ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 33, Number 1, Spring 2003

THE HARNACK ESTIMATE FOR THE MODIFIED RICCI FLOW ON COMPLETE $\mathbf{R}^2$

SHU-CHENG CHANG, SHI-KUO HONG AND CHIN-TUNG WU

ABSTRACT. In this paper we prove the Harnack estimate for the evolved curvature R of the modified Ricci flow on complete $\mathbf{R}^2$.

1. Introduction. The Ricci flow is a second-order parabolic equation which deforms metric $g(t)$ in the direction of minus the Ricci curvature tensor $\operatorname{Ric}(g)$. That is, due to Hamilton ([**3**]), a family of Riemannian metrics $g(t)$, $t \in [0,T]$, is called a solution to the Ricci flow if

$$\frac{\partial}{\partial t} g(t) = -2 \operatorname{Ric}(g).$$

In particular, for a surface (Σ, g_0), the Ricci flow is given by

$$\begin{cases} \dfrac{\partial}{\partial t} g = -R \cdot g, \\ g(0) = g_0. \end{cases} \tag{1.1}$$

In [**4**], Hamilton studied the Ricci flow (1.1) on a compact Riemann surface S. He proved, among other results, that if the Riemann surface S is diffeomorphic to 2-sphere and the initial metric g_0 has positive curvature, that the solution of the normalized Ricci flow:

$$\begin{cases} \dfrac{\partial}{\partial t} g = (r-R) \cdot g, \\ g(0) = g_0, \end{cases}$$

converges to the limiting metric of positive constant curvature. Here r is the average value of the scalar curvature R. On the other hand, in [**7**], L.-F. Wu considered the Ricci flow (1.1) on a complete noncompact $\mathbf{R}^2$. She proved, if the covariant derivative of u_0 and the curvature of

1991 *Mathematics Subject Classification.* 53C21.
Key words and phrases. Modified Ricci flow, Harnack inequality, aperture.
Research of the first author supported in part by NSC.

the initial metric $g_0 = e^{u_0} g_{\mathbf{R}^2}$ are bounded, that the Ricci flow (1.1) has modified subsequence convergence at time infinity to a limiting metric. Furthermore, in the case when the curvature is positive at time zero, the limiting metric is a flat metric if the aperture $A(g_0) > 0$ (see [**7**] for details). Among their works, one of the important ingredients is the Harnack inequality for the evolved curvature R ([**5**]).

In this paper, we will consider $(\mathbf{R}^2, e^u g_{\mathbf{R}^2})$ as a complete noncompact surface. Instead of the Ricci flow, we will consider the so-called modified Ricci flow on $(\mathbf{R}^2, e^u g_{\mathbf{R}^2})$ as follows:

$$(1.2) \qquad \begin{cases} \dfrac{\partial}{\partial t} g = -\dfrac{R}{1+R} \cdot g, \\ g(t) = e^{u(t)} g_{\mathbf{R}^2}, \\ g(0) = g_0 = e^{u_0} g_{\mathbf{R}^2}, \end{cases}$$

where R is the curvature of the metric g and $g_{\mathbf{R}^2}$ is the standard Euclidean metric on $\mathbf{R}^2$. We will assume that the solutions of (1.2) exist for $t \in [0, T]$, for some $T > 0$. Our goal is to do the Harnack estimate for R under the modified Ricci flow (1.2).

Remark 1.1. (i) The modified Ricci flow (1.2) is proposed by S.-T. Yau. It leads to understanding the uniformization theorem on complete noncompact surfaces.

(ii) For the short time existence of solution of the Ricci flow (1.1) on $(\mathbf{R}^2, e^u g_{\mathbf{R}^2})$, one needs the curvature of the initial metric g_0 to be bounded. However, in the case of the modified Ricci flow (1.2) on $(\mathbf{R}^2, e^u g_{\mathbf{R}^2})$, we do not need this assumption. We expect that the solution of the modified Ricci flow (1.2) exists for all time when the initial metric has positive curvature ([**2**]), and are interested in the asymptotic behavior of the solution of (1.2). Furthermore, we conjecture that the solution converges to a flat metric when the initial metric has positive aperture. We believe that the Harnack inequality will play an important role in understanding this problem as in [**4**] and [**7**].

The first step is to obtain a geometric quantity, usually called the Harnack quantity. Following the method of Hamilton in [**5**], we can derive the Harnack quantity:

$$Z(g, X) = \frac{\partial}{\partial t} R + \langle \nabla R, X \rangle + \frac{1}{4} R(1+R)^2 |X|^2 + \frac{R}{t},$$

in which X is a vector field. Then we have the following differential Harnack inequality for the modified Ricci flow:

Theorem 1.1. *Let $g(t)$ be a complete solution of the modified Ricci flow (1.2) with curvature positive at $t = 0$ and bounded on $(0, T]$. Then for any vector field X, we have*

$$Z(g, X) = \frac{\partial}{\partial t} R + \langle \nabla R, X \rangle + \frac{1}{4} R(1 + R)^2 |X|^2 + \frac{R}{t} \geq 0.$$

Corollary 1.2. *Let $g(t)$ be a complete solution of the modified Ricci flow (1.2) with curvature positive at $t = 0$ and bounded on $(0, T]$. Then for any two points $x_1, x_2 \in \mathbf{R}^2$ and any two times with $0 < t_1 < t_2$, we have*

$$R(x_2, t_2) \geq \frac{t_1}{t_2} \exp\left(-\frac{1}{4}\Omega\right) R(x_1, t_1),$$

where

$$\Omega = \inf_{\gamma} \int_{t_1}^{t_2} R^2 (1 + R)^2 \left| \frac{d\gamma}{dt} \right|^2_{g(t)} dt$$

and the infimum is taken over all paths γ whose graphs $(\gamma(t), t)$ joining (x_1, t_1) to (x_2, t_2).

Furthermore, based on Theorem 1.1, we have the following matrix Harnack inequality:

Theorem 1.3. *Let $g(t)$ be a complete solution of the modified Ricci flow (1.2) with curvature positive at $t = 0$ and bounded on $(0, T]$, which also bounds on the covariant derivatives of R on $(0, T]$. Then for any vector field X, we have*

$$\begin{aligned} Z_{ij}(g, X) = {} & \frac{\nabla_i \nabla_j R}{(R+1)^2} - \frac{2}{(R+1)^3} \nabla_i R \cdot \nabla_j R + \frac{1}{2}(\nabla_i R \cdot X_j + \nabla_j R \cdot X_i) \\ & + \frac{1}{4} R(R+1)^2 X_i X_j + \frac{1}{2} R \left(\frac{R}{1+R} + \frac{1}{t} \right) g_{ij} \geq 0. \end{aligned}$$

In Section 2, we describe some basic properties of the modified Ricci flow (1.2). In Section 3, we follow the method in [**5**] to obtain the so-called Harnack quantity. In Sections 4 and 5, we prove the trace Harnack and the matrix Harnack by using the methods in [**1**] and [**5**].

2. Basic properties of the modified Ricci flow. In this section, we will derive some properties of the modified Ricci flow.

Lemma 2.1. *Under the modified Ricci flow* (1.2)*, the evolution equations for the scalar curvature R and f are*

$$\frac{\partial}{\partial t} R = \frac{\Delta R}{(1+R)^2} - \frac{2}{(1+R)^3} |\nabla R|^2 + \frac{R^2}{1+R},$$

and

$$\frac{\partial}{\partial t} f = (1-f)^2 \Delta f + (1-f) f^2,$$

where $f = R/(1+R)$.

