 173–197 .

6. A.L.S. Corner, *Endomorphism algebras of large modules with distinguished submodules*, J. Algebra **11** (1969), 155–185.

7. A.L.S. Corner and R. Göbel, *Prescribing endomorphism algebras, a unified treatment*, Proc. London Math. Soc. (3) **50** (1985), 447–479.

8. D. Cutler and R. Dimitric, *Valuated vector spaces, Kurepa's hypothesis and abelian p-groups*, Rocky Mountain J. Math. **23** (1993), 1253–1266.

9. K.J. Devlin and S. Shelah, *A weak version of* $\diamond$ *which follows from* $2^{\aleph_0} < 2^{\aleph_1}$, Israel J. Math. **29** (1978), 239–247.

10. M. Dugas, *Fast freie abelsche Gruppen mit Endomorphismenring Z*, J. Algebra **71** (1981), 314–321.

11. M. Dugas and R. Göbel, *On endomorphism rings of primary abelian groups*, Math. Ann. **261** (1982), 359–385.

12. ———, *Every cotorsion-free ring is an endomorphism ring*, Proc. London Math. Soc. (3) **45** (1982), 319–336.

13. M. Dugas and R. Göbel, *Every cotorsion-free algebra is an endomorphism algebra*, Math. Z. **181** (1982), 451–470.

14. M. Dugas, P. Hill and K.M. Rangaswamy, *Butler groups of infinite rank*, II, Trans. Amer. Math. Soc. **320** (1990), 643–664.

15. M. Dugas and B. Thomé, *The functor* Bext *under the negation of* CH, Forum Math. **3** (1991), 23–33.

16. M. Dugas and R. Vergohsen, *On socles of abelian p-groups in L*, Rocky Mountain J. Math. **18** (1988), 733–752.

17. K. Eda, *A boolean power and a direct product of abelian groups*, Tsukuba J. Math. **6** (1982), 187–193.

18. ———, *A note on subgroups of* Z^N, Abelian Group Theory, Lecture Notes in Math., vol. 1006, Springer, New York, 1983, pp. 371–374.

19. ———, *On a Boolean power of a torsion free abelian group*, J. Algebra **82** (1983), 84–93.

20. A. Ehrenfeucht and J. Łoś, *Sur les produits cartésiens des groupes cycliques infinis*, Bull. Acad. Polon. Sci., Sér. Sci. Math. **2** (1954), 261–263.

21. P. Eklof, *On the existence of* κ*-free abelian groups*, Proc. Amer. Math. Soc. **47** (1975), 65–72.

22. ———, *Whitehead's problem is undecidable*, Amer. Math. Monthly **83** (1976), 775–788.

23. ———, *Homological algebra and set theory*, Trans. Amer. Math. Soc. **227** (1977), 207–225.

24. ———, *Methods of logic in abelian group theory*, Abelian Group Theory, Lecture Notes in Math., vol. 616, Springer, New York, 1977, pp. 251–269.

25. ———, *Set theoretic methods in homological algebra and abelian groups*, Les Presses de L'Université de Montréal, 1980.

26. ———, *The structure of* ω_1*-separable groups*, Trans. Amer. Math. Soc. **279** (1983), 497–523.

27. ———, *Set theory generated by abelian group theory*, Bull. Symbolic Logic **3** (1997), 1–16.

28. P. Eklof and L. Fuchs, *Baer modules over valuation domains*, Ann. Mat. Pura Appl. (4) **150** (1988), 363–374.

29. P. Eklof, L. Fuchs and S. Shelah, *Baer modules over domains*, Trans. Amer. Math. Soc. **322** (1990), 547–560.

30. P. Eklof and A. Mekler, *On constructing indecomposable groups in L*, J. Algebra **49** (1977), 96–103.

31. ———, *Almost free modules*, North-Holland Math. Library, vol. 46, Elsevier Sci. Publ., New York, 1990.

32. ———, *Almost free modules*, revised and expanded ed., Elsevier Sci. Publ., New York, 2002.

33. P. Eklof, A. Mekler and S. Shelah, *On strongly nonreflexive groups*, Israel J. Math. **59** (1987), 283–298.

34. P. Eklof and S. Shelah, *A combinatorial principle equivalent to the existence of non-free Whitehead groups*, Contemp. Math., vol. 171, 1994, Amer. Math. Soc., pp. 79–98.

