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WEIGHTED COMPOSITION OPERATORS BETWEEN BLOCH-TYPE SPACES

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ABSTRACT. We discuss boundedness and compactness of composition operators followed by multiplication as operators between Bloch-type spaces of analytic functions on the unit disk.

1. Introduction. In this paper $\mathbf{D}$ denotes the open unit disk, i.e., $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$. For $0 < \alpha < \infty$, let $\mathcal{B}^\alpha$ consist of all analytic functions f on $\mathbf{D}$ satisfying the condition

$$\sup_{z\in\mathbf{D}}(1-|z|^2)^\alpha|f'(z)| < \infty.$$

Note that $\mathcal{B}^1 = \mathcal{B}$, the usual Bloch space. For $f \in \mathcal{B}^\alpha$, define

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z\in\mathbf{D}}(1-|z|^2)^\alpha|f'(z)| < \infty.$$

With this norm $\mathcal{B}^\alpha$ is a Banach space.

For $0 < \alpha < 1$ and an analytic map $\varphi : \mathbf{D} \to \mathbf{D}$ the question when the composition operator C_φ given by

$$C_\varphi f = f \circ \varphi, \quad \text{for } f \in \mathcal{B}^\alpha,$$

is a bounded operator on $\mathcal{B}^\alpha$ was considered and solved first by Roan [**8**] and later by Madigan [**5**]. Boundedness and compactness of C_φ on the Bloch space $\mathcal{B} = \mathcal{B}^1$ were described by Madigan and Matheson [**6**].

The multipliers of the Bloch space $\mathcal{B} = \mathcal{B}^1$ were first characterized by Arazy [**1**]. For $\alpha \neq 1$, Zhu [**12**] characterized the multipliers of the spaces $\mathcal{B}^\alpha$.

We will consider the question for which analytic functions u on $\mathbf{D}$ and for which analytic mappings $\varphi : \mathbf{D} \to \mathbf{D}$ the weighted composition

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operator uC_φ maps the space $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$. We furthermore consider the question when these weighted composition operators are compact.

The results in this article supplement those recently obtained by the first and third author [**7**].

In the remainder of this section, we will prove some estimates for functions in $\mathcal{B}^\alpha$ needed in the sequel.

Integrating the estimate $|f'(z)| \leq \|f\|_{\mathcal{B}^\alpha}/(1-|z|)^\alpha$, we obtain

$$|f(z)-f(0)| \leq \int_0^1 |z|\,|f'(tz)|\,dt \leq \|f\|_{\mathcal{B}^\alpha} \int_0^1 \frac{|z|}{(1-t|z|)^\alpha}\,dt,$$

for all $z \in \mathbf{D}$. In case $0 < \alpha < 1$ the integral at the right is uniformly bounded by the constant $\int_0^1 (1-t)^{-\alpha}\,dt$, and it follows that $\mathcal{B}^\alpha \subset H^\infty$. It is easy to check that in this case the linear space $\mathcal{B}^\alpha$ is an algebra. In fact, Hardy and Littlewood [**4**] have shown that for $0 < \alpha < 1$ the space $\mathcal{B}^\alpha$ consists of all functions f analytic on $\mathbf{D}$ satisfying the Lipschitz condition

$$|f(z)-f(w)| \leq M|z-w|^{1-\alpha}, \tag{1.1}$$

for all $z, w \in \mathbf{D}$ (see also Duren [**3**]).

In case $1 < \alpha < \infty$, the above estimate implies

$$|f(z)-f(0)| \leq \frac{\|f\|_{\mathcal{B}^\alpha}}{\alpha-1}\frac{1}{(1-|z|)^{\alpha-1}}, \tag{1.2}$$

while for $\alpha = 1$ we have

$$|f(z)-f(0)| \leq \|f\|_{\mathcal{B}} \log\frac{1}{1-|z|}, \tag{1.3}$$

for all $z \in \mathbf{D}$.

2. Weighted composition operators between Bloch-type spaces. The following result completely characterizes the bounded weighted composition operators from one Bloch-type space into another.

Theorem 2.1. *Let φ and u be analytic on $\mathbf{D}$, φ a self-map of $\mathbf{D}$. Let α and β be positive real numbers.*

(i) *If $0 < \alpha < 1$, then uC_φ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$ if and only if $u \in \mathcal{B}^\beta$ and*

$$\sup_{w\in\mathbf{D}} |u(w)| \frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^\alpha} |\varphi'(w)| < \infty.$$

(ii) *The operator uC_φ maps $\mathcal{B}$ boundedly into $\mathcal{B}^\beta$ if and only if*

$$\sup_{w\in\mathbf{D}} |u'(w)|(1-|w|^2)^\beta \log\frac{1}{1-|\varphi(w)|^2} < \infty$$

and

$$\sup_{w\in\mathbf{D}} |u(w)| \frac{(1-|w|^2)^\beta}{1-|\varphi(w)|^2} |\varphi'(w)| < \infty.$$

(iii) *If $\alpha > 1$, then uC_φ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$ if and only if*

$$\sup_{w\in\mathbf{D}} |u'(w)| \frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^{\alpha-1}} < \infty$$

and

$$\sup_{w\in\mathbf{D}} |u(w)| \frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^\alpha} |\varphi'(w)| < \infty.$$

Proof. First we consider the case that $\alpha > 1$. We begin by proving that the conditions in (iii) are necessary for the operator uC_φ to map $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$. Suppose that $uC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded. Fix $w \in \mathbf{D}$ and assume that $\varphi(w) \neq 0$. Consider the function f defined by

$$f_w(z) = \frac{1}{\overline{\varphi(w)}} \left\{ \frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}z)^\alpha} - \frac{1}{(1-\overline{\varphi(w)}z)^{\alpha-1}} \right\}, \tag{2.2}$$

for $z \in \mathbf{D}$. Then

$$f_w'(z) = \frac{\alpha(1-|\varphi(w)|^2)}{(1-\overline{\varphi(w)}z)^{\alpha+1}} - \frac{(\alpha-1)}{(1-\overline{\varphi(w)}z)^\alpha},$$

for $z \in \mathbf{D}$. Hence

$$|f_w'(z)| \leq \frac{\alpha(1-|\varphi(w)|^2)}{(1-|\varphi(w)|\,|z|)^{\alpha+1}} + \frac{\alpha}{(1-|z|)^\alpha} \leq \frac{\alpha 2^{\alpha+2}}{(1-|z|^2)^\alpha},$$

for all $z \in \mathbf{D}$, so $f_w \in \mathcal{B}^\alpha$. Since $f_w(0) = -\varphi(w)$, we have $|f_w(0)| \leq 1$. Thus $M = \sup\{\|f_w\|_{\mathcal{B}^\alpha} : w \in \mathbf{D}\} \leq 1+\alpha 2^{\alpha+2}$. Note that $f_w(\varphi(w)) = 0$ and $f_w'(\varphi(w)) = 1/(1-|\varphi(w)|^2)^\alpha$. For this function f_w and this point w we have $(uC_\varphi f_w)'(w) = u(w)\varphi'(w)/(1-|\varphi(w)|^2)^\alpha$, thus

$$\begin{aligned}(1-|w|^2)^\beta\,|u(w)|\,\frac{|\varphi'(w)|}{(1-|\varphi(w)|^2)^\alpha} &= (1-|w|^2)^\beta\,|(uC_\varphi f_w)'(w)| \\ &\leq \|uC_\varphi f_w\|_{\mathcal{B}^\beta} \leq \|uC_\varphi\| M,\end{aligned}$$

for all $w \in \mathbf{D}$. By continuity the above estimate remains valid also if $\varphi(w) = 0$. This proves that the second condition in (iii) is necessary.