Proof. For $g = e^u g_{\mathbf{R}^2}$, the scalar curvature of g is given by

$$R_g = -\Delta_g u = -e^{-u} \Delta_{g_{\mathbf{R}^2}} u. \tag{2.1}$$

Then, from (1.2), we have

$$-fg = \frac{\partial}{\partial t} g = \frac{\partial}{\partial t} (e^u g_{\mathbf{R}^2}) = g \frac{\partial}{\partial t} u.$$

This implies that

$$\frac{\partial}{\partial t} u = -f.$$

From (2.1), we compute

$$\begin{aligned} \frac{\partial}{\partial t} R &= -e^{-u} \left(\frac{\partial}{\partial t} u \right) (-\Delta_{g_{\mathbf{R}^2}} u) - e^{-u} \Delta_{g_{\mathbf{R}^2}} \left(\frac{\partial}{\partial t} u \right) \\ &= f \cdot R + \Delta_g f. \end{aligned}$$

Thus,

$$\frac{\partial}{\partial t} R = \frac{\Delta R}{(1+R)^2} - \frac{2}{(1+R)^3} |\nabla R|^2 + \frac{R^2}{1+R}. \tag{2.2}$$

Moreover, since $(\partial/\partial t)f = (1/(1+R)^2)(\partial/\partial t)R$ and $1/(1+R)^2 = (1-f)^2$, we have

$$\frac{\partial}{\partial t} f = (1-f)^2 \Delta f + (1-f) f^2. \qquad \square \tag{2.3}$$

Now we show that the positivity of the scalar curvature R is preserved under the modified Ricci flow (1.2). By using the local technique, which is exactly the same as the one used in the compact case, we only need to prove the following lemma:

Lemma 2.2. *If $R \geq 0$ at $t = 0$, under the modified Ricci flow* (1.2)*, we have $R \geq 0$ for all $t \in [0, T]$.*

Proof. It is equivalent to show $f = R/(1+R) \geq 0$ for all $t \in [0, T]$. Let $B(0, r)$ be the open ball of radius r centered at the origin. For any integer $n > 0$, let $D_n = B(0, n)$. Then we get a family of open sets, $\{D_n\}$, such that

$$D_n \subset D_{n+1}, \quad \overline{D_n} \text{ is a compact subset of } \mathbf{R}^2, \text{ and } \mathbf{R}^2 = \bigcup_{n=1}^{\infty} D_n.$$

Choose a cut-off function $\chi_n(x) \in C^\infty(\mathbf{R}^2)$, such that

$$\begin{aligned} \chi_n(x) = 1 &\quad \text{if } x \in D_n, \\ 0 \leq \chi_n(x) \leq 1 &\quad \text{if } x \in D_{n+1} \setminus D_n \end{aligned}$$

and

$$\chi_n(x) = 0 \quad \text{if } x \in \mathbf{R}^2 \setminus D_{n+1}.$$

Now we consider the local modified Ricci flow,

$$\frac{\partial}{\partial t} g = -\chi_n^2 f \cdot g,$$

then

$$\chi_n^2(x) \cdot f(x, 0) \geq 0 \quad \text{if } x \in D_{n+1}$$

and

$$\chi_n^2(x) \cdot f(x,t) = 0 \quad \text{if } x \in \mathbf{R}^2 \setminus D_{n+1}, \quad t \in [0,T],$$

and the evolution equation of $\chi_n^2 f$ is given by

$$\begin{aligned} \frac{\partial}{\partial t} \quad \text{if } (\chi_n^2 f) &= (1-f)^2 \chi_n^2 \Delta(\chi_n^2 f) + (1-f) f^2 \chi_n^4 \\ &\geq (1-f)^2 \chi_n^2 \Delta(\chi_n^2 f). \end{aligned}$$

Define $h = e^{-t}(\chi_n^2 f)$, then

$$\begin{aligned} h(x,0) &\geq 0 \quad \text{if } x \in D_{n+1}, \\ h(x,t) &= 0 \quad \text{if } x \in \mathbf{R}^2 \setminus D_{n+1}, \quad t \in [0,T], \end{aligned}$$

and

$$\Delta h = e^{-t} \Delta(\chi_n^2 f),$$

and

$$\begin{aligned} \frac{\partial}{\partial t} h &= e^{-t} \frac{\partial}{\partial t} (\chi_n^2 f) - e^{-t}(\chi_n^2 f) \geq e^{-t}(1-f)^2 \chi_n^2 \Delta(\chi_n^2 f) - h \\ &= (1-f)^2 \chi_n^2 \Delta h - h. \end{aligned}$$

Let (x_0, t_0) be a point where h assumes its minimum. If $\chi_n^2 f$ is negative somewhere, then $h(x_0,t_0) < 0$. We will show that this leads to a contradiction. We must have $x_0 \notin \mathbf{R}^2 \setminus D_{n+1}$. In local coordinates, h is smooth at (x_0,t_0) and

$$\frac{\partial h}{\partial x^i}(x_0,t_0) = 0, \qquad \frac{\partial^2 h}{\partial x^i \partial x^j}(x_0,t_0) \geq 0 \text{ as a matrix},$$

and

$$\frac{\partial h}{\partial t}(x_0,t_0) \leq 0.$$

Moreover, at (x_0,t_0),

$$\frac{\partial h}{\partial t} \geq (1-f)^2 \chi_n^2 \left[g^{ij} \left(\frac{\partial^2}{\partial x^i \partial x^j} h \right) - \Gamma_{ij}^k \frac{\partial}{\partial x^k} h \right] - h \geq 0.$$

This implies $h(x_0, t_0) \geq 0$, which is a contradiction.

Hence $\chi_n^2 f \geq 0$ for all $t \in [0, T]$. Let n approach to infinity, then we have $f \geq 0$ for all $t \in [0, T]$. □

3. The Harnack quantity. In this section, we apply Hamilton's general method to obtain the Harnack quantity for the modified Ricci flow (1.2).

We first need to know what the gradient soliton equation is supposed to be for our flow (1.2). A solution $g(t)$ to the modified Ricci flow is a soliton if a one-parameter family of diffeomorphisms $\varphi(t)$ exists such that

$$g(t) = \varphi(t)^* g(0).$$

Differentiating this equation with respect to time implies

$$\frac{\partial}{\partial t} g = L_{-X} g,$$

or equivalently

$$\frac{R}{1+R} g = L_X g, \tag{3.1}$$

where $\{-X(t)\}$ is the one-parameter family of vector fields generated by $\varphi(t)$. In local coordinates, (3.1) becomes

$$\frac{R}{1+R} g_{ij} = \nabla_i X_j + \nabla_j X_i. \tag{3.2}$$

If $X = \nabla f$ is the gradient of some time independent function f, then (3.2) implies

$$\frac{R}{1+R} g_{ij} = 2\nabla_i X_j.$$

In this case, we say that $g(t)$ is a gradient soliton.

