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MODIFICATIONS OF THE MOMENTS IN FREQUENCY ANALYSIS

VIGDIS BREVIK PETERSEN

Dedicated to Professor William B. Jones on the occasion of his 70th birthday

1. The frequency analysis problem. The frequency analysis problem is to determine the unknown frequencies and amplitudes in a trigonometric signal when the signal values are known. A method for solving this problem by using the asymptotic properties of the zeros of certain orthogonal polynomials, has been established in [**2, 7**]. The method has its root back to Wiener-Levinson [**14, 4**]. The method starts with measured signal values from a sample of observations of the signal. In discrete form the signal is

$$x(m) = \sum_{j=-I}^{I} \alpha_j e^{\mathrm{i}m\omega_j} \tag{1.1}$$

$$\text{for} \quad m = 0, 1, \dots, N-1, \quad \alpha_0 = 0, \quad 0 \text{ otherwise.}$$

Here $|\alpha_j|$ are the amplitudes, ω_j are the unknown normalized frequencies and N denotes the number of observed values in $\{x(m)\}$.

From the observations is constructed an absolutely continuous measure $\psi_N(\theta)$ defined by

$$\frac{d\psi_N(\theta)}{d\theta} = \frac{1}{2\pi} \left| \sum_{m=0}^{N-1} x(m) e^{-\mathrm{i}m\theta} \right|^2, \quad \theta \in [-\pi, \pi]. \tag{1.2}$$

For any fixed N the measure gives rise to a sequence of moments $\{\mu_m^{(N)}\}$. For practical purpose we use

$$\widetilde{\mu}_m^{(N)} := \frac{\mu_m^{(N)}}{N} \tag{1.3}$$

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since we are interested in the asymptotic behavior.

The measure also gives rise to a sequence of monic orthogonal polynomials; the Szegö polynomials $\{\rho_n(\psi_N;z)\}$. All the zeros of $\{\rho_n(\psi_N;z)\}$ are located in the open unit disc.

The number of frequencies is denoted $n_0 := 2I$. Since n_0 is unknown, we choose a number n which we believe is such that $n > n_0$. In this situation there are two important properties:

(i) If we go to certain subsequence $\{N_k\}$ of $\{N\}$, we obtain convergence of the limit polynomial when $k \to \infty$. In each convergence case the limit polynomial is

$$\begin{aligned}\lim_{k\to\infty}\{\rho_n(\psi_{N_k};z)\} &= \prod_{j=1}^{I}\left(z-e^{i\omega_j}\right)\left(z-e^{-i\omega_j}\right)\\ &\quad\cdot\prod_{p=n_0+1}^{n}\left(z-z_p^{(k,n)}\right).\end{aligned} \tag{1.4}$$

Thus n_0 of the zeros in the limit polynomial tend to the frequency points $e^{\pm i\omega_j}$. In addition we have $(n-n_0)$ "uninteresting" zeros. Those zeros depend upon the degree n of the polynomial and the subsequence $\{N_k\}$ [**2, 7**].

(ii) For a given $n > n_0$ there exists a number $K_n < 1$ such that

$$|z_p^{(n)}| \le K_n < 1 \quad \text{for } p = n_0+1,\dots,n. \tag{1.5}$$

Hence the "uninterested" zeros can be separated from the frequency points [**7**].

The method briefly described above is called the N-process.

2. Modifications.

Several modifications of the N-process are established during the last ten years. There are two main approaches. One way is to construct *new modified measures* which can be used to find the frequency points, another is to *modify the moments* in different ways. In this paper we deal with the *second* approach.

In this situation the moments

$$\widetilde{\mu}_m^{(N)} := \frac{\mu_m^{(N)}}{N}$$

from the $N-$process or

$$\mu_m = \lim_{N\to\infty} \widetilde{\mu}_m^{(N)} = \sum_{j=1}^{I} |\alpha_j|^2 \cos m\omega_j \tag{2.1}$$

are multiplied by *certain factors*.