Next, for fixed $w \in \mathbf{D}$, consider the function

$$g_w(z) = \frac{\alpha}{(1-\overline{\varphi(w)}z)^{\alpha-1}} - \frac{(\alpha-1)(1-|\varphi(w)|^2)}{(1-\overline{\varphi(w)}z)^\alpha}. \tag{2.3}$$

Then

$$g_w'(z) = \frac{\alpha(\alpha-1)\overline{\varphi(w)}}{(1-\overline{\varphi(w)}z)^\alpha} - \frac{\alpha(\alpha-1)\overline{\varphi(w)}(1-|\varphi(w)|^2)}{(1-\overline{\varphi(w)}z)^{\alpha+1}}.$$

As before $|g_w'(z)| \leq \alpha^2 2^{\alpha+2}/(1-|z|^2)^\alpha$, for all $z \in \mathbf{D}$, so that $g_w \in \mathcal{B}^\alpha$ and $L = \sup\{\|g_w\|_{\mathcal{B}^\alpha} : w \in \mathbf{D}\} < \infty$. Note that now $g_w'(\varphi(w)) = 0$ and $g_w(\varphi(w)) = 1/(1-|\varphi(w)|^2)^{\alpha-1}$. Thus we have $(uC_\varphi g_w)'(w) = u'(w)/(1-|\varphi(w)|^2)^{\alpha-1}$. Hence

$$\begin{aligned}|u'(w)|\,\frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^{\alpha-1}} &\leq (1-|w|^2)^\beta\,|(uC_\varphi g_w)'(w)| \\ &\leq \|uC_\varphi g_w\|_{\mathcal{B}^\beta} \leq \|uC_\varphi\| L,\end{aligned}$$

for all $w \in \mathbf{D}$, showing that the first condition in (iii) is also necessary.

The fact that the conditions in (iii) are sufficient for boundedness of operator uC_φ as an operator from $\mathcal{B}^\alpha$ into $\mathcal{B}^\beta$ is shown as follows.

Suppose that $f \in \mathcal{B}^\alpha$ is in the unit ball of $\mathcal{B}^\alpha$ and that u and φ satisfy the conditions in (iii). Note that $u = uC_\varphi 1 \in \mathcal{B}^\beta$. We have

$$\begin{aligned}(1-|z|^2)^\beta\,|(uC_\varphi f)'(z)| &\le (1-|z|^2)^\beta\,|u'(z)|\,|f(\varphi(z))| \\ &\quad + (1-|z|^2)^\beta\,|u(z)|\,|f'(\varphi(z))|\,|\varphi'(z)| \\ &\le (1-|z|^2)^\beta\,|u'(z)|\,|f(\varphi(z)) - f(\varphi(0))| \\ &\quad + (1-|z|^2)^\beta\,|u'(z)|\,|f(\varphi(0))| \\ &\quad + (1-|z|^2)^\beta\,|u(z)|\,|f'(\varphi(z))|\,|\varphi'(z)| \\ &\le (1-|z|^2)^\beta\,|u'(z)|\,\frac{C}{(1-|\varphi(z)|^2)^{\alpha-1}} \\ &\quad + C(1-|z|^2)^\beta\,|u'(z)| \\ &\quad + (1-|z|^2)^\beta\,|u(z)|\,\frac{1}{(1-|\varphi(z)|^2)^\alpha}\,|\varphi'(z)|,\end{aligned}$$

since $\|f\|_{\mathcal{B}^\alpha} \le 1$ and by the above conditions we conclude that $\|uC_\varphi f\|_{\mathcal{B}^\beta} \le C$, thus uC_φ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$. This completes the proof of (iii).

Next we will prove (ii). Suppose that uC_φ maps $\mathcal{B}$ boundedly into $\mathcal{B}^\beta$. For given $w \in \mathbf{D}$ instead of the function g_w consider the function h_w given by

$$h_w(z) = 2\log\frac{1}{1-\overline{\varphi(w)}z} - \left(\log\frac{1}{(1-\overline{\varphi(w)}z}\right)^2 \Big/ \log\frac{1}{1-|\varphi(w)|^2},$$

for $z \in \mathbf{D}$. Then

$$h_w'(z) = \frac{2\overline{\varphi(w)}}{1-\overline{\varphi(w)}z} - \frac{2\overline{\varphi(w)}}{1-\overline{\varphi(w)}z}\left(\log\frac{1}{1-\overline{\varphi(w)}z}\right)\Big/\log\frac{1}{1-|\varphi(w)|^2},$$

for $z \in \mathbf{D}$, so that $h_w \in \mathcal{B}$ and $L = \sup\{\|h_w\|_{\mathcal{B}} : w \in \mathbf{D}\} < \infty$. Note that $h_w'(\varphi(w)) = 0$ and $h_w(\varphi(w)) = \log(1/(1-|\varphi(w)|^2))$. Thus

$$\begin{aligned}(1-|w|^2)^\beta\,|u'(w)|\,\log\frac{1}{1-|\varphi(w)|^2} &= (1-|w|^2)^\beta\,|(uC_\varphi h_w)'(w)| \\ &\le \|uC_\varphi h_w\|_{\mathcal{B}^\beta} \le \|uC_\varphi\|L,\end{aligned}$$

for all $w \in \mathbf{D}$, proving that the first condition in (ii) is necessary. That also the second condition in (ii) is necessary is proved as before. It is easy to see that the two conditions in (ii) are also sufficient.

Finally we will prove (i). Suppose $0 < \alpha < 1$ and that uC_φ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$. As in the proof of (iii), it is necessary that

$$\sup_{w\in\mathbf{D}} |u(w)| \frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^\alpha} |\varphi'(w)| < \infty.$$

It is also necessary that $u = uC_\varphi 1 \in \mathcal{B}^\beta$. We will show that these conditions are also sufficient. Let $f \in \mathcal{B}^\alpha$. Then

$$\begin{aligned}(1-|z|^2)^\beta\,|(uC_\varphi f)'(z)| \le{} & (1-|z|^2)^\beta\,|u'(z)f(\varphi(z))| \\ & + (1-|z|^2)^\beta\,|u(z)f'(\varphi(z))\varphi'(z)|.\end{aligned}$$

Note that

$$\begin{aligned}(1-|z|^2)^\beta\,&|u(z)f'(\varphi(z))\varphi'(z)| \\ &= \frac{(1-|z|^2)^\beta\,|\varphi'(z)|}{(1-|\varphi(z)|^2)^\alpha}\,|u(z)|\,(1-|\varphi(z)|^2)^\alpha\,|f'(\varphi(z))| \\ &\le M(1-|\varphi(z)|^2)^\alpha\,|f'(\varphi(z))| \le M\|f\|_{\mathcal{B}^\alpha}.\end{aligned}$$

Also,

$$(1-|z|^2)^\beta\,|u'(z)f(\varphi(z))| \le (1-|z|^2)^\beta\,|u'(z)|\,\|f\|_\infty \le \|u\|_{\mathcal{B}^\beta}\,\|f\|_{\mathcal{B}^\alpha}.$$

Thus the weighted composition operator $uC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded. □

The above theorem has the following two corollaries, the first of which is a recent result of Jie Xiao [**10**].

Corollary 2.4. *Let $\varphi : \mathbf{D} \to \mathbf{D}$ be analytic, and let α and β be positive real numbers. Then C_φ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$ if and only if*

$$\sup_{w\in\mathbf{D}} \frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^\alpha} |\varphi'(w)| < \infty.$$

In the formulation of the next corollary, we will use the notation $\mathcal{M}(\mathcal{X},\mathcal{Y})$ to denote the set of all multipliers of $\mathcal{X}$ into $\mathcal{Y}$:

$$\mathcal{M}(\mathcal{X},\mathcal{Y}) = \{u : fu \in \mathcal{Y},\ \text{for all } f \in \mathcal{X}\}.$$

We write $\mathcal{LB}^\beta$ for the space of analytic functions f on $\mathbf{D}$ for which

$$\sup_{w\in\mathbf{D}} |u'(w)|\,(1-|w|^2)^\beta \log\frac{1}{1-|w|^2} < \infty.$$

We simply write $\mathcal{LB}$ for $\mathcal{LB}^1$. The functions in $\mathcal{LB}^\beta$ will be referred to as *logarithmic* β*-Bloch functions*. The following corollary of Theorem 2.1 describes the multipliers from one Bloch-type space into another.