Now we consider the equation for an expanding gradient soliton:

$$\left(\frac{R}{1+R} + \frac{1}{t}\right) g_{ij} = \nabla_i X_j, \tag{3.3}$$

where X_j is a vector field on $\mathbf{R}^2$. Taking the divergence of (3.3), we have

$$\nabla_j\left(\frac{R}{1+R}\right) + R_{jk}X^k = 0. \tag{3.4}$$

In $\mathbf{R}^2$, we have $R_{jk} = Rg_{jk}/2$, and substitute this into (3.4). We get

$$\frac{\nabla_j R}{(1+R)^2} + \frac{1}{2}\, RX_j = 0. \tag{3.5}$$

Apply the covariant derivative ∇_i to (3.5), and from (3.3) this implies

$$\begin{aligned}\frac{\nabla_i\nabla_j R}{(R+1)^2} - \frac{2}{(R+1)^3}\,\nabla_i R\cdot\nabla_j R \\ + \frac{1}{2}\,\nabla_i R\cdot X_j + \frac{1}{2}\, R\left(\frac{R}{1+R} + \frac{1}{t}\right)g_{ij} = 0.\end{aligned} \tag{3.6}$$

We modify the expression (3.6) by adding the product of (3.5) with $(1+R)^2X_i/2$, and define for any metric g and vector field X_i:

$$\begin{aligned}Z_{ij}(g,X) = {}& \frac{\nabla_i\nabla_j R}{(R+1)^2} - \frac{2}{(R+1)^3}\,\nabla_i R\cdot\nabla_j R \\ & + \frac{1}{2}\,(\nabla_i R\cdot X_j + \nabla_j R\cdot X_i) + \frac{1}{4}\, R(R+1)^2X_iX_j \\ & + \frac{1}{2}\, R\left(\frac{R}{1+R} + \frac{1}{t}\right)g_{ij}.\end{aligned} \tag{3.7}$$

Trace (3.7) and define

$$\begin{aligned}Z(g,X) = {}& \frac{\Delta R}{(1+R)^2} - \frac{2}{(1+R)^3}\,|\nabla R|^2 + \langle\nabla R, X\rangle \\ & + \frac{1}{4}\, R(1+R)^2|X|^2 + R\left(\frac{R}{1+R} + \frac{1}{t}\right).\end{aligned} \tag{3.8}$$

From (2.2), it follows that

$$Z(g,X) = \frac{\partial}{\partial t}\, R + \langle\nabla R, X\rangle + \frac{1}{4}\, R(1+R)^2|X|^2 + \frac{R}{t}. \tag{3.9}$$

This is the Harnack quantity for the modified Ricci flow (1.2). We remark that (3.9) is equivalent to

$$Z(g,X) = \frac{\partial}{\partial t} f + \langle \nabla f, X\rangle + \frac{1}{4}\frac{f}{1-f}|X|^2 + (1-f)f\frac{1}{t}, \tag{3.10}$$

where $f = R/(1+R)$.

By (3.5) and (3.7), if g is a gradient soliton flowing along X, then

$$Z(g,X) = 0. \tag{3.11}$$

Moreover, in this case, using (3.5) we find

$$\frac{\partial Z}{\partial X}(g,X) = 0. \tag{3.12}$$

The vanishing of (3.11) and (3.12) for gradient solitons are necessary for any Harnack quantity $Z(g,X)$ which could possibly be nonnegative for any solution g to the modified Ricci flow and any vector field X_i, and zero on solitons.

4. The trace Harnack estimate. In this section, following the methods in [**5**] and [**1**], we will prove Theorem 1.1. It should be noticed that Theorem 1.1 is equivalent to the following theorem:

Theorem 4.1. *Let $g(t)$ be a complete solution of the modified Ricci flow* (1.2) *with curvature positive at $t=0$ and bounded on* $(0,T]$. *Then, for any vector field X, we have*

$$Z(g,X) = \frac{\partial}{\partial t} f + \langle \nabla f, X\rangle + \frac{1}{4}\frac{f}{1-f}|X|^2 + (1-f)f\frac{1}{t} \geq 0. \tag{4.1}$$

The proof of Theorem 4.1 relies on the computation of the evolution equation for $Z(g,X)$, which we compute term by term. For convenience, we define $\square = (\partial/\partial t) - (1-f)^2\Delta$.

Lemma 4.2. *Under the modified Ricci flow* (1.2), *we have the following evolution equations*:

(4.2)

$$\square\left(\frac{\partial}{\partial t} f\right) = f(1-f)^2\Delta f - [2(1-f)\Delta f - (2-3f)f]\frac{\partial}{\partial t} f,$$

(4.3)
$$\Box\langle\nabla f, X\rangle = f\langle\nabla f, X\rangle - \left[2(1-f)\Delta f - \frac{1}{2}\,f(3-5f)\right]\langle\nabla f, X\rangle + \langle\nabla f, \Box X\rangle - 2(1-f)^2\langle\nabla\nabla f, \nabla X\rangle,$$

(4.4)
$$\Box\left(\frac{1}{4}\,\frac{f}{1-f}|X|^2\right) = \frac{1}{4}\,\frac{f}{1-f}\,[f|X|^2 + 2\langle X, \Box X\rangle] + \frac{1}{4}\,\frac{|X|^2}{1-f}\,[f^2 - 2|\nabla f|^2] - \frac{1}{2}\,f(1-f)|\nabla X|^2 - \frac{1}{2}\,\langle\nabla f, \nabla|X|^2\rangle,$$

(4.5)
$$\Box\left[(1-f)f\,\frac{1}{t}\right] = \frac{1}{t}\,(1-f)\left[(1-2f)f^2 + 2(1-f)|\nabla f|^2 - \frac{1}{t}\,f\right].$$

Proof. Equation (4.2) follows directly from (2.3), while (4.3) follows from the equation

$$\frac{\partial}{\partial t}\,\langle\nabla f, X\rangle = f\langle\nabla f, X\rangle + \left\langle\nabla\left(\frac{\partial}{\partial t}\,f\right), X\right\rangle + \left\langle\nabla f, \frac{\partial}{\partial t}\,X\right\rangle,$$

and the product formula

$$\Delta\langle\nabla f, X\rangle = \langle D\nabla f, X\rangle + \langle\nabla f, \Delta X\rangle + 2\langle\nabla\nabla f, \nabla X\rangle.$$

Similarly, (4.4) and (4.5) follow from a direct computation using (2.3) and the above product formula. $\Box$

By combining each term of Lemma 4.2, we find the evolution equation for Z is given by

(4.6)
$$\begin{aligned}\Box Z = {} & [(1-f)(3f-2\Delta f)]Z - f^3(1-f) - 2(1-f)^2\langle\nabla\nabla f, \nabla X\rangle \\ & + \left\langle\Box X, \nabla f + \frac{1}{2}\,\frac{f}{1-f}X\right\rangle - \frac{1}{2}\,f(1-f)[\,|\nabla X|^2 + \langle\nabla f, X\rangle] \\ & - \frac{1}{2}\,\langle\nabla f, \nabla|X|^2\rangle + \frac{1}{4}\left[2f\Delta f + \frac{f^2(3f-1)}{1-f} - \frac{2}{1-f}|\nabla f|^2\right]|X|^2 \\ & + \frac{1}{t}(1-f)\left[2(1-f)\Delta f - f^2(2+f) + 2(1-f)|\nabla f|^2 - \frac{1}{t}\,f\right].\end{aligned}$$