The first modification of that type is called the R-process [**3**]. In the R-process we multiply (2.1) by $R^{|m|}$ where $R \in (0,1)$. Then we have a new sequence

$$\mu_m^{(R)} = \mu_m R^{|m|} = \left(\sum_{j=1}^{I} |\alpha_j|^2 \cos m\omega_j \right) R^{|m|}. \tag{2.2}$$

We know that the sequence (2.2) is a moment-sequence and that the properties (1.4)–(1.5) hold [**3, 5**].

3. Notation. Before the next sections we introduce some determinant formulas for the N-process: The Szegö polynomials

$$\rho_n(\psi_N; z) = \frac{1}{\Delta_{n-1}^{(N)}} \begin{vmatrix} \widetilde{\mu}_0^{(N)} & \widetilde{\mu}_{-1}^{(N)} & \cdots & \widetilde{\mu}_{-n}^{(N)} \\ \widetilde{\mu}_1^{(N)} & \widetilde{\mu}_0^{(N)} & \cdots & \widetilde{\mu}_{-n+1}^{(N)} \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{\mu}_{n-1}^{(N)} & \widetilde{\mu}_{n-2}^{(N)} & \cdots & \widetilde{\mu}_{-1}^{(N)} \\ 1 & z & \cdots & z^n \end{vmatrix} \tag{3.1}$$

where $\Delta_{n-1}^{(N)}$ is the Toeplitz determinant of order n.

The Toeplitz determinant of order n

$$\Delta_{n-1}^{(N)} = \begin{vmatrix} \widetilde{\mu}_0^{(N)} & \widetilde{\mu}_{-1}^{(N)} & \widetilde{\mu}_{-2}^{(N)} & \cdots & \widetilde{\mu}_{-n+1}^{(N)} \\ \widetilde{\mu}_1^{(N)} & \widetilde{\mu}_0^{(N)} & \widetilde{\mu}_{-1}^{(N)} & \cdots & \widetilde{\mu}_{-n+2}^{(N)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \widetilde{\mu}_{n-1}^{(N)} & \widetilde{\mu}_{n-2}^{(N)} & \widetilde{\mu}_{n-3}^{(N)} & \cdots & \widetilde{\mu}_0^{(N)}. \end{vmatrix} \tag{3.2}$$

The reflection coefficients

$$\delta_n^{(N)} = \frac{(-1)^n}{\Delta_{n-1}^{(N)}} \begin{vmatrix} \widetilde{\mu}_{-1}^{(N)} & \widetilde{\mu}_{-2}^{(N)} & \widetilde{\mu}_{-3}^{(N)} & \cdots & \widetilde{\mu}_{-n}^{(N)} \\ \widetilde{\mu}_0^{(N)} & \widetilde{\mu}_{-1}^{(N)} & \widetilde{\mu}_{-2}^{(N)} & \cdots & \widetilde{\mu}_{-n+1}^{(N)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \widetilde{\mu}_{n-2}^{(N)} & \widetilde{\mu}_{n-3}^{(N)} & \widetilde{\mu}_{n-4}^{(N)} & \cdots & \widetilde{\mu}_{-1}^{(N)} \end{vmatrix}. \tag{3.3}$$

The identities

$$\cos m\omega_j = 2\cos\omega_j \cos(m-1)\,\omega_j - \cos(m-2)\,\omega_j. \tag{3.4}$$

and

$$\cos(m-2)\,\omega_j - 2\cos m\omega_j + \cos(m+2)\,\omega_j = -4\sin^2\omega_j \cos m\omega_j \tag{3.5}$$

will play an important role in the following section. For the sake of simplicity we will use the notation $\cos m\omega_j = T_m(x_j) =: T_m$. Hence we may write

$$T_m - 2x_j T_{m-1} + T_{m-2} = 0 \tag{3.6}$$

and

$$T_{m-2} - 2T_m + T_{m+2} = -4(1-x_j^2)T_m. \tag{3.7}$$

4. Modifications of the moments. A new type of modification of the moments was inspired from the R-process. The idea was to multiply the sequence (2.1)

$$\mu_m = \sum_{j=1}^{I} |\alpha_j|^2 \cos m\omega_j$$

by R^{m^2} where $R \in (0,1)$. Then we have a new sequence

$$\mu_m^{(V)} = \mu_m R^{m^2} = \left(\sum_{j=1}^{I} |\alpha_j|^2 \cos m\omega_j\right) R^{m^2} \tag{4.1}$$

In this situation we can pick up the frequency points (1.4) but the property (1.5) does not hold [**9**].