Corollary 2.5. *Let u be analytic on $\mathbf{D}$, and let α and β be positive real numbers.*

(i) *If $0<\alpha<1$, then*

$$\mathcal{M}(\mathcal{B}^\alpha,\mathcal{B}^\beta) = \begin{cases} \mathcal{B}^\beta & \text{if } \beta\ge\alpha,\\ \{0\} & \text{if } 0<\beta<\alpha.\end{cases}$$

(ii) *We have*

$$\mathcal{M}(\mathcal{B},\mathcal{B}^\beta) = \begin{cases} \mathcal{LB}^\beta & \text{if } \beta>1,\\ \mathcal{LB}\cap H^\infty & \text{if } \beta=1,\\ \{0\} & \text{if } 0<\beta<1.\end{cases}$$

(iii) *If $\alpha>1$, then*

$$\mathcal{M}(\mathcal{B}^\alpha,\mathcal{B}^\beta) = \begin{cases} \mathcal{B}^{\beta-\alpha+1} & \text{if } \beta>\alpha,\\ H^\infty & \text{if } \beta=\alpha,\\ \{0\} & \text{if } 0<\beta<\alpha.\end{cases}$$

Proof. By the Closed Graph Theorem an analytic function $u\in\mathcal{M}(\mathcal{B}^\alpha,\mathcal{B}^\beta)$ if and only if the multiplication operator M_u maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$.

(i) Suppose $0<\alpha<1$. Taking φ to be the identity function on $\mathbf{D}$, the conditions that multiplication operator M_u map $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$ are $u\in\mathcal{B}^\beta$ and $\sup_{w\in\mathbf{D}}|u(w)|(1-|w|^2)^{\beta-\alpha}<\infty$. If $\beta<\alpha$, then the second condition implies that there exists a constant C such that $|u(w)| \le C(1-|w|^2)^{\alpha-\beta}$ for all $w \in \mathbf{D}$. It follows that $|u(w)| \to 0$ as $|w| \to 1^-$ so that the Maximum Modulus Principle implies that $u \equiv 0$. For $\alpha \le \beta < 1$ the second condition follows from the first, since $\mathcal{B}^\beta \subset H^\infty$.

For $\beta = 1$ we have $u \in \mathcal{B}$. Using (1.3) and the inequality $\log(1/(1-|w|)) \le (1-|w|)^{\alpha-1}/(1-\alpha)$, we see that $|u(w)|(1-|w|^2)^{1-\alpha} \le C'$ for all $w \in \mathbf{D}$, so the first condition implies the second.

For $\beta > 1$ the first condition implies that $|u(w)| \le C/(1-|w|^2)^{\beta-1}$ for all $w \in \mathbf{D}$, so that $|u(w)|(1-|w|^2)^{\beta-\alpha} \le C(1-|w|^2)^{1-\alpha} \le C$ for all $w \in \mathbf{D}$. So again the second condition follows from the first.

(ii) Suppose $\alpha = 1$. Taking φ to be the identity function on $\mathbf{D}$, the conditions that M_u map $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$ are $\sup_{w\in\mathbf{D}} |u'(w)|(1-|w|^2)^\beta \log(1/(1-|w|^2)) < \infty$ and $\sup_{w\in\mathbf{D}} |u(w)|(1-|w|^2)^{\beta-1} < \infty$. If $\beta > 1$, then the first condition implies the second. If $\beta = 1$, the second condition says that $u \in H^\infty$. If $\beta < 1$, then as above the second condition implies that $u \equiv 0$.

(iii) Suppose $\alpha > 1$. Then the operator M_u maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$ if and only if $\sup_{w\in\mathbf{D}} |u'(w)|(1-|w|^2)^{\beta-\alpha+1} < \infty$ and $\sup_{w\in\mathbf{D}} |u(w)|(1-|w|^2)^{\beta-\alpha} < \infty$. If $\beta < \alpha$, then the second condition implies that $u \equiv 0$. If $\beta = \alpha$, then the conditions mean containment in $\mathcal{B}$ and H^∞, respectively, and since $H^\infty \subset \mathcal{B}$, we obtain $u \in H^\infty$. If $\beta > \alpha$ then the second condition is implied by the first, which says that $u \in \mathcal{B}^{\beta-\alpha+1}$. □

3. Compact weighted composition operators. Analogous to Theorem 2.1 the following result holds:

Theorem 3.1. *Let φ and u be analytic on $\mathbf{D}$, φ a self-map of $\mathbf{D}$. Let α and β be positive real numbers, and suppose that uC_φ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$.*

(i) *If $0 < \alpha < 1$, then uC_φ maps $\mathcal{B}^\alpha$ compactly into $\mathcal{B}^\beta$ if and only if*

$$\lim_{|\varphi(w)|\to 1^-} |u(w)| \frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^\alpha} |\varphi'(w)| = 0.$$

(ii) *The operator* uC_φ *maps* $\mathcal{B}$ *compactly into* $\mathcal{B}^\beta$ *if and only if*

$$\lim_{|\varphi(w)|\to 1^-} |u'(w)|\,(1-|w|^2)^\beta \,\log\frac{1}{1-|\varphi(w)|^2} = 0$$

and

$$\lim_{|\varphi(w)|\to 1^-} |u(w)|\,\frac{(1-|w|^2)^\beta}{1-|\varphi(w)|^2}|\,\varphi'(w)| = 0.$$

(iii) *If* $\alpha > 1$, *then* uC_φ *maps* $\mathcal{B}^\alpha$ *compactly into* $\mathcal{B}^\beta$ *if and only if*

$$\lim_{|\varphi(w)|\to 1^-} |u'(w)|\,\frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^{\alpha-1}} = 0$$

and

$$\lim_{|\varphi(w)|\to 1^-} |u(w)|\,\frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^{\alpha}}|\,\varphi'(w)| = 0.$$

Before we prove the above result, we discuss three corollaries. If we take u to be the constant function 1, we obtain Xiao's [**10**] recent characterization of compactness of composition operators between Bloch type spaces:

Corollary 3.2. *Let* $\varphi : \mathbf{D} \to \mathbf{D}$ *be analytic, and let* α *and* β *be positive real numbers, and suppose that* C_φ *maps* $\mathcal{B}^\alpha$ *boundedly into* $\mathcal{B}^\beta$. *Then* C_φ *maps* $\mathcal{B}^\alpha$ *compactly into* $\mathcal{B}^\beta$ *if and only if*

$$\lim_{|\varphi(w)|\to 1^-} \frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^{\alpha}}\,|\varphi'(w)| = 0.$$

In the formulation of the next corollary, we will use the notation $\mathcal{M}_0(\mathcal{X},\mathcal{Y})$ to denote the set of all compact multipliers of $\mathcal{X}$ into $\mathcal{Y}$:

$$\mathcal{M}_0(\mathcal{X},\mathcal{Y}) = \{u \in \mathcal{M}(\mathcal{X},\mathcal{Y}) : M_u \text{ is compact}\}.$$

We write $\mathcal{LB}_0^\beta$ for the space of analytic functions f on $\mathbf{D}$ for which

$$\lim_{|w|\to 1^-} |u'(w)|\,(1-|w|^2)^\beta \,\log\frac{1}{1-|w|^2} = 0.$$

We simply write $\mathcal{LB}_0$ for $\mathcal{LB}_0^1$. We will refer to the functions in $\mathcal{LB}_0^\alpha$ as *logarithmic little α-Bloch functions.*

Corollary 3.3. *Let $\alpha > 0$ and $\beta > 0$.*

(i) *If $0 < \alpha < 1$, then*

$$\mathcal{M}_0(\mathcal{B}^\alpha, \mathcal{B}^\beta) = \begin{cases} \mathcal{B}^\beta & \text{if } \beta > \alpha, \\ \{0\} & \text{if } 0 < \beta \le \alpha. \end{cases}$$

(ii) *We have*

$$\mathcal{M}_0(\mathcal{B}, \mathcal{B}^\beta) = \begin{cases} \mathcal{LB}_0^\beta & \text{if } \beta > 1, \\ \{0\} & \text{if } 0 < \beta \le 1. \end{cases}$$

(iii) *If $\alpha > 1$, then*

$$\mathcal{M}_0(\mathcal{B}^\alpha, \mathcal{B}^\beta) = \begin{cases} \mathcal{B}_0^{\beta-\alpha+1} & \text{if } \beta > \alpha, \\ \{0\} & \text{if } 0 < \beta \le \alpha. \end{cases}$$

Proof. Suppose $0 < \alpha < 1$. By Theorem 3.1 the operator M_u maps $\mathcal{B}^\alpha$ compactly into $\mathcal{B}^\beta$ if and only if $u \in \mathcal{B}^\beta$ and $\lim_{|w|\to 1^-} |u(w)|(1-|w|^2)^{\beta-\alpha} = 0$. If $\beta \le \alpha$ the latter condition implies that $\lim_{|w|\to 1^-} |u(w)| = 0$, and by the Maximum Modulus Principle, $u \equiv 0$. For $\beta > \alpha$ the second condition is automatically satisfied for $u \in \mathcal{B}^\beta$.