To simplify the equation above, we prescribe at a point the covariant derivative of X and the heat operator $\Box$ of X. This is always possible by extending X suitably in space and time. At a point (x,t) where the vector field X is extended to satisfy

$$\nabla_i X_j = 2\Big(f + \frac{1}{t}\Big) g_{ij}, \tag{4.7}$$

the evolution equation becomes

(4.8)

$$\begin{aligned}\Box Z = &-\Big[\frac{2}{1-f} Z + f(1+f) + \frac{4}{t}(1-f)\Big] Z + \Big\langle \Box X, \nabla f + \frac{1}{2}\frac{f}{1-f} X\Big\rangle + f^3(1-f) \\ &+ \langle \nabla f, X\rangle \Big[\frac{1}{2} f(3+f) + \frac{2}{t}(1-f) - \frac{1}{2}\frac{f}{(1-f)^2}|X|^2 + \frac{2}{1-f} Z\Big] \\ &+ \frac{|X|^2}{4}\Big[\frac{f^2(3+f)}{1-f} - \frac{1}{2}\frac{f^2}{(1-f)^3}|X|^2 - \frac{2}{1-f}|\nabla f|^2 + \frac{4}{t} f + \frac{4f}{(1-f)^2} Z\Big] \\ &+ \frac{1}{t}(1-f)\Big[f^2(2-f) + 2(1-f)|\nabla f|^2 + \frac{1}{t} f(1-2f)\Big].\end{aligned}$$

If at the same point (x,t), we also extend X in time such that

$$\begin{aligned}\Box X = &-\Big[\frac{1}{2} f(3+f) + \frac{2}{t}(1-f) - \frac{1}{4}\frac{f}{(1-f)^2}|X|^2 + \frac{2}{1-f} Z\Big] X \\ &+ \frac{|X|^2}{2}\frac{\nabla f}{1-f},\end{aligned} \tag{4.9}$$

then the equation for Z simplifies to

$$\begin{aligned}\Box Z = &-\Big[\frac{2}{1-f} Z + f(1+f) + \frac{4}{t}(1-f)\Big] Z + f^3(1-f) \\ &+ \frac{1}{t}(1-f)\Big[f^2(2-f) + 2(1-f)|\nabla f|^2 + \frac{1}{t} f(1-2f)\Big].\end{aligned}$$

The idea of the proof of Theorem 4.1 is to perturb the expression Z slightly to $\tilde{Z}$ so as to make $\tilde{Z}$ very positive if $t \to 0$, or if the point $x \to \infty$, and wherever $\tilde{Z}$ first acquires a zero it is strictly increasing. It then follows that $\tilde{Z}$ never could make it to zero after all. Since we can take $\tilde{Z}$ as close to Z as we like on compact sets in space-time avoiding $t = 0$, we get $Z \geq 0$ as desired.

We also need the following lemma:

Lemma 4.3 [**5**, Lemma 5.2]. *Given any constant $L > 0$, any $\eta > 0$ and any compact set K in space-time we can find a function $\varphi = \varphi(x, t)$ depending on both space and time such that*

(1) *$\varphi \leq \eta$ on the set K and $\varphi \geq \alpha$ for some $\alpha > 0$,*

(2) *$\varphi(x, t) \to \infty$ as $x \to \infty$,*

(3) *$\Box\varphi > L\varphi$,*

(4) *$|\nabla\varphi| \leq C\varphi$ for some constant C.*

Proof of Theorem 4.1. Given $\varepsilon > 0$, let $\tilde{Z}(g, X) = Z(g, X) + \varepsilon e^{kt}\varphi$, where φ as in Lemma 4.3. First, we observe that f is always positive, and for all X, $Z(g, X) \geq Y(g)$ where

$$Y(g) = (1 - f)^2 \Delta f + (1 - f)f\left(f + \frac{1}{t}\right) - \frac{1 - f}{f}\,|\nabla f|^2.$$

Also, there exists a constant $\delta > 0$ such that $Y(g) > 0$ for $t < \delta$. Hence $\tilde{Z}(g, X) > 0$ for $t < \delta$. Furthermore, we can choose a compact set K such that $\tilde{Z}(g, X)$ is strictly positive outside the set K for $t > 0$.

We show that $\tilde{Z}(g, X)$ is strictly positive everywhere for $t > 0$. Suppose $\tilde{Z}(g, X) \leq 0$ at some space-time point for some X. Then there exists a first time $\tau > 0$, a point $\xi \in \mathbf{R}^2$ and a tangent vector X at ξ such that at (ξ, τ),

$$\tilde{Z}(g, X) = 0.$$

If X is extended in space and time satisfies (4.7) and (4.9), then the evolution equation for $\tilde{Z}$ is

$$\begin{aligned}\frac{\partial}{\partial t}\,\tilde{Z} = (1-f)^2\Delta\tilde{Z} - \left[\frac{2}{1-f}\,\tilde{Z} - \frac{4}{1-f}\,\varepsilon e^{kt}\varphi + f(1+f) + \frac{4}{t}(1-f)\right]\tilde{Z}\\ + \varepsilon e^{kt}\left[\left(k - \frac{2}{1-f}\,\varepsilon e^{kt}\varphi + f(1+f) + \frac{4}{t}(1-f)\right)\varphi + \Box\varphi\right]\\ + \frac{1}{t}\,(1-f)\left[f^2(2-f) + 2(1-f)|\nabla f|^2 + \frac{1}{t}\,f(1-2f)\right] + f^3(1-f).\end{aligned}$$

Since R is bounded on $(0,T]$, we can choose constants L, k and η such that $k\varepsilon\alpha > (2/\delta^2)$ and $L > 2/(1-f)\varepsilon e^{kT}\eta$ in Lemma 4.3. On the other hand, since (ξ,τ) is contained in the compact set K, we have, at (ξ,τ),

$$\Box\varphi > L\varphi > \frac{2}{1-f}\,\varepsilon e^{kt}\varphi^2,$$

and

$$\varepsilon e^{kt}\,k\varphi > \frac{2}{t^2}\,f^2(1-f).$$

Hence the maximum principle implies that, at (ξ,τ),

$$\begin{aligned} 0 \geq\ & \varepsilon e^{kt}\varphi\left[f(1+f) + \frac{4}{t}\,(1-f)\right] + f^3(1-f) \\ & + \frac{1}{t}\,(1-f)\left[f^2(2-f) + 2(1-f)|\nabla f|^2 + \frac{1}{t}\,f\right], \end{aligned}$$

which is a contradiction. Therefore, for all $\varepsilon > 0$, we have $Z + \varepsilon e^{kt}\varphi > 0$.

This implies that

$$Z \geq 0,$$

which completes the proof of Theorem 4.1. □

Taking $X = -2f/(1-f)\nabla f$ in Theorem 4.1 implies that

$$Y(g) = (1-f)^2\Delta f + (1-f)f\left(f + \frac{1}{t}\right) - \frac{1-f}{f}\,|\nabla f|^2 \geq 0, \tag{4.10}$$

which is equivalent to

$$Y(g) = \frac{\partial}{\partial t}\,R + \frac{R}{t} - \frac{1}{R(1+R)^2}\,|\nabla R|^2 \geq 0. \tag{4.11}$$

Integrating (4.11) over paths in M yields the following Harnack inequality:

Theorem 4.4. *Let $g(t)$ be a complete solution of the modified Ricci flow* (1.2) *with curvature positive at* $t = 0$ *and bounded on* $(0,T]$. *Then,* *for any two points $x_1, x_2 \in \mathbf{R}^2$ and any two times with $0 < t_1 < t_2$, we have*

$$R(x_2, t_2) \geq \frac{t_1}{t_2} \exp\left(-\frac{1}{4}\Omega\right) R(x_1, t_1),$$

where

$$\Omega = \inf_{\gamma} \int_{t_1}^{t_2} R^2(1+R)^2 \left|\frac{d\gamma}{dt}\right|^2_{g(t)} dt$$

and the infimum is taken over all paths γ whose graphs $(\gamma(t), t)$ joins (x_1, t_1) to (x_2, t_2).