A more general idea, introduced in [**13**], was to multiply the sequence (2.1) by $R^{|m|^\alpha}$ where $R \in (0,1)$ and $\alpha > 0$. Then we have a new sequence

$$\mu_m^{(\alpha)} = \mu_m R^{|m|^\alpha} = \left(\sum_{j=-I}^{I} |\alpha_j|^2 \cos m\omega_j\right) R^{|m|^\alpha}. \tag{4.2}$$

In [**13**] we find the following result

Proposition 1. *For any $\omega \in [0, \pi]$ and $R \in (0,1)$ and $\alpha > 2$, the function*

$$F_R(z) = 1 + 2\sum_{m=1}^{\infty} R^{m^\alpha} \cos(m\omega) z^m$$

is not always a Carathéodory-function, or equivalently

$$\{R^{|m|^\alpha} \cos(m\omega)\}_{-\infty}^{\infty}$$

is not always positive definite.

For the reflection coefficients we have the lemma

Lemma 2. *For the reflections coefficients $\delta_n^{(N)}$ the following holds*:

$$\text{(i)} \quad \lim_{N\to\infty} \left|\delta_n^{(N)}\right| = 1 \quad \text{for } n = n_0$$

$$\text{(ii)} \quad \lim_{N\to\infty} \left|\delta_n^{(N)}\right| \neq 1 \quad \text{for } n > n_0$$

The first part of the lemma means that the product of the zeros in the limit polynomial $\rho_{n_0}(\psi_{N_k}; z)$ equals one if $n = n_0$. In this situation we get the n_0 frequency points.

The second part means that the product of the zeros in the limit polynomial is different from *one* if $n > n_0$. In this situation we get the n_0 frequency points, and in addition the $(n - n_0)$ zeros which are located *inside* the unit circle.

4.1 The reflection coefficients for $n = 2$. In this section we consider the reflection coefficients. It is known that the following property holds:

A sequence $\{\mu_m\}_{-\infty}^{\infty}$ of complex numbers, $\mu_n = \overline{\mu}_{-n}$, is positive definite, if and only if, $\delta_0 > 0$ and $|\delta_n| < 1$ for every $n \geq 1$, where the

$\{\delta_n\}$ are the corresponding reflection coefficients, (see Theorem 3.2 in [**1**]).

This property is the reason why we study the reflection coefficients for a given signal and a given $n \geq n_0$, and the results obtained are the reason to state a later conjecture.

We use the simple signal (1.1):

$$x(m) = \frac{1}{\sqrt{2}}\, e^{im\omega} + \frac{1}{\sqrt{2}}\, e^{-im\omega} = \sqrt{2}\cos m\omega. \tag{4.1.1}$$

In our situation the "moments" are

$$\mu_m^{(\alpha)} = \mu_{-m}^{(\alpha)} = \cos m\omega R^{|m|^\alpha}.$$

An important property regarding the reflection coefficients [**1**] is that for $n \neq n_0$ we have

$$\left|\delta_n^{(N)}\right|^2 < 1.$$

Let $x := cos\omega$, and consider $\delta_2^{(\alpha,R)}$. We know that $x^2 \in [0,1]$ and $R \in (0,1)$. From (3.3) we have

$$\begin{aligned} \delta_2^{(\alpha,R)} &= (-1)^2 \frac{\begin{vmatrix} xR & \left(2x^2-1\right) R^{2^\alpha} \\ 1 & xR \end{vmatrix}}{\begin{vmatrix} 1 & xR \\ xR & 1 \end{vmatrix}} \\ &= \frac{R^{(2^\alpha)} - 2R^{(2^\alpha)}x^2 + x^2R^2}{1-x^2R^2} \end{aligned} \tag{4.1.2}$$

We want to find out when the product of the zeros is less than one, so we look at $\delta_2^{(\alpha,R)} < 1$.