Suppose $\alpha = 1$. By Theorem 3.1, the multiplication operator M_u maps $\mathcal{B}^\alpha$ compactly into $\mathcal{B}^\beta$ if and only if $\lim_{|w|\to 1^-} |u'(w)|(1-|w|^2)^\beta \log(1/(1-|w|^2)) = 0$, and $\lim_{|w|\to 1^-} |u(w)|(1-|w|^2)^{\beta-1} = 0$. For $\beta \le 1$ the second condition implies that $\lim_{|w|\to 1^-} |u(w)| = 0$, hence $u \equiv 0$. For $\beta > 1$, the first condition implies that $\lim_{|w|\to 1^-} |u'(w)|(1-|w|^2)^\beta = 0$, which implies the second condition.

Suppose $\alpha > 1$. By Theorem 3.1, the multiplication operator M_u maps $\mathcal{B}^\alpha$ compactly into $\mathcal{B}^\beta$ if and only if $\lim_{|w|\to 1^-} |u'(w)|(1-|w|^2)^{\beta-\alpha+1} = 0$ and $\lim_{|w|\to 1^-} |u(w)|(1-|w|^2)^{\beta-\alpha} = 0$. For $\beta \le \alpha$ the second condition implies that $u \equiv 0$. For $\beta > \alpha$ the first condition is that $u \in \mathcal{B}_0^{\beta-\alpha+1}$, and such u automatically satisfies the second condition. $\square$

The following result gives a sufficient condition for compactness of weighted composition operators on Lipschitz spaces.

Corollary 3.4. *Let φ and u be analytic on $\mathbf{D}$, φ a self-map of $\mathbf{D}$. Let α and β be positive real numbers with $\alpha < 1$. If $u \in \mathcal{B}^\beta$ and*

$$\sup_{w\in\mathbf{D}} (1-|w|^2)^{\beta-1}\,|u(w)| < \infty,$$

then the operator uC_φ maps $\mathcal{B}^\alpha$ compactly into $\mathcal{B}^\beta$.

Proof. By the Schwarz-Pick lemma, $|\varphi'(w)| \le (1-|\varphi(w)|^2)/(1-|w|^2)$. Thus

$$|u(w)|\,|\varphi'(w)|\,\frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^\alpha} \le M(1-|\varphi(w)|^2)^{1-\alpha},$$

for all $w \in \mathbf{D}$ and by Theorem 2.1 (i) the weighted composition operator uC_φ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$. The above inequality shows that the condition in (i) of Theorem 3.1 is satisfied, so uC_φ maps $\mathcal{B}^\alpha$ compactly into $\mathcal{B}^\beta$. $\square$

Remark. Let $0 < \alpha < 1$. If $\beta > 1$ then, for every analytic $\varphi : \mathbf{D} \to \mathbf{D}$ and $u \in \mathcal{B}^\beta$, the operator uC_φ maps $\mathcal{B}^\alpha$ compactly into $\mathcal{B}^\beta$. Also, for every analytic $\varphi : \mathbf{D} \to \mathbf{D}$ and $u \in H^\infty$ the operator uC_φ maps $\mathcal{B}^\alpha$ compactly into $\mathcal{B}$.

Proof of Theorem 3.1. Suppose that uC_φ is compact from $\mathcal{B}^\alpha$ into $\mathcal{B}^\beta$. We will first show that the condition

$$\lim_{|\varphi(w)|\to 1^-} |u(w)|\,\frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^\alpha}\,|\varphi'(w)| = 0 \tag{3.5}$$

is necessary. Suppose that (w_n) is a sequence in $\mathbf{D}$ such that $|\varphi(w_n)| \to 1$ as $n \to \infty$ (if there are no such sequences, then the above condition holds vacuously). Consider the functions f_n defined by

$$f_n(z) = \frac{(1-|\varphi(w_n)|^2)^2}{(1-\overline{\varphi(w_n)}z)^{\alpha+1}} - \frac{1-|\varphi(w_n)|^2}{(1-\overline{\varphi(w_n)}z)^{\alpha}},$$

for $z \in \mathbf{D}$. Clearly $f_n(z) \to 0$ uniformly on compact subsets of $\mathbf{D}$. Note that

$$f_n'(z) = (\alpha+1)\overline{\varphi(w_n)}\,\frac{(1-|\varphi(w_n)|^2)^2}{(1-\overline{\varphi(w_n)}z)^{\alpha+2}} - \alpha\overline{\varphi(w_n)}\,\frac{1-|\varphi(w_n)|^2}{(1-\overline{\varphi(w_n)}z)^{\alpha+1}},$$

thus

$$\begin{aligned} |f_n'(z)| &\le \frac{(\alpha+1)(1-|\varphi(w_n)|^2)^2}{(1-|z|)^\alpha(1-|\varphi(w_n)|)^2} + \frac{\alpha(1-|\varphi(w_n)|^2)}{(1-|z|)^\alpha(1-|\varphi(w_n)|)} \\ &\le \frac{6\alpha+4}{(1-|z|)^\alpha}, \end{aligned}$$

so the $\|f_n\|_{\mathcal{B}^\alpha}$ are uniformly bounded. Note that $f_n(\varphi(w_n)) = 0$ and $f_n'(\varphi(w_n)) = \overline{\varphi(w_n)}/(1-|\varphi(w_n)|^2)^\alpha$, so that $(uC_\varphi f_n)'(w_n) = u(w_n)\overline{\varphi(w_n)}\varphi'(w_n)/(1-|\varphi(w_n)|^2)^\alpha$. Thus

$$\begin{aligned} (1-|w_n|^2)^\beta\,|u(w_n)|\,\frac{|\varphi(w_n)|\,|\varphi'(w_n)|}{(1-|\varphi(w_n)|^2)^\alpha} &= (1-|w_n|^2)^\beta\,|(uC_\varphi f_n)'(w_n)| \\ &\le \|uC_\varphi f_n\|_{\mathcal{B}^\beta}. \end{aligned}$$

Since $uC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact, it follows from the proof of the Weak Convergence Theorem in [**9**] that $\|uC_\varphi f_n\|_{\mathcal{B}^\beta} \to 0$, and therefore

$$(1-|w_n|^2)^\beta\,|u(w_n)|\,\frac{|\varphi(w_n)|}{(1-|\varphi(w_n)|^2)^\alpha}\,|\varphi'(w_n)| \to 0,$$

as $n \to \infty$. Because $|\varphi(w_n)| \to 1$ as $n \to \infty$, the necessity of condition (3.5) follows.

Next consider the functions g_n defined by

$$g_n(z) = (\alpha+1)\,\frac{1-|\varphi(w_n)|^2}{(1-\overline{\varphi(w_n)}z)^\alpha} - \alpha\,\frac{(1-|\varphi(w_n)|^2)^2}{(1-\overline{\varphi(w_n)}z)^{\alpha+1}},$$

for $z \in \mathbf{D}$. Then $g_n'(\varphi(w_n)) = 0$ so that $(uC_\varphi g_n)'(w_n) = u'(w_n)/(1 - |\varphi(w_n)|^2)^{\alpha-1}$. Since $\sup\{\|g_n\|_{\mathcal{B}^\alpha} : n = 1, 2, \dots\} < \infty$, it follows that

$$\frac{(1-|w_n|^2)^\beta\,|u'(w_n)|}{(1-|\varphi(w_n)|^2)^{\alpha-1}} = (1-|w_n|^2)^\beta\,|(uC_\varphi g_n)'(w_n)| \to 0,$$

as $n \to \infty$. Thus the first condition in (iii) is necessary so that the operator uC_φ acting from $\mathcal{B}^\alpha$ into $\mathcal{B}^\beta$ is compact. To prove that the conditions in (iii) are sufficient for compactness of the aforementioned operator, it is enough to show that if $\|f_n\|_{\mathcal{B}^\alpha} \le 1$ for all n and $f_n \to 0$ uniformly on compact subsets of $\mathbf{D}$, then $\|uC_\varphi f_n\|_{\mathcal{B}^\beta} \to 0$ as $n \to \infty$. This amounts to showing that both

$$\sup_{w\in\mathbf{D}}(1-|w|^2)^\beta\,|u(w)f_n'(\varphi(w))\varphi'(w)| \longrightarrow 0,$$

and

$$\sup_{w\in\mathbf{D}}(1-|w|^2)^\beta\,|u'(w)f_n(\varphi(w))| \longrightarrow 0.$$