Proof. Let $x_1, x_2 \in \mathbf{R}^2$ be any two points and t_1, t_2 be two times. If $\gamma : [t_1, t_2] \to \mathbf{R}^2$ is a C^1-path joining x_1 and x_2, by the fundamental theorem of calculus, we have

$$\begin{aligned}
\log \frac{R(x_2, t_2)}{R(x_1, t_1)} &= \int_{t_1}^{t_2} \frac{d}{dt} \log R(\gamma(t), t)\, dt \\
&= \int_{t_1}^{t_2} \frac{1}{R(\gamma(t), t)} \left[\frac{\partial R}{\partial t}(\gamma(t), t) + \left\langle \nabla R(\gamma(t), t), \frac{d\gamma}{dt}(t)\right\rangle\right] dt \\
&\geq \int_{t_1}^{t_2} \left[-\frac{1}{t} - \frac{1}{4} R^2(1+R)^2 \left|\frac{d\gamma}{dt}\right|^2_{g(t)}\right] dt.
\end{aligned}$$

Define $\Omega = \Omega(x_1, t_1, x_2, t_2) = \inf_{\gamma} \int_{t_1}^{t_2} R^2(1+R)^2 |d\gamma/dt|^2_{g(t)}\, dt$ where the infimum is taken over all C^1-paths $\gamma : [t_1, t_2] \to \mathbf{R}^2$ joining x_1 and x_2. Exponentiating the equation implies the theorem. □

5. The matrix Harnack estimate. In this section, based on the trace Harnack estimate in Theorem 4.1, we prove the matrix Harnack estimate for the modified Ricci flow (1.2) as seen in Theorem 1.3. We first assume that we have a complete solution to (1.2) with bounded curvature and bounds on the covariant derivatives of f on $(0, T]$. We recall from Section 3 that we have the following matrix Harnack quantity:

$$
\begin{aligned}
Z_{ij}(g,X) = {} & \frac{\nabla_i\nabla_j R}{(R+1)^2} - \frac{2}{(R+1)^3}\,\nabla_i R\cdot\nabla_j R \\
& + \frac{1}{2}\,(\nabla_i R\cdot X_j + \nabla_j R\cdot X_i) + \frac{1}{4}\,R(R+1)^2 X_i X_j \\
& + \frac{1}{2}\,R\left(\frac{R}{1+R} + \frac{1}{t}\right) g_{ij},
\end{aligned}
$$

where X_i is a vector field on $\mathbf{R}^2$, which is equivalent to

$$
\begin{aligned}
Z_{ij}(g,X) = {} & (1-f)^2\nabla_i\nabla_j f + (\nabla_i f\cdot X_j + \nabla_j f\cdot X_i) \\
& + \frac{f}{1-f}\,X_i X_j + \frac{1}{2}\,f(1-f)\left(f + \frac{1}{t}\right) g_{ij},
\end{aligned}
\tag{5.1}
$$

where $f = R/(R+1)$ (and replacing X by $2X$). Then Theorem 1.3 is equivalent to the following matrix Harnack estimate for the modified Ricci flow:

Theorem 5.1. *Let $g(t)$ be a complete solution of the modified Ricci flow* (1.2) *with curvature positive at $t = 0$ and bounded on* $(0, T]$*, which also bounds on the covariant derivatives of f on* $(0, T]$*. Then, for any vector field X, we have*

$$
\begin{aligned}
Z_{ij}(g,X) = {} & (1-f)^2\nabla_i\nabla_j f + (\nabla_i f\cdot X_j + \nabla_j f\cdot X_i) \\
& + \frac{f}{1-f}\,X_i X_j + \frac{1}{2}\,f(1-f)\left(f + \frac{1}{t}\right) g_{ij} \geq 0.
\end{aligned}
$$

The proof of Theorem 5.1 follows essentially from calculations plus the perturbation method of Hamilton for strong maximum principle as seen in [**5**].

First we compute the evolution equation for $Z_{ij}(g, X)$ term by term.

Lemma 5.2. *Under the modified Ricci flow* (1.2), *we have the following evolution equations*:

(5.2)
$$\begin{aligned}\square[(1-f)^2\nabla_i\nabla_j f] = &-2(1-f)^3[\nabla_i f\cdot\nabla_j\Delta f+\nabla_j f\cdot\nabla_i\Delta f]\\&-(1-f)^2[2(1-f)\Delta f+3f^2+2|\nabla f|^2]\nabla_i\nabla_j f\\&+f(1-f)^3\Delta f g_{ij}+4(1-f)^3\nabla_k f\cdot\nabla_k\nabla_i\nabla_j f\\&+2(1-f)^2[\Delta f+(1-3f)]\nabla_i f\nabla_j f,\end{aligned}$$

(5.3)
$$\begin{aligned}\square[\nabla_i f\cdot X_j+\nabla_j f\cdot X_i] = &\left[-2(1-f)\Delta f+\frac{1}{2}f(3-5f)\right]\\&\times[\nabla_i f\cdot X_j+\nabla_j f\cdot X_i]\\&+(\nabla_i f\cdot\square X_j+\nabla_j f\cdot\square X_i)\\&-2(1-f)^2[\nabla_k\nabla_i f\cdot\nabla_k X_j+\nabla_k\nabla_j f\nabla_k X_i],\end{aligned}$$

(5.4)
$$\begin{aligned}\square\left[\frac{f}{1-f}X_iX_j\right] = &\frac{1}{1-f}(f^2-2|\nabla f|^2)X_iX_j\\&-2f(1-f)\nabla_k X_i\cdot\nabla_k X_j\\&+\frac{f}{1-f}(\square X_i\cdot X_j+\square X_j\cdot X_i)\\&-2\nabla_k f\cdot(X_j\nabla_k X_i+X_i\nabla_k X_j),\end{aligned}$$

(5.5)
$$\begin{aligned}\square\left[\frac{1}{2}f(1-f)\left(f+\frac{1}{t}\right)g_{ij}\right] = &(1-f)\Bigg\{f^3\left[\frac{1}{2}(1-3f)-\frac{1}{t}\right]\\&+(1-f)|\nabla f|^2\left[\frac{1}{t}-(1-3f)\right]-\frac{f}{2t^2}\Bigg\}g_{ij}.\end{aligned}$$

Proof. In order to show the evolution equation (5.2) by using

$$\frac{\partial}{\partial t}\Gamma_{ij}^k=\frac{1}{2}g^{kl}(-\nabla_i f\cdot g_{jl}-\nabla_j f\cdot g_{il}+\nabla_l f\cdot g_{ij}),$$

we can compute the evolution equation for $\nabla_i\nabla_j f$:

$$\begin{aligned}\frac{\partial}{\partial t}\nabla_i\nabla_j f &= \nabla_i\nabla_j\left(\frac{\partial}{\partial t} f\right) - \left(\frac{\partial}{\partial t}\Gamma_{ij}^k\right)\nabla_k f\\ &= \nabla_i\nabla_j[(1-f)^2\Delta f + f^2(1-f)] + \left(\nabla_i f\cdot\nabla_j f - \frac{1}{2}|\nabla f|^2 g_{ij}\right)\\ &= (1-f)^2\nabla_i\nabla_j\Delta f - 2(1-f)[\nabla_i f\cdot\nabla_j\Delta f + \nabla_j f\cdot\nabla_i\Delta f]\\ &\quad + [-2(1-f)\Delta f + (2-3f)f]\nabla_i\nabla_j f\\ &\quad + 2[\Delta f + (1-3f)]\nabla_i f\nabla_j f + \left(\nabla_i f\cdot\nabla_j f - \frac{1}{2}|\nabla f|^2 g_{ij}\right).\end{aligned}$$