$$\begin{aligned} \frac{R^{2^\alpha} - 2R^{2^\alpha}x^2 + x^2R^2}{1-x^2R^2} &< 1 \\ R^{2^\alpha} - 2R^{2^\alpha}x^2 + x^2R^2 &< 1 - x^2R^2 \\ 2x^2R^2\left(1 - R^{2^\alpha-2}\right) &< 1 - R^{2^\alpha}. \end{aligned}$$

The "worst" case is for $x^2 = 1$

$$\begin{aligned} R^{2^\alpha} - 2R^2 + 1 &> 0 \\ \alpha &\leq 2 \end{aligned}$$

Hence for $\alpha \leq 2$, the product of the zeros is *less* than *one*. This means that the zeros tend to the frequency points from the *inside* of the unit circle.

For $\alpha > 2$ the product of the zeros is *greater* than *one* for any fixed R in this interval and sufficiently small $\left(1 - x^2\right)$. This means that the zeros may tend to the frequency points from the *outside* of the unit circle for special values of $x \in [-1, 1]$.

An alternative way is to consider the situation where R is replaced by $R = 1 - d$. We use the power series expansion, and look at the influence of the terms $O\left(d^2\right)$. Here $x^2 \in [0, 1]$ and $R \in (0, 1)$. From (4.1.2) we get

$$\delta_2^{(\alpha,d)} = \frac{(1-d)^{(2^\alpha)} - 2\,(1-d)^{(2^\alpha)}\,x^2 + x^2\,(1-d)^2}{1 - x^2\,(1-d)^2}$$

We look at the case $\delta_2^{(\alpha,d)} < 1$

$$\frac{(1-d)^{(2^\alpha)} - 2\,(1-d)^{(2^\alpha)}\,x^2 + x^2\,(1-d)^2}{1 - x^2\,(1-d)^2} < 1 \tag{4.1.3}$$

The power series expansion in d of (4.1.3) is

$$1 - \frac{\left(4 - 2^{\alpha+1}\right)x^2 + 2^\alpha}{1 - x^2}\,d + O\left(d^2\right).$$

Thus we consider

$$\begin{aligned} 1 - \frac{\left(4 - 2^{\alpha+1}\right)x^2 + 2^\alpha}{1 - x^2} d &< 1 \\ -\frac{\left(4 - 2^{\alpha+1}\right)x^2 + 2^\alpha}{1 - x^2} d &< 0 \\ \left(4 - 2^{\alpha+1}\right)x^2 + 2^\alpha &> 0 \end{aligned} \tag{4.1.4}$$

The "worst" case is for $x^2 = 1$

$$\begin{gathered}\left(2^{\alpha-1} - 1\right) < 2^{\alpha-2}\\ \alpha < 2\end{gathered}$$

Hence for $\alpha < 2$ the product of the zeros is *less* than *one*. This means that the zeros tend to the frequency points from the *inside* of the unit circle.

For $\alpha > 2$ the product of the zeros is *greater* than *one* for any fixed R in this interval and $\left(1 - x^2\right)$ sufficiently small. This means that the zeros may tend to the frequency points from the *outside* of the unit circle for special values of $x \in [-1, 1]$.

For $\alpha = 2$ we observe from (4.1.4) that the zeros tend to the frequency points from the *inside* of the unit circle for all $x \in [-1, 1]$.

The value $\alpha = 2$ is such a border case.

4.2 The reflection coefficients for $n = 3$. The next step is to look at the absolute value of the "uninteresting" zero where $n_0 = 2$ and $n = 3$. The reflection coefficients are

$$\begin{aligned}\delta_3^{(\alpha)} &= \frac{(-1)^3}{\Delta_2^{(\alpha)}}\begin{vmatrix} x(1-d) & (2x^2-1)(1-2^\alpha d) & (4x^3-3x)(1-3^\alpha d)\\ 1 & x(1-d) & (2x^2-1)(1-2^\alpha d)\\ x(1-d) & 1 & x(1-d)\end{vmatrix} + O(d)\\ &= -\frac{\begin{vmatrix} x(1-d) & (2x^2-1)(1-2^\alpha d & (4x^3-3x)(1-3^\alpha d)\\ 1 & x(1-d) & (2x^2-1)(1-2^\alpha d)\\ x(1-d) & 1 & x(1-d)\end{vmatrix}}{\begin{vmatrix} 1 & x(1-d) & (2x^2-1)(1-2^\alpha d)\\ x(1-d) & 1 & x(1-d)\\ (2x^2-1)(1-2^\alpha d) & x(1-d) & 1\end{vmatrix}} + O(d).\end{aligned}$$