If $|\varphi(w)| \le r < 1$, then

$$(1-|w|^2)^\beta\,|u(w)f_n'(\varphi(w))\varphi'(w)| \le M_r \max_{|z|\le r}|f_n'(z)|.$$

If $|\varphi(w)| > r$, then

$$(1-|w|^2)^\beta\,|u(w)f_n'(\varphi(w))\varphi'(w)| \le (1-|w|^2)^\beta\,\frac{|u(w)\varphi'(w)|}{(1-|\varphi(w)|^2)^\alpha}.$$

Thus

$$\sup_{w\in\mathbf{D}}(1-|w|^2)^\beta\,|u(w)f_n'(\varphi(w))\varphi'(w)| \le M_r \max_{|z|\le r}|f_n'(z)| + \sup_{|\varphi(w)|>r}(1-|w|^2)^\beta\,\frac{|u(w)\varphi'(w)|}{(1-|\varphi(w)|^2)^\alpha}.$$

First letting n tend to infinity and subsequently letting r increase to 1, one obtains that

$$\sup_{w\in\mathbf{D}}(1-|w|^2)^\beta\,|u(w)f_n'(\varphi(w))\varphi'(w)| \longrightarrow 0,$$

as $n \to \infty$. The other statement is proved similarly, now using the following estimates: if $|\varphi(w)| \le r < 1$, then

$$(1-|w|^2)^\beta\,|u'(w) f_n(\varphi(w))| \le M_r \max_{|z|\le r} |f_n(z)|,$$

while for $|\varphi(w)| > r$, we have

$$(1-|w|^2)^\beta\,|u'(w) f_n(\varphi(w))| \le M(1-|w|^2)^\beta\,\frac{|u'(w)|}{(1-|\varphi(w)|^2)^{\alpha-1}}.$$

Thus

$$\begin{aligned}\sup_{w\in\mathbf{D}} &(1-|w|^2)^\beta\,|u'(w) f_n(\varphi(w))| \\ &\le M_r \max_{|z|\le r}|f_n(z)| + M \sup_{|\varphi(w)|>r}(1-|w|^2)^\beta\,\frac{|u'(w)|}{(1-|\varphi(w)|^2)^{\alpha-1}},\end{aligned}$$

which implies that also

$$\sup_{w\in\mathbf{D}}(1-|w|^2)^\beta\,|u'(w) f_n(\varphi(w))| \longrightarrow 0,$$

as $n \to \infty$. This completes the proof of (iii).

To prove (ii), assuming that (w_n) is a sequence in $\mathbf{D}$ for which $|\varphi(w_n)| \to 1$, instead of the above functions g_n consider the functions h_n defined by

$$h_n(z) = \frac{3}{\mu_n}\left(\log\frac{1}{1-\overline{\varphi(w_n)}z}\right)^2 - \frac{2}{\mu_n^2}\left(\log\frac{1}{1-\overline{\varphi(w_n)}z}\right)^3,$$

for $z \in \mathbf{D}$, where $\mu_n = \log(1/(1-|\varphi(w_n)|^2))$. Then

$$h_n'(z) = \frac{6\overline{\varphi(w_n)}}{1-\overline{\varphi(w_n)}z}\left\{\frac{1}{\mu_n}\left(\log\frac{1}{1-\overline{\varphi(w_n)}z}\right) - \frac{1}{\mu_n^2}\left(\log\frac{1}{1-\overline{\varphi(w_n)}z}\right)^2\right\},$$

for $z \in \mathbf{D}$, so that $\sup\{\|h_n\|_{\mathcal{B}} : n = 1,2,\dots\} < \infty$. Clearly $h_n \to 0$ uniformly on compact subsets of $\mathbf{D}$. Note that $h_n'(\varphi(w_n)) = 0$ and $h_n(\varphi(w_n)) = \mu_n = \log(1/(1-|\varphi(w_n)|^2))$. So if $uC_\varphi : \mathcal{B} \to \mathcal{B}^\beta$ is compact, then $\|(uC_\varphi)h_n\|_{\mathcal{B}^\beta} \to 0$, and consequently,

$$(1-|w_n|^2)^\beta\,|u'(w_n)|\log\frac{1}{1-|\varphi(w_n)|^2} = (1-|w_n|^2)^\beta|(uC_\varphi h_n)'(w_n)| \to 0$$

as $n \to \infty$. Thus the first condition in (ii) is necessary, so that the operator uC_φ acting from $\mathcal{B}$ into $\mathcal{B}^\beta$ is compact. The proof of the sufficiency is similar to that of the above case and is omitted.

If $0 < \alpha < 1$ and uC_φ is a compact operator from $\mathcal{B}^\alpha$ into $\mathcal{B}^\beta$, then clearly $u = uC_\varphi 1 \in \mathcal{B}^\beta$ so the conditions in (i) are necessary. To prove that in this case they are also sufficient, we need some preliminary results.

Lemma 3.6. *Let $0 < \alpha < 1$. Every norm-bounded sequence in $\mathcal{B}^\alpha$ has a subsequence that converges uniformly on $\bar{\mathbf{D}}$.*

Proof. Let (f_n) be a sequence in $\mathcal{B}^\alpha$, and let M be a positive number for which $\|f_n\|_{\mathcal{B}^\alpha} \le M$ for $n = 1, 2, \dots$. Using (1.1) we have $|f_n(z) - f_n(w)| \le AM|z-w|^{1-\alpha}$ for all $z, w \in \bar{\mathbf{D}}$ and every $n = 1, 2, \dots$. Thus the family $\{f_n : n = 1, 2, \dots\}$ is equicontinuous. Since there also is a constant L such that $\|f\|_\infty \le L\|f\|_{\mathcal{B}^\alpha}$ for every analytic function f on $\mathbf{D}$, the family $\{f_n : n = 1, 2, \dots\}$ is bounded in $C(\bar{\mathbf{D}})$ (in fact, $\sup\{|f_n(z)| : |z| \le 1\} \le LM$ for all $n = 1, 2, \dots$). The statement of the lemma now follows from the Arzela-Ascoli Theorem. $\square$

Lemma 3.7. *Let $0 < \alpha < 1$, and let T be a bounded linear operator from $\mathcal{B}^\alpha$ into a normed linear space $\mathcal{Y}$. Then T is compact if and only if $\|Tf_n\|_{\mathcal{Y}} \to 0$ whenever (f_n) is a norm-bounded sequence in $\mathcal{B}^\alpha$ that converges to 0 uniformly on $\bar{\mathbf{D}}$.*

Proof. The necessity being obvious we will only prove the sufficiency part of the equivalence above. Suppose that T is *not* compact on $\mathcal{B}^\alpha$. Then there is a bounded sequence (g_n) in $\mathcal{B}^\alpha$ such that (Tg_n) has no convergent subsequence. By the above lemma (g_n) has a subsequence (f_n) such that $f_n \to f$ uniformly on $\bar{\mathbf{D}}$. Then $f_n'(z) \to f'(z)$ for every $z \in \mathbf{D}$, and because $|f_n'(z)| \le M/(1-|z|^2)^\alpha$ for all $n = 1, 2, \dots$, we obtain $|f'(z)| \le M/(1-|z|^2)^\alpha$. Thus $f \in \mathcal{B}^\alpha$. The sequence $(f_n - f)$ is bounded in $\mathcal{B}^\alpha$ and converges to 0 uniformly on $\bar{\mathbf{D}}$. By assumption $\|T(f_n - f)\|_{\mathcal{Y}} \to 0$ as $n \to \infty$. This implies that the subsequence (Tf_n) of (Tg_n) converges in $\mathcal{Y}$ (to Tf), a contradiction. $\square$

After these preliminaries, we are now ready to complete the proof of Theorem 3.1.