Now, by using the equation

$$\begin{aligned}\nabla_i\nabla_j\Delta f &= \Delta\nabla_i\nabla_j f - \frac{2f}{1-f}\nabla_i\nabla_j f\\ &\quad + \frac{1}{(1-f)^2}\left\{\left[\frac{1}{2}|\nabla f|^2 + f(1-f)\right]g_{ij} - \nabla_i f\nabla_j f\right\},\end{aligned}$$

we have

$$\begin{aligned}\frac{\partial}{\partial t}\nabla_i\nabla_j f &= (1-f)^2\Delta\nabla_i\nabla_j f - 2(1-f)[\nabla_i f\cdot\nabla_j\Delta f + \nabla_j f\cdot\nabla_i\Delta f]\\ &\quad - [2(1-f)\Delta f + f^2]\nabla_i\nabla_j f + f(1-f)\Delta f g_{ij}\\ &\quad + 2[\Delta f + (1-3f)]\nabla_i f\nabla_j f.\end{aligned}$$

Thus,

$$\begin{aligned}\Box[(1-f)^2\nabla_i\nabla_j f] &= -2(1-f)\frac{\partial f}{\partial t}\nabla_i\nabla_j f + (1-f)^2\frac{\partial}{\partial t}\nabla_i\nabla_j f\\ &\quad -(1-f)^2\Delta[(1-f)^2\nabla_i\nabla_j f]\\ &= -2(1-f)^3[\nabla_i f\cdot\nabla_j\Delta f + \nabla_j f\cdot\nabla_i\Delta f]\\ &\quad -(1-f)^2[2(1-f)\Delta f + 3f^2 + 2|\nabla f|^2]\nabla_i\nabla_j f\\ &\quad + f(1-f)^3\Delta f g_{ij} + 4(1-f)^3\nabla_k f\cdot\nabla_k\nabla_i\nabla_j f\\ &\quad + 2(1-f)^2[\Delta f + (1-3f)]\nabla_i f\nabla_j f.\end{aligned}$$

Equations (5.3), (5.4) and (5.5) follow from a direct computation and using the evolution equation (2.3) of f. $\Box$

By combining each term in the above Lemma 5.2, we find that the evolution equation for Z_{ij} is given by

(5.6)

$$\begin{aligned}\Box Z_{ij} =& -2(1-f)^3\,[\nabla_i f\cdot\nabla_j\Delta f+\nabla_j f\cdot\nabla_i\Delta f]\\ &-(1-f)^2\,[2(1-f)\Delta f+3f^2+2|\nabla f|^2]\nabla_i\nabla_j f\\ &+4(1-f)^3\,\nabla_k f\cdot\nabla_k\nabla_i\nabla_j f+2(1-f)^2[\Delta f+(1-3f)]\nabla_i f\nabla_j f\\ &+(1-f)\left\{f(1-f)^2\Delta f+(1-3f)\left[\frac{1}{2}\,f^3-(1-f)|\nabla f|^2\right]\right\}g_{ij}\\ &+\frac{1}{t}\,(1-f)\left[(1-f)|\nabla f|^2-f^3-\frac{1}{2t}\,f\right]g_{ij}\\ &+\left[\Box X_i\cdot\left(\nabla_j f+\frac{f}{1-f}X_j\right)+\Box X_j\cdot\left(\nabla_i f+\frac{f}{1-f}X_i\right)\right]\\ &+\left[-2(1-f)\Delta f+\frac{1}{2}\,f(3-5f)\right]\left[\nabla_i f\cdot X_j+\nabla_j f\cdot X_i\right]\\ &+\frac{1}{1-f}\,(f^2-2|\nabla f|^2)X_iX_j-2f(1-f)\nabla_k X_i\cdot\nabla_k X_j\\ &-2\nabla_k f\cdot(X_j\nabla_k X_i+X_i\nabla_k X_j).\end{aligned}$$

In order to eliminate the higher three-derivative terms, we substitute the following equations into (5.6):

$$\begin{aligned}&4(1-f)^3\nabla_k f\cdot\nabla_k\nabla_i\nabla_j f\\ &\quad=4(1-f)\nabla_k Z_{ij}\cdot\nabla_k f+8(1-f)^2|\nabla f|^2\nabla_i\nabla_j f-\frac{4}{1-f}\,|\nabla f|^2X_iX_j\\ &\qquad-4(1-f)\nabla_k f\cdot(\nabla_k\nabla_i fX_j+\nabla_k\nabla_j fX_i+\nabla_i f\nabla_k X_j+\nabla_j f\nabla_k X_i)\\ &\qquad-4f\nabla_k f\cdot(X_j\nabla_k X_i+X_i\nabla_k X_j)-2f(1-f)(2-3f)|\nabla f|^2g_{ij}\\ &\qquad-\frac{2}{t}\,(1-f)(1-2f)|\nabla f|^2g_{ij},\end{aligned}$$

and

$$\begin{aligned}&-2(1-f)^3[\nabla_i f\cdot\nabla_j\Delta f+\nabla_j f\cdot\nabla_i\Delta f]\\ &\qquad=-2(1-f)[\nabla_i f\cdot\nabla_j Y+\nabla_j f\cdot\nabla_i Y]\\ &\qquad\quad-\frac{2}{f}\,(1-f)^2[\nabla_i f\cdot\nabla_j|\nabla f|^2+\nabla_j f\cdot\nabla_i|\nabla f|^2]\\ &\qquad\quad+4\left\{-2Y+(1-f)\left[f(2-f)+\frac{1}{t}+\frac{1-2f}{f^2}|\nabla f|^2\right]\right\}\nabla_i f\nabla_j f,\end{aligned}$$

where, from (4.10),

$$Y(g) = (1-f)^2\Delta f + f(1-f)\left(f+\frac{1}{t}\right) - \frac{1-f}{f}|\nabla f|^2 \geq 0,$$

we yield

$$\begin{aligned}
\square Z_{ij} = {} & 4(1-f)\nabla_k Z_{ij}\cdot\nabla_k f - 2(1-f)[\nabla_i f\cdot\nabla_j Y + \nabla_j f\cdot\nabla_i Y] \\
& - (1-f)^2[2(1-f)\Delta f + 3f^2 - 6|\nabla f|^2]\nabla_i\nabla_j f \\
& + 2\{(1-f)^2[\Delta f + (1-3f)] - 4Y\}\nabla_i f\nabla_j f \\
& + 4(1-f)\left[f(2-f) + \frac{1}{t} + \frac{1-2f}{f^2}|\nabla f|^2\right]\nabla_i f\nabla_j f \\
& + (1-f)[f(1-f)^2\Delta f + \frac{1}{2}(1-3f)f^3 - (1-3f^2)|\nabla f|^2]g_{ij} \\
& + \frac{1}{t}(1-f)\left[(3f-1)|\nabla f|^2 - f^3 - \frac{f}{2t}\right]g_{ij} \\
& \times\left[\square X_i\cdot\left(\nabla_j f + \frac{f}{1-f}\,X_j\right) + \square X_j\cdot\left(\nabla_i f + \frac{f}{1-f}\,X_i\right)\right] \\
& + \left[-2(1-f)\Delta f + \frac{1}{2}\,f(3-5f)\right]\left[\nabla_i f\cdot X_j + \nabla_j f\cdot X_i\right] \\
& - \frac{2}{f}(1-f)^2\left[\left(\nabla_i f + \frac{f}{1-f}\,X_i\right)\cdot\nabla_j|\nabla f|^2\right. \\
& \qquad\qquad \left. + \left(\nabla_j f + \frac{f}{1-f}\,X_j\right)\cdot\nabla_i|\nabla f|^2\right] \\
& + \frac{1}{1-f}\,(f^2 - 6|\nabla f|^2)X_iX_j - 2f(1-f)\nabla_k X_i\cdot\nabla_k X_j \\
& - 2(1+2f)\nabla_k f\cdot(X_i\nabla_k X_j + X_j\nabla_k X_i) \\
& - 4(1-f)\nabla_k f\cdot(\nabla_i f\nabla_k X_j + \nabla_j f\nabla_k X_i) \\
& - 2(1-f)^2(\nabla_k\nabla_i f\nabla_k X_j + \nabla_k\nabla_j f\nabla_k X_i).
\end{aligned} \tag{5.7}$$