The expression leads to

(4.2.1)

$$\left| -\frac{(4 \bullet 3^\alpha - 2^{\alpha+3} + 4)x^5 - (7 \bullet 3^\alpha + 3 \bullet 2^{\alpha+2} + 1)x^3 + (3^{\alpha+1} - 2^{\alpha+2} - 3)x}{(2^{\alpha+2} - 8)x^4 - (3 \bullet 2^{\alpha+1} - 8)x^2 + 2^{\alpha+1}} \right| + O(d)$$

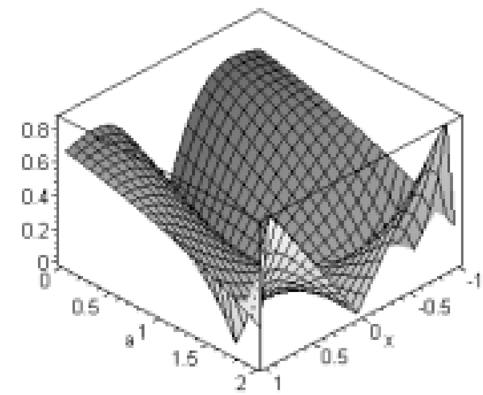

FIGURE 1. $0 < \alpha < 1.99$.

Neglecting the O-terms, rewriting (4.2.1) and cancelling the factor $\left(1 - x^2\right)$, we may write

$$\text{(4.2.2)}\quad \lim_{d\to 0} |\delta_3| = \left|\frac{x}{2}\right| \left|\frac{(8 \bullet 2^{\alpha} - 4 \bullet 3^{\alpha} - 4)\, x^2 - \left(4 \bullet 2^{\alpha} - 3^{\alpha+1} + 3\right)}{(2^{\alpha+1} - 4)\, x^2 - 2^{\alpha}}\right|.$$

We consider (4.2.2) for different α-values.

In Figure 1 we observe that the absolute value of the "uninteresting" zero is less than one for all $x \in [-1, 1]$.

In Figure 2 we observe that the absolute value of the "uninteresting" zero can be greater than one for some $x \in [-1, 1]$.

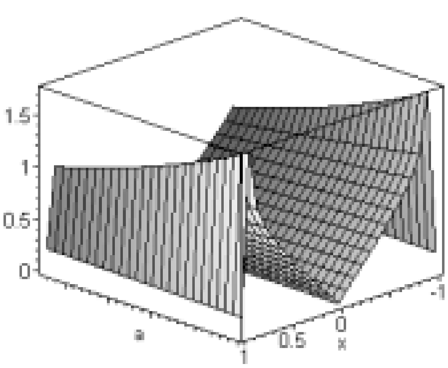

FIGURE 2. $2.01 < \alpha < 2.10$.

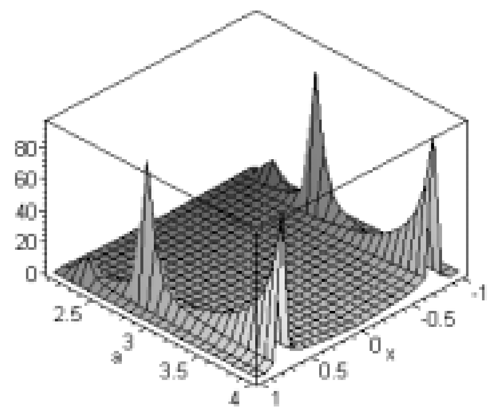

FIGURE 3. $2.1 < \alpha < 4.0$.

In Figure 3 we observe that the absolute value of the "uninteresting" zero can be much greater than *one* for some $x \in [-1, 1]$.