Completion of the proof of Theorem 3.1. We have already shown that the conditions in (i) are necessary for the weighted composition operator $uC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ to be compact. We will use Lemma 3.7 to show that these conditions are also sufficient. Suppose that $\|f_n\|_{\mathcal{B}^\alpha} \le 1$ for all $n = 1, 2, \ldots$, and that $f_n \to 0$ uniformly on $\bar{\mathbf{D}}$. Then

$$\sup_{w\in\mathbf{D}} (1-|w|^2)^\beta \, |u'(w)| \, |f_n(\varphi(w))| \le \|u\|_{\mathcal{B}^\beta} \sup_{|z|\le 1} |f_n(z)| \to 0. \tag{3.8}$$

If $|\varphi(w)| \le r < 1$, then

$$\begin{aligned}(1-|w|^2)^\beta \, &|u(w) f_n'(\varphi(w))\varphi'(w)| \\ &\le \max_{|z|\le r} |f_n'(z)| (1-|w|^2)^{\beta-1} \, |u(w)| (1-|w|^2) \, |\varphi'(w)| \\ &\le M \max_{|z|\le r} |f_n'(z)|.\end{aligned}$$

If $|\varphi(w)| > r$, then

$$(1-|w|^2)^\beta \, |u(w) f_n'(\varphi(w))\varphi'(w)| \le (1-|w|^2)^\beta \, \frac{|u(w)\varphi'(w)|}{(1-|\varphi(w)|^2)^\alpha}.$$

Thus

$$\begin{aligned}\sup_{w\in\mathbf{D}} &(1-|w|^2)^\beta \, |u(w) f_n'(\varphi(w))\varphi'(w)| \\ &\le M \max_{|z|\le r} |f_n'(z)| + \sup_{|\varphi(w)|>r} (1-|w|^2)^\beta \, \frac{|u(w)\varphi'(w)|}{(1-|\varphi(w)|^2)^\alpha}.\end{aligned}$$

First letting n tend to infinity, and subsequently letting r increase to 1, one obtains that

$$\sup_{w\in\mathbf{D}} (1-|w|^2)^\beta \, |u(w) f_n'(\varphi(w))\varphi'(w)| \longrightarrow 0, \tag{3.9}$$

as $n \to \infty$. Combining (3.8) and (3.9), we get

$$\sup_{w\in\mathbf{D}} (1-|w|^2)^\beta \, |u'(w)(f_n \circ \varphi)(w) + u(w)(f_n \circ \varphi)'(w)| \longrightarrow 0,$$

as $n \to \infty$, that is, $\|uC_\varphi f_n\|_{\mathcal{B}^\beta} \to 0$ as $n \to \infty$. It follows from Lemma 3.7 that the operator uC_φ maps $\mathcal{B}^\alpha$ compactly into $\mathcal{B}^\beta$. This completes the proof of Theorem 3.1. $\square$

4. Boundedness of weighted composition operators on $\mathcal{B}_0^\alpha$. Let $\mathcal{B}_0^\alpha$ denote the subspace of $\mathcal{B}^\alpha$ consisting of all analytic functions f on $\mathbf{D}$ for which $(1-|z|^2)^\alpha f'(z) \to 0$ as $|z| \to 1^-$. This space will be referred to as the *little* $\mathcal{B}^\alpha$ space. If $f \in \mathcal{B}_0^\alpha$ for $\alpha > 1$, then estimates similar to those in Section 1 can be used to prove that $(1-|z|^2)^{\alpha-1}|f(z)| \to 0$ as $|z| \to 1^-$. For $f \in \mathcal{B}_0$, we have $\log(1/(1-|z|^2))|f(z)| \to 0$ as $|z| \to 1^-$.

The following theorem describes boundedness of weighted composition operators from one little Bloch-type space into another.

Theorem 4.1. *Let φ and u be analytic on $\mathbf{D}$, φ a self-map of $\mathbf{D}$. Let α and β be positive real numbers. Under these assumptions uC_φ maps $\mathcal{B}_0^\alpha$ boundedly into $\mathcal{B}_0^\beta$ if and only if uC_φ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$, $u \in \mathcal{B}_0^\beta$ and*

$$\lim_{|z|\to 1^-}(1-|z|^2)^\beta u(z)\varphi'(z) = 0. \tag{4.2}$$

Proof. Suppose that uC_φ maps $\mathcal{B}_0^\alpha$ into $\mathcal{B}_0^\beta$. Then $u = uC_\varphi 1 \in \mathcal{B}_0^\beta$. Also $u\varphi = uC_\varphi z \in \mathcal{B}_0^\beta$, thus $(1-|z|^2)^\beta(u(z)\varphi'(z) + u'(z)\varphi(z)) \to 0$ as $|z| \to 1^-$. Since φ is bounded and $u \in \mathcal{B}_0^\beta$, we have $(1-|z|^2)^\beta u'(z)\varphi(z) \to 0$ as $|z| \to 1^-$, and thus (4.2) holds. For fixed $w \in \mathbf{D}$ the functions defined in (2.2) and (2.3) are in fact in $\mathcal{B}_0^\alpha$, so the proof of Theorem 2.1 shows that if uC_φ maps $\mathcal{B}_0^\alpha$ boundedly into $\mathcal{B}_0^\beta$, then the conditions in (iii) of Theorem 2.1 hold, and thus uC_φ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$.

Conversely, suppose u and φ are such that uC_φ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$, $u \in \mathcal{B}_0^\beta$ and (4.2) holds. We will show that uC_φ maps $\mathcal{B}_0^\alpha$ into $\mathcal{B}_0^\beta$. By Theorem 2.1 there is a finite M such that $|u'(w)|(1-|w|^2)^\beta/(1-|\varphi(w)|^2)^{\alpha-1} \le M$ and $|u(w)||\varphi'(w)|(1-|w|^2)^\beta/(1-|\varphi(w)|^2)^\alpha \le M$, for all $w \in \mathbf{D}$. For $f \in \mathcal{B}_0^\alpha$, we will show that $uC_\varphi f \in \mathcal{B}_0^\beta$. Since $f \in \mathcal{B}_0^\alpha$ both $(1-|z|^2)^\alpha|f'(z)| \to 0$ and $(1-|z|^2)^{\alpha-1}|f(z)| \to 0$ as $|z| \to 1^-$, so given $\varepsilon > 0$ there is a $0 < \delta < 1$ such that $(1-|z|^2)^\alpha |f'(z)| < \varepsilon/(2M)$ and $(1-|z|^2)^{\alpha-1}|f(z)| < \varepsilon/(2M)$, for all z with $\delta < |z| < 1$. If $|\varphi(w)| > \delta$, it follows that

$$\begin{aligned}&(1-|w|^2)^\beta\,|(uC_\varphi f)'(w)| \\ &\quad\le (1-|w|^2)^\beta\,|u'(w)|\,|f(\varphi(w))| + (1-|w|^2)^\beta\,|u(w)|\,|f'(\varphi(w))|\,|\varphi'(w)| \\ &\quad\le M(1-|\varphi(w)|^2)^{\alpha-1}\,|f(\varphi(w))| + M(1-|\varphi(w)|^2)^\alpha\,|f'(\varphi(w))| < \varepsilon.\end{aligned}$$

Choose a constant L such that $|f(z)| \le L$ and $|f'(z)| \le L$ for all $|z| \le \delta$. If $|\varphi(w)| \le \delta$, then

$$\begin{aligned}&(1-|w|^2)^\beta\,|(uC_\varphi f)'(w)| \\ &\quad\le (1-|w|^2)^\beta\,|u'(w)|\,|f(\varphi(w))| + (1-|w|^2)^\beta\,|u(w)|\,|f'(\varphi(w))|\,|\varphi'(w)| \\ &\quad\le L(1-|w|^2)^\beta\,|u'(w)| + L(1-|w|^2)^\beta\,|u(w)|\,|\varphi'(w)|,\end{aligned}$$

and using (4.2) and the fact that $u \in \mathcal{B}_0$, we conclude that $(1-|w|^2)^\beta|(uC_\varphi f)'(w)| \to 0$ as $|w| \to 1^-$. Hence $uC_\varphi f \in \mathcal{B}_0^\beta$, for all $f \in \mathcal{B}_0^\alpha$. It follows from the Closed Graph Theorem that $uC_\varphi : \mathcal{B}_0^\alpha \to \mathcal{B}_0^\beta$ is bounded. This completes the proof in case $\alpha > 1$. The cases $\alpha = 1$ and $0 < \alpha < 1$ have similar proofs. □

5. Compactness of weighted composition operators on $\mathcal{B}_0^\alpha$. We have the following result for the compactness of weighted composition operators on $\mathcal{B}_0^\alpha$.