To simplify the above equation, we prescribe at a point the covariant derivative of X_l and the heat operator $\square$ of X_l. This is always possible by extending X_l suitably in space and time. At a point (x,t) where the vector field X_l is extended to satisfy

$$\nabla_k X_l = \frac{1}{2}\left(f+\frac{1}{t}\right)g_{kl}, \tag{5.8}$$

the evolution equation (5.7) becomes

(5.9)

$$
\begin{aligned}
\square Z_{ij} &= 4(1-f)\nabla_k Z_{ij}\cdot\nabla_k f - 2(1-f)[\nabla_i f\cdot\nabla_j Y + \nabla_j f\cdot\nabla_i Y] \\
&\quad +(1-f)^2\left[6|\nabla f|^2 - 2(1-f)\Delta f - 3f^2 - 2\left(f+\frac{1}{t}\right)\right]\nabla_i\nabla_j f \\
&\quad +2\left\{(1-f)^2[\Delta f + (1-3f)] - 2(1-f)\left(f+\frac{1}{t}\right) - 4Y\right\}\nabla_i f\nabla_j f \\
&\quad +4(1-f)\left[f(2-f)+\frac{1}{t}+\frac{1-2f}{f^2}|\nabla f|^2\right]\nabla_i f\nabla_j f \\
&\quad +(1-f)[f(1-f)\Delta f + f^3 - |\nabla f|^2]g_{ij} \\
&\quad +\frac{1}{t}(1-f)\left[f(1-f)\Delta f + \frac{1}{2}f^2(f+2) - |\nabla f|^2\right]g_{ij} \\
&\quad +\left[\square X_i\cdot\left(\nabla_j f + \frac{f}{1-f}X_j\right) + \square X_j\cdot\left(\nabla_i f + \frac{f}{1-f}X_i\right)\right] \\
&\quad +\left[\frac{1}{2}f(f+3)+(1-2f)\left(f+\frac{1}{t}\right)-6|\nabla f|^2\right]\left[\nabla_i f\cdot X_j + \nabla_j f\cdot X_i\right] \\
&\quad -\frac{2}{f}(1-f)^2\left[\left(\nabla_i f + \frac{f}{1-f}X_i\right)\cdot\nabla_j|\nabla f|^2\right. \\
&\qquad\qquad \left. +\left(\nabla_j f + \frac{f}{1-f}X_j\right)\cdot\nabla_i|\nabla f|^2\right] \\
&\quad +\left\{2f\Delta f + \frac{1}{1-f}\left[f^2(3f+1) + 2f\left(f+\frac{1}{t}\right) - 6(f+1)|\nabla f|^2\right]\right\}X_i X_j.
\end{aligned}
$$

Now substituting

$$
\begin{aligned}
&(1-f)^2\,\nabla_i\nabla_j f \\
&\quad = Z_{ij} - \left[(\nabla_i f\cdot X_j + \nabla_j f\cdot X_i) + \frac{f}{1-f}X_i X_j + \frac{1}{2}f(1-f)\left(f+\frac{1}{t}\right)g_{ij}\right],
\end{aligned}
$$

we get

$$
\begin{aligned}
\square Z_{ij} = &-\left[\frac{2}{1-f}\,Y + 2(1-f)\left(f+\frac{1}{t}\right) + 3f^2 + \frac{2}{f}\,(1-f)|\nabla f|^2\right] Z_{ij} \\
&+ 4(1-f)\nabla_k Z_{ij}\cdot\nabla_k f - 2(1-f)[\nabla_i f\cdot\nabla_j Y + \nabla_j f\cdot\nabla_i Y] \\
&+ \left[fY\left(1+\frac{1}{t}\right) - \frac{1}{t}\,f^2(1-f)\left(\frac{f}{2}+1\right)\right] g_{ij} \\
&+ 2\Bigg[(1-f)(1-2f) - 3Y - \frac{f}{t}\,(1-f) \\
&\qquad + \frac{1}{f^2}\,(1-f)(2-3f)|\nabla f|^2\Bigg]\nabla_i f\nabla_j f \\
&+ \left[\square X_i\cdot\left(\nabla_j f + \frac{f}{1-f}\,X_j\right) + \square X_j\cdot\left(\nabla_i f + \frac{f}{1-f}\,X_i\right)\right] \\
&+ \left[\frac{1}{2}\,f(5-3f) + \frac{1}{t}\,(1-2f) - 6|\nabla f|^2\right]\left[\nabla_i f\cdot X_j + \nabla_j f\cdot X_i\right] \\
&- \frac{2}{f}(1-f)^2\Bigg[\left(\nabla_i f + \frac{f}{1-f}\,X_i\right)\cdot\nabla_j|\nabla f|^2 \\
&\qquad\qquad + \left(\nabla_j f + \frac{f}{1-f}\,X_j\right)\cdot\nabla_i|\nabla f|^2\Bigg] \\
&+ \frac{1}{1-f}\left[\frac{2f}{1-f}Y + f^2(f+3) + \frac{2}{t}f(1-f) - (6f+4)|\nabla f|^2\right] X_i X_j.
\end{aligned}
$$

If at the same point (x,t), we also extend X_l in time such that

$$
\begin{aligned}
\square X_l = &-\left[\frac{1}{1-f}\,Y + \frac{1}{t}(1-f) + \frac{1}{2}f(f+3) - \frac{3f+2}{f}\,|\nabla f|^2\right] X_l \\
&+ \left[\frac{1}{f}\,Y + (1-f)\left(\frac{1}{t} - (1-2f) - \frac{1}{f^2}(2-3f)|\nabla f|^2\right)\right]\nabla_l f \qquad (5.11) \\
&+ \frac{2}{f}\,(1-f)^2\,\nabla_l|\nabla f|^2,
\end{aligned}
$$

then the equation (5.10) for Z_{ij} simplifies to

$$\begin{aligned}\Box Z_{ij} = &-\left[\frac{2}{1-f}\,Y + 2(1-f)\left(f+\frac{1}{t}\right) + 3f^2 + \frac{2}{f}\,(1-f)|\nabla f|^2\right] Z_{ij}\\ &+ 4(1-f)\nabla_k Z_{ij}\cdot\nabla_k f - 2(1-f)[\nabla_i f\cdot\nabla_j Y + \nabla_j f\cdot\nabla_i Y]\\ &+ \left[Y\left(1+\frac{1}{t}\right) - \frac{1}{t}\,f(1-f)\left(\frac{f}{2}+1\right)\right] f\, g_{ij}\\ &+ \left[\frac{2}{f}\,(1-3f)Y + \frac{2}{t}\,(1-f)^2\right]\nabla_i f\nabla_j f.\end{aligned}$$

In what follows we will let C denote various constants which depend only on the time interval T and bounds on the $|f|$ and its derivatives $|\nabla f|$ and $|\nabla^2 f|$. The constants will vary from line to line, and to be precise could be indexed by the order of occurrence.