35. M. Foreman and M. Magidor, *A very weak square principle*, J. Symbolic Logic **62** (1997), 175–196.

36. L. Fuchs, *The existence of indecomposable abelian groups of arbitrary power*, Acta Math. Acad. Sci. Hungar. **10** (1959), 453–457.

37. ———, *Abelian Groups*, Publishing House of the Hungarian Academy of Science, Budapest (1958); reprinted by Pergamon Press, Oxford, 1960.

38. ———, *Infinite abelian groups*, Pure Appl. Math., vol. 36-I, Acad. Press, New York, 1970.

39. ———, *Infinite abelian groups*, Pure Appl. Math., vol. 36-II, Acad. Press, New York, 1973.

40. ———, *Indecomposable abelian groups of measurable cardinalities*, Sympos. Math., vol. 13, Acad. Press, London, 1974, pp. 233–244.

41. L. Fuchs and M. Magidor, *Butler groups of arbitrary cardinality*, Israel J. Math. **84** (1993), 239–263.

42. R. Göbel and S. Shelah, *On the existence of rigid $\aleph_1$-free abelian groups of cardinality $\aleph_1$*, Abelian Groups and Modules (Padova, 1994), Math. Appl., vol. 343, Kluwer, 1995, pp. 227–237.

43. R. Göbel and J. Trlifaj, *Approximations and endomorphism algebras of modules*, deGruyter, to appear.

44. R. Göbel and B. Wald, *Martin's axiom implies the existence of certain slender groups*, Math. Z. **172** (1980), 107–121.

45. J. Gregory, *Abelian groups infinitarily equivalent to free ones*, Notices Amer. Math. Soc. **20** (1973), A–500.

46. P.A. Griffith, *$\aleph_n$-free abelian groups*, Quart. J. Math. Oxford Ser. (2) **23** (1972), 417–425.

47. P. Hill, *On the decomposition of groups*, Canad. J. Math. **21** (1969), 762–768.

48. ———, *On the freeness of abelian groups*: *A generalization of Pontryagin's theorem*, Bull. Amer. Math. Soc. **76** (1970), 1118–1120.

49. ———, *New criteria for freeness in abelian groups* II, Trans. Amer. Math. Soc. **196** (1974), 191–201.

50. ———, *A special criterion for freeness*, Sympos. Math., vol. 13, Acad. Press, London, 1974, pp. 311–314.

51. H. Hiller, M. Huber and S. Shelah, *The structure of Ext*$(A, \mathbf{Z})$ *and* $V = L$, Math. Z. **162** (1978), 39–50.

52. H. Hiller and S. Shelah, *Singular cohomology in* L, Israel J. Math. **26** (1977), 313–319.

53. M. Huber, *Sur le problème de Whitehead concernant les groupes abeliens libres*, C.R. Acad. Sci. Paris Sér. A-B **284** (1977), A471–A472.

54. ———, *Caracterisation des groupes abeliens libres, et cardinalités*, C.R. Acad. Sci. Paris Sér. A-B **285** (1977), A1–A2.

55. ———, *Sur les groupes abeliens de la forme* Ext (A, G), C.R. Acad. Sci. Paris Sér. A-B **285** (1977), A529–A531.

56. D. Kan and G. Whithead, *On the realizability of singular cohomology groups*, Proc. Amer. Math. Soc. **12** (1961), 24–25.

57. A. Kanamori, *The higher infinite*, Perspect. Math. Logic, Springer, New York, 1994.

58. P. Keef, *On set theory and the balanced projective dimension of* $C_\vee$*-groups*, Contemp. Math., vol. 87, Amer. Math. Soc., 1989, pp.31–41.

59. ———, *On iterated torsion products of abelian p-groups*, Rocky Mountain J. Math. **21** (1991), 1035–1055.

60. ———, *An application of set theory to the torsion product of abelian groups*, Proc. Amer. Math. Soc. **118** (1993), 735–738.

61. ———, *Abelian groups and the torsion product*, Abelian Groups and Modules (Colorado Springs, CO, 1995), Lecture Notes in Pure and Appl. Math., vol. 182, Dekker, New York, 1996, pp. 45–66.

62. M. Magidor and S. Shelah, *When does almost free imply free? (For groups, transversals, etc.)*, J. Amer. Math. Soc. **7** (1994), 769–830.