Theorem 5.1. *Let φ and u be analytic on* $\mathbf{D}$*, φ a self-map of* $\mathbf{D}$*. Let α and β be positive numbers.*

(i) *If $0 < \alpha < 1$, then uC_φ is a compact operator from $\mathcal{B}_0^\alpha$ into $\mathcal{B}_0^\beta$ if and only if $u \in \mathcal{B}_0^\beta$ and*

$$\lim_{|w|\to 1^-} |u(w)|\,\frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^\alpha}\,|\varphi'(w)| = 0.$$

(ii) *The operator uC_φ is a compact operator from $\mathcal{B}_0$ into $\mathcal{B}_0^\beta$ if and only if*

$$\lim_{|w|\to 1^-} |u'(w)|\,(1-|w|^2)^\beta \log\frac{1}{1-|\varphi(w)|^2} = 0$$

and

$$\lim_{|w|\to 1^-} |u(w)| \, \frac{(1-|w|^2)^\beta}{1-|\varphi(w)|^2} \, |\varphi'(w)| = 0.$$

(iii) *If $\alpha > 1$, then uC_φ is a compact operator from $\mathcal{B}_0^\alpha$ into $\mathcal{B}_0^\beta$ if and only if*

$$\lim_{|w|\to 1^-} |u'(w)| \, \frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^{\alpha-1}} = 0$$

and

$$\lim_{|w|\to 1^-} |u(w)| \, \frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^\alpha} \, |\varphi'(w)| = 0.$$

Our proof of the above theorem requires the following lemma which describes the compact subsets of $\mathcal{B}_0^\alpha$.

Lemma 5.2. *Let $K \subset \mathcal{B}_0^\alpha$. Then K is compact if and only if K is closed, bounded, and satisfies*

$$\lim_{|z|\to 1^-} \sup_{f\in K} (1-|z|^2)^\alpha \, |f'(z)| = 0. \tag{5.3}$$

Proof. Suppose K is compact. If $\varepsilon > 0$, then the balls centered at the elements of K with radii $\varepsilon/2$ cover K, so by compactness there exist $f_1, \dots, f_n \in K$ such that for every $f \in K$ we have $\|f - f_j\|_{\mathcal{B}^\alpha} < \varepsilon/2$ for some $1 \le j \le n$, and consequently

$$(1-|z|^2)^\alpha \, |f'(z)| \le (1-|z|^2)^\alpha \, |f_j'(z)| + \varepsilon/2,$$

for all $z \in \mathbf{D}$. For each j, there exists an $r_j \in (0,1)$ such that $(1-|z|^2)^\alpha |f_j'(z)| < \varepsilon/2$ whenever $r_j < |z| < 1$. Setting $r = \max\{r_1, \dots, r_n\}$ we have $(1-|z|^2)^\alpha |f'(z)| \le \varepsilon$ whenever $r < |z| < 1$ and $f \in K$. This proves that (5.3) holds.

Now suppose that $K \subset \mathcal{B}_0^\alpha$ is closed, bounded and satisfies (5.3). Then K is a normal family. If (f_n) is a sequence in K, by passing to a subsequence (which we do not relabel) we may assume that $f_n \to f$ uniformly on compact subsets of $\mathbf{D}$. We are done once we show that $f_n \to f$ in $\mathcal{B}_0^\alpha$. Let $\varepsilon > 0$ be given. By (5.3) there exists an $r \in (0,1)$ such that $(1-|z|^2)^\alpha |g'(z)| \le \varepsilon/2$, for all $r < |z| < 1$ and all $g \in K$. Since $f_n' \to f'$ uniformly on compact subsets of $\mathbf{D}$, it follows that $f_n' \to f'$ pointwise on $\mathbf{D}$, and thus also $(1-|z|^2)^\alpha |f'(z)| \le \varepsilon/2$, for all $r < |z| < 1$. Hence $(1-|z|^2)^\alpha |f_n'(z) - f'(z)| \le \varepsilon$, for all $r < |z| < 1$. Since $f_n' \to f'$ uniformly on $r\bar{\mathbf{D}}$, there exists an N such that $|f_n'(z) - f'(z)| \le \varepsilon$ for all $|z| \le r$ and $n \ge N$. It follows that $(1-|z|^2)^\alpha |f_n'(z) - f'(z)| \le \varepsilon$ for all $z \in \mathbf{D}$ and all $n \ge N$. Thus $f_n \to f$ in $\mathcal{B}^\alpha$. Since K is closed, it follows that $f \in K$. This proves that the set K is compact. □

Proof of Theorem 5.1. By Lemma 5.2 the set $\{uC_\varphi f : f \in \mathcal{B}_0^\alpha, \|f\|_{\mathcal{B}^\alpha} \le 1\}$ has compact closure in $\mathcal{B}_0^\beta$ if and only if

$$(5.4) \quad \lim_{|w|\to 1^-} \sup\{(1-|w|^2)^\beta \, |(uC_\varphi f)'(w)| : f \in \mathcal{B}_0^\alpha, \|f\|_{\mathcal{B}^\alpha} \le M\} = 0,$$

for some $M > 0$. If (5.4) is satisfied, then it follows by the proof of Theorem 2.1 and the fact that the functions given in (2.2) are in $\mathcal{B}_0^\alpha$ and have norms bounded independently of w, that

$$\lim_{|w|\to 1^-} |u(w)| \, \frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^\alpha} \, |\varphi'(w)| = 0.$$

Similarly, it is shown that the other conditions in Theorem 5.1 are necessary. It remains to show that the conditions are also sufficient. We will first prove this for $\alpha > 1$. Suppose that $f \in \mathcal{B}_0^\alpha$ is such that $\|f\|_{\mathcal{B}^\alpha} \le 1$ and that u and ϕ satisfy the conditions in (iii). Then we have

$$\begin{aligned} |(1-|z|^2)^\beta (uC_\varphi f)'(z)| &\le (1-|z|^2)^\beta \, |u'(z)| \, |f(\varphi(z))| \\ &\quad + (1-|z|^2)^\beta \, |u(z)| \, |f'(\varphi(z))| \, |\varphi'(z)| \\ &\le (1-|z|^2)^\beta \, |u'(z)| \, \frac{C}{(1-|\varphi(z)|^2)^{\alpha-1}} \\ &\quad + (1-|z|^2)^\beta \, |u(z)| \, \frac{1}{(1-|\varphi(z)|^2)^\alpha} |\varphi'(z)|, \end{aligned}$$

thus

$$\begin{aligned}\sup\{|(1-|z|^2)^\beta(uC_\varphi f)'(z)| : f\in\mathcal{B}_0^\alpha,\ \|f\|_{\mathcal{B}^\alpha}\le 1\}\\ \le (1-|z|^2)^\beta\,|u'(z)|\,\frac{C}{(1-|\varphi(z)|^2)^{\alpha-1}}\\ + (1-|z|^2)^\beta\,|u(z)|\,\frac{1}{(1-|\varphi(z)|^2)^{\alpha}}\,|\varphi'(z)|,\end{aligned}$$

and it follows that

$$\lim_{|z|\to 1^-}\sup\{|(1-|z|^2)^\beta(uC_\varphi f)'(z)| : f\in\mathcal{B}_0^\alpha,\ \|f\|_{\mathcal{B}^\alpha}\le 1\}=0,$$

so that uC_φ is compact on $\mathcal{B}_0^\alpha$. The proof is complete in the case $\alpha>1$.