Proof of Theorem 5.1. Given $\varepsilon > 0$, let $\tilde{Z}_{ij}(g,X) = Z_{ij}(g,X) + \varepsilon e^{kt}\varphi g_{ij}$, where φ as in Lemma 4.3. Since $0 < f < 1$, we have $Z_{ij}(g,X) \geq Q_{ij}(g)$ where

$$Q_{ij}(g) = (1-f)^2\,\nabla_i\nabla_j f - \frac{1-f}{f}\,\nabla_i f\nabla_j f + \frac{1}{2}\,f(1-f)\left(f+\frac{1}{t}\right) f_{ij}.$$

Then there exists a constant $\delta > 0$ such that $Q_{ij}(g) > 0$ for $t < \delta$; hence $\tilde{Z}_{ij}(g,X) > 0$ for $t < \delta$. And, by Lemma 4.3, we can also choose a compact set K such that $\tilde{Z}_{ij}(g,X)$ is strictly positive outside K for $t > 0$.

Now suppose that $\tilde{Z}_{ij}(g,X) \leq 0$ at some point for some X_l. Then there exists a first time $\tau > 0$, a point $\xi \in \mathbf{R}^2$ and a tangent vector X_l at ξ such that at (ξ,τ),

$$\tilde{Z}_{ij}(g,X) = 0 \quad \text{and} \quad \nabla_{X_l}\tilde{Z}_{ij}(g,X) = 0.$$

If X_l is extended in space and time to satisfy (5.8) and (5.11), then the evolution equation for $\tilde{Z}_{ij}$ is given by

(5.12)

$$\begin{aligned}
&\frac{\partial}{\partial t}\tilde{Z}_{ij}\\
&= (1-f)^2\Delta\tilde{Z}_{ij} + 4(1-f)\nabla_k\tilde{Z}_{ij}\cdot\nabla_k f\\
&- \left[\frac{2}{1-f}\,Y + 2(1-f)\left(f+\frac{1}{t}\right) + 3f^2 + \frac{2}{f}(1-3f)\,|\nabla f|^2\right]\tilde{Z}_{ij}\\
&- 2(1-f)[\nabla_i f\cdot\nabla_j(Y+2\varepsilon e^{kt}\varphi) + \nabla_j f\cdot\nabla_i(Y+2\varepsilon e^{kt}\varphi)]\\
&+ \left\{\left[k+\frac{2}{1-f}\,Y+\frac{2}{t}(1-f)+f(1+f)+\frac{2}{f}(1-3f)|\nabla f|^2\right]\varphi+\Box\varphi\right\}\varepsilon e^{kt}g_{ij}\\
&+ 4(1-f)\varepsilon e^{kt}\left[(\nabla_i f\cdot\nabla_j\varphi+\nabla_j f\cdot\nabla_i\varphi) - \langle\nabla f,\nabla\varphi\rangle g_{ij}\right]\\
&+ fY\left(1+\frac{1}{t}\right)g_{ij} + \left[\frac{2}{f}\,(1-3f)Y + \frac{2}{t}\,(1-f)^2\right]\nabla_i f\nabla_j f\\
&- \frac{1}{t}\,f^2(1-f)\left(\frac{1}{2}\,f+\frac{1}{t}\right)g_{ij}.
\end{aligned}$$

Since R is bounded and the covariant derivatives of f are also bounded on $(0,T]$. Hence, we have

$$Y(g) = (1-f)^2\Delta f+f(1-f)\left(f+\frac{1}{t}\right)-\frac{1-f}{f}|\nabla f|^2 \le C_1 \quad \text{on } (\delta,T],$$

for some constant C_1, also depends on δ. From Lemma 4.3, we have

$$\nabla_i f\cdot\nabla_j\varphi+\nabla_j f\cdot\nabla_i\varphi-\langle\nabla f,\nabla\varphi\rangle g_{ij} \le 3|\nabla f||\nabla\varphi|g_{ij} \le C_2\varphi g_{ij},$$

for some constant C_2. Now we can choose constants L, k and η as in Lemma 4.3 such that $k > 6|\nabla f|^2 + 4C_2$ and $L\varepsilon > (1/\delta)[1+(1/\delta)] + 6C_1|\nabla f|^2$ on $(\delta,T]$. While the matrix maximum principle implies that, at (ξ,τ),

$$\frac{\partial}{\partial t}\tilde{Z}_{ij} \le 0, \quad \Delta\tilde{Z}_{ij} \ge 0 \quad \text{and} \quad \nabla\tilde{Z}_{ij} = 0.$$

Since $\nabla_{X_l}\tilde{Z}_{ij} = 0$ yields $X_l = -(1-f/f)\nabla_l f$. Hence, at (ξ,τ), we have

$$\nabla(Y+2\varepsilon e^{kt}\varphi) = \nabla(g^{ij}\tilde{Z}_{ij}) = g^{ij}\nabla(\tilde{Z}_{ij}) = 0,$$

for the special tangent vector $X_l = -(1-f/f)\nabla_l f$ at ξ. Combining all of these we had done, at (ξ, τ), implies that

$$0 \geq \left[f(1+f)+\frac{2}{t}(1-f)\right]\varepsilon e^{kt}\varphi g_{ij}+fY\left(1+\frac{1}{t}\right)g_{ij}+\left[\frac{2}{f}Y+\frac{2}{t}(1-f)^2\right]\nabla_i f\nabla_j f,$$

which is a contradiction. Therefore, for all $\varepsilon > 0$, we have $Z_{ij} + \varepsilon e^{kt}\varphi g_{ij} > 0$. This implies

$$Z_{ij} \geq 0,$$

which completes the proof of Theorem 5.1. □

Acknowledgments. The first author would like to express his thanks to Prof. S.-T. Yau for bringing his attention to this modified Ricci flow and for constant encouragement.

REFERENCES

1. B. Chow, *The Yamabe flow on locally conformally flat manifolds with positive Ricci curvature*, Comm. Pure Appl. Math. **45** (1992), 1003–1014.

2. S.-C. Chang and C.-T. Wu, *The modified Ricci flow on complete* $\mathbf{R}^2$, in preparation.

3. R.S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), 255–306.

4. ———, *The Ricci flow on surface*, *Mathematics and General Relativity* (J. Isenberg, ed.), Contemp. Math., vol. 71, Amer. Math. Soc., Providence, RI, 1988, pp. 237–262.

5. ———, *The Harnack estimate for the Ricci flow*, J. Differential Geom. **37** (1993), 225–243.

6. W.-X. Shi, *Ricci deformation of the metric on complete noncompact Riemannian manifolds*, J. Differential Geom. **30** (1989), 303–394.

7. L.-F. Wu, *The Ricci flow on complete* $\mathbf{R}^2$, Comm. Anal. Geom. **1** (1993), 439–472.

Department of Mathematics, National Tsing Hua University, Hsinchu, Taiwan 30043, R.O.C.
E-mail address: `scchang@math.nthu.edu.tw`

Department of Mathematics, National Tsing Hua University, Hsinchu, Taiwan 30043, R.O.C.

Department of Mathematics, National Tsing Hua University, Hsinchu, Taiwan 30043, R.O.C.
E-mail address: `ctwu@math.cts.nthu.edu.tw`