63. ———, $\mathrm{Bext}^2(G, T)$ *can be nontrivial, even assuming* GCH, Abelian Group Theory and Related Topics (Oberwolfach, 1993), Contemp. Math., vol. 171, Amer. Math. Soc., 1994, pp. 287–294.

64. C. Megibben, *Crawley's problem on the unique* ω*-elongation of p-groups is undecidable*, Pacific J. Math. **107** (1983), 205–212.

65. ———, ω_1*-separable p-groups*, Abelian Group Theory (Oberwolfach, 1985), Gordon and Breach, New York, 1987, pp. 117–136.

66. A. Mekler, *The number of* κ*-free abelian groups and the size of Ext*, Abelian Group Theory, Lecture Notes in Math., vol. 616, Springer, New York, 1977, pp. 323–331.

67. ———, *How to construct almost free groups*, Canad. J. Math. **32** (1980), 1206–1228.

68. ———, *On Shelah's Whitehead groups and CH*, Rocky Mountain J. Math. **12** (1982), 271–278.

69. ———, *Proper forcing and abelian groups*, Abelian Group Theory, Lecture Notes in Math., vol. 1006, Springer, New York, 1983, pp. 285–303.

70. A. Mekler and S. Shelah, ω*-elongations and Crawley's problem*, Pacific J. Math. **121** (1986), 121–132.

71. ———, *The solution to Crawley's problem*, Pacific J. Math. **121** (1986), 133–134.

72. A. Mekler and S. Shelah, *Uniformization principles*, J. Symbolic Logic **54** (1989), 441–459.

73. R. Nunke, *Whitehead's problem*, Abelian Group Theory, Lecture Notes in Math., vol. 616, Springer, New York, 1977, pp. 240–250.

74. K.M. Rangaswamy, *A homological characterization of abelian B_2-groups*, Proc. Amer. Math. Soc. **121** (1994), 409–415.

75. G. Sageev and S. Shelah, *Weak compactness and the structure of $Ext(A, \mathbf{Z})$*, Abelian Group Theory, Lecture Notes in Math., vol. 874, Springer, New York, 1981, pp. 87–92.

76. ———, *On the structure of $Ext(A, \mathbf{Z})$ in ZFC^+*, J. Symbolic Logic **50** (1985), 302–315.

77. S. Shelah, *Infinite abelian groups, Whitehead problem and some constructions*, Israel J. Math. **18** (1974), 243–325.

78. ———, *A compactness theorem for singular cardinals, free algebras, Whitehead problem and transversals*, Israel J. Math. **21** (1975), 319–349.

79. ———, *Existence of rigid-like families of abelian p-groups*, Model Theory and Algebra, Lecture Notes in Math., vol. 498, Springer, New York, 1975, pp. 384–402.

80. ———, *Whitehead groups may not be free even assuming CH*, I, Israel J. Math. **28** (1977), 193–203.

81. ———, *Classification theory and the number of non-isomorphic models*, North-Holland, Amsterdam, 1978.

82. ———, *On uncountable abelian groups*, Israel J. Math. **32** (1979), 311–330.

83. ———, *Whitehead groups may not be free even assuming CH*, II, Israel J. Math. **35** (1980), 257–285.

84. ———, *On endo-rigid strongly $\aleph_1$-free abelian groups in $\aleph_1$*, Israel J. Math. **40** (1981), 291–295.

85. ———, *The consistency of $Ext(G, \mathbf{Z}) = \mathbf{Q}$*, Israel J. Math. **39** (1981), 74–82.

86. ———, *A combinatorial principle and endomorphism rings* I: *On p-groups*, Israel J. Math. **49** (1984), 239–257.

87. ———, *A combinatorial theorem and endomorphism rings of Abelian groups* II, Abelian Groups and Modules, CISM Courses and Lectures, vol. 287, Springer, New York, 1984, pp. 37–86.

88. ———, *Incompactness in regular cardinals*, Notre Dame J. Formal Logic **26** (1985), 195–228.

89. ———, *On reconstructing separable reduced p-groups with a given socle*, Israel J. Math. **60** (1987), 146–166.

90. S. Shelah and L. Strüngmann, *It is consistent with ZFC that B_1 groups are not B_2*, preprint.

91. S. Ulam, *Zur Masstheorie in der allgemeinen Mengenlehre*, Fund. Math. **16** (1930), 140–150.

92. E.C. Zeeman, *On direct sums of free cycles*, J. London Math. Soc. **30** (1955), 195–212.

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