The case $\alpha=1$ has a similar proof. Suppose $0<\alpha<1$, $u\in\mathcal{B}_0^\alpha$, and

$$\lim_{|w|\to 1^-}|u(w)|\,\frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^\alpha}\,|\varphi'(w)|<\infty.$$

If $f\in\mathcal{B}_0^\alpha$ is such that $\|f\|_{\mathcal{B}^\alpha}\le 1$, then

$$\begin{aligned}(1-|z|^2)^\beta\,|(uC_\varphi f)'(z)| &\le (1-|z|^2)^\beta\,|u'(z)f(\varphi(z))|\\ &\quad + (1-|z|^2)^\beta\,|u(z)f'(\varphi(z))\varphi'(z)|.\end{aligned}$$

Note that

$$\begin{aligned}&(1-|z|^2)^\beta\,|u(z)f'(\varphi(z))\varphi'(z)|\\ &\qquad=\frac{(1-|z|^2)^\beta|\varphi'(z)|}{(1-|\varphi(z)|^2)^\alpha}\,|u(z)|\,(1-|\varphi(z)|^2)^\alpha\,|f'(\varphi(z))|\\ &\qquad\le\frac{(1-|z|^2)^\beta|\varphi'(z)|}{(1-|\varphi(z)|^2)^\alpha}\,|u(z)|.\end{aligned}$$

Also,

$$\begin{aligned}(1-|z|^2)^\beta\,|u'(z)f(\varphi(z))| &\le (1-|z|^2)^\beta\,|u'(z)|\,\|f\|_\infty\\ &\le (1-|z|^2)^\beta\,|u'(z)|.\end{aligned}$$

It follows that

$$\lim_{|z|\to 1^-}\sup\{(1-|z|^2)^\beta\,|(uC_\varphi f)'(z)| : f\in\mathcal{B}_0^\alpha,\ \|f\|_{\mathcal{B}^\alpha}\le 1\}=0,$$

and by Lemma 5.2, $uC_\varphi : \mathcal{B}_0^\alpha \to \mathcal{B}_0^\beta$ is compact. □

6. Examples. In this final section we give some examples of functions u and φ for which the weighted composition operator uC_φ between the various spaces considered is bounded, compact or noncompact.

We first give some examples of bounded weighted composition operators uC_φ.

Example 6.1. Let α and β be positive real numbers such that $\alpha \le \beta$. First we consider the case that $\alpha > 1$. Then the composition operator C_φ is bounded on $\mathcal{B}^\alpha$. If $u \in \mathcal{B}^{\beta-\alpha+1}$, then the multiplication operator M_u maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$ and thus $uC_\varphi = M_u C_\varphi$ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$ too.

We will give examples of analytic self mappings φ on the unit disk and analytic functions u on the unit disk not belonging to $\mathcal{B}^{\beta-\alpha+1}$ such that C_φ is bounded on $\mathcal{B}^\alpha$, but not from $\mathcal{B}^\alpha$ into $\mathcal{B}^\gamma$, for any $0 < \gamma < \alpha$, and such that uC_φ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$.

Let $\varphi(z) = (1-z)/2$ and let $u(z) = 1/(1-z)^\tau$ for $z \in \mathbf{D}$ where $\beta - \alpha < \tau < \beta - 1$. It is easy to check that $u \notin \mathcal{B}^{\beta-\alpha+1}$. Using $1 - |\varphi(z)|^2 \ge (1-|z|^2)/4$, we see that

$$|u'(z)| \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}} \le \frac{\tau}{4^{\alpha-1}} \frac{(1-|z|^2)^{\beta-\alpha+1}}{|1-z|^{\tau+1}} \le \frac{\tau}{4^{\alpha-1}},$$

for all $z \in \mathbf{D}$ with $|z-1| \ge 1$. If $z \in \mathbf{D}$ is such that $|z-1| < 1$, then $|\varphi(z)| < 1/2$, so that $1 - |\varphi(z)|^2 > 3/4$, and thus

$$|u'(z)| \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}} \le \tau \left(\frac{4}{3}\right)^{\alpha-1} \frac{(1-|z|^2)^\beta}{|1-z|^{\tau+1}} \le \tau 2^\beta \left(\frac{4}{3}\right)^{\alpha-1}.$$

Hence

$$\sup_{z\in\mathbf{D}} |u'(z)| \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}} < \infty.$$

Similarly it is shown that

$$\sup_{z\in\mathbf{D}} |u(z)| \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha}} |\varphi'(z)| < \infty.$$

By Theorem 2.1 the operator uC_φ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$ (in fact compactly if $\alpha < \beta$).

For $\alpha = 1$ a similar example was given in [**7**].

Next suppose $\alpha \le \beta < 1$. Take any number τ such that $\max\{\beta, 1-\beta\} < \tau < 1$, and consider $\varphi(z) = ((1-z)/2)^{1-\tau}$ and $u(z) = (1-z)^\tau$. Then it can be checked that $u \in \mathcal{B}^\beta$,

$$\sup_{z\in\mathbf{D}} |\varphi'(z)| \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} = \infty,$$

and

$$\sup_{z\in\mathbf{D}} |u(z)|\,|\varphi'(z)| \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} < \infty.$$

Thus the weighted composition operator uC_φ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$ (in fact compactly if $\alpha < \beta$), while the composition operator C_φ does not map $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$.

Our second example shows that the condition in Theorem 3.1 that uC_φ is bounded is necessary.

Example 6.2. Let α and β be positive real numbers with $\beta \ge \alpha$. In this example we will show that there exist analytic functions $\varphi : \mathbf{D} \to \mathbf{D}$ and u on $\mathbf{D}$ such that the conditions (i), (ii) or (iii) in Theorem 3.1 are satisfied while uC_φ does not map $\mathcal{B}^\alpha$ compactly into $\mathcal{B}^\beta$. Let $\varphi(z) = (1-z)/2$ for $z \in \mathbf{D}$.

First suppose that $0 < \alpha < 1$. Consider the function $u(z) = (1+z)/(1-z)^{1+\beta}$ for $z \in \mathbf{D}$. Since

$$|u(w)| \frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^\alpha} |\varphi'(w)| \le 4^\alpha |u(w)| (1-|w|^2)^{\beta-\alpha}$$

and $|u(w)| \to 0$ as $|\varphi(w)| \to 1^-$, condition (i) in Theorem 3.1 is satisfied. For $w = x$, where $0 < x < 1$, we have

$$|u(w)| \frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^\alpha} |\varphi'(w)| \ge \frac{1}{2} \frac{(1+x)(1-x)^\beta}{(1-x)^{1+\beta}} = \frac{1}{2}\frac{1+x}{1-x},$$

so that

$$\sup_{w\in\mathbf{D}} |u(w)| \frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^\alpha} |\varphi'(w)| = \infty.$$

By Theorem 2.1 the weighted composition operator uC_φ does not map $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$, so uC_φ cannot be compact.

In the case that $\alpha \geq 1$, consider the function $u(z) = (1+z)/(1-z)^{1+\alpha+\beta}$ for $z \in \mathbf{D}$. That the second condition of (ii) or (iii) in Theorem 3.1 is satisfied is shown as before. Using

$$|u'(w)| \frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^{\alpha-1}} \leq 4^{\alpha-1} |u'(w)| (1-|w|^2)^{\beta-\alpha+1},$$

and $|u'(w)| \to 1/2^{1+\alpha+\beta}$ as $|\varphi(w)| \to 1^-$, we see that also the first condition of (iii) in Theorem 3.1 is satisfied. If $w = x$ where $0 < x < 1$, then

$$\begin{aligned} |u'(w)| \frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^{\alpha-1}} &\geq \frac{((\alpha+\beta)x + 2 + \alpha + \beta)(1-x^2)^\beta}{(1-x)^{2+\alpha+\beta}} \\ &\geq \frac{2+\alpha+\beta}{(1-x)^{2+\alpha}}. \end{aligned}$$

This is clearly unbounded, so

$$\sup_{w \in \mathbf{D}} (1-|w|^2)^\beta \, |u'(w)|/(1-|\varphi(w)|^2)^{\alpha-1} = \infty.$$

If $\alpha = 1$ it can be shown that $\sup_{w\in\mathbf{D}}(1-|w|^2)^\beta |u'(w)| \log(1/(1-|\varphi(w)|^2)) = \infty$. By Theorem 2.1, the weighted composition operator uC_φ does not map $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$, so uC_φ cannot be compact.

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