Conjecture 3. *For* $\alpha \in (0, 2]$ *the sequence* $\{\cos m\omega R^{|m|^\alpha}\}_{m=-\infty}^{\infty}$ *is positive definite.*

We consider some known results:

(i) *The R-process* ($\alpha = 1$) *with moments*

$$\mu_m^{(R)} = \mu_m R^{|m|} = \mu_m (1-d)^{|m|}.$$

In the R-processes n_0 of the zeros tend to the frequency points $e^{\pm i\omega_j}$ [**3**]. The remaining zeros are such that $\left|z_p^{(n)}\right| \le K_n < 1$ [**5**].

For $n_0 = 2$ we have the special result $\lim_{d\to 0} |\delta_3^{(R)}| = z_3^{(R)} = |x/2|$ [**3**].

(ii) *The V-process* ($\alpha = 2$) *with moments*

$$\mu_m^{(V)} = \mu_m R^{m^2} = \mu_m (1-d)^{m^2}.$$

Let $n = \beta n_0 + \gamma$. Then βn_0 of the zeros in the V-process tend to the frequency points $e^{\pm i\omega_j}$. The remaining zeros are such that $|z_p^{(n)}| \le K_n \le 1$ [**9**].

For $n_0 = 2$ we have the special result $\lim_{d\to 0} |\delta_3^{(V)}| = z_3^{(V)} = |x|$.

Hence we know that the conjecture holds for $\alpha = 1, 2$.

4.3 How important is the power of d in the Toeplitz determinant? The power of d tells us at which rate the Toeplitz determinants tend to zero. Let $n = \beta n_0 + \gamma$.

In the R-process ($\alpha = 1$) the power of d is [**8**]

$$d^{n-n_0} = d^{(\beta-1)n_0+\gamma}. \tag{4.3.1}$$

In the V-processes ($\alpha = 2$) the power of d is [**9**]

$$d^{\frac{1}{2}\beta(\beta-1)n_0+\beta\gamma}. \tag{4.3.2}$$

The power of d is closely related to α and hence to the power of m.

Proposition 4. *For $\alpha \in (0, 2)$ and $n = \beta n_0 + \gamma$, the rate at which the Toeplitz determinant tends to zero is*

$$d^{n-n_0} = d^{(\beta-1)n_0+\gamma}.$$

Outline of proof. We use the simple signal (4.1.1) and consider the Toeplitz determinant of degree $n_0 = 2$ and $n = 5$, i.e., $n = 2\bullet 2+1 = 5$.

$$\Delta_4^{(\alpha)} = \begin{vmatrix} \mu_0^{(\alpha)} & \mu_{-1}^{(\alpha)} & \mu_{-2}^{(\alpha)} & \mu_{-3}^{(\alpha)} & \mu_{-4}^{(\alpha)} \\ \mu_1^{(\alpha)} & \mu_0^{(\alpha)} & \mu_{-1}^{(\alpha)} & \mu_{-2}^{(\alpha)} & \mu_{-3}^{(\alpha)} \\ \mu_2^{(\alpha)} & \mu_1^{(\alpha)} & \mu_0^{(\alpha)} & \mu_{-1}^{(\alpha)} & \mu_{-2}^{(\alpha)} \\ \mu_3^{(\alpha)} & \mu_2^{(\alpha)} & \mu_1^{(\alpha)} & \mu_0^{(\alpha)} & \mu_{-1}^{(\alpha)} \\ \mu_4^{(\alpha)} & \mu_3^{(\alpha)} & \mu_2^{(\alpha)} & \mu_1^{(\alpha)} & \mu_0^{(\alpha)} \end{vmatrix} \tag{4.3.3}$$

We *first* simplify the determinant (4.3.3) by using *row* operations. Notice that the indices of the "moments" are increasing in the columns. Replace row k by

$$\text{row } (k)\text{ -2x row}\,(k+1)\ \text{ +row } (k+2)\,.$$

We notice that *all* the terms without m vanish because of (3.6). The remaining terms are:

(4.3.4)

$$\begin{aligned}&(T_m(1-|m|^\alpha d) - 2xT_{m+1}(1-|m+1|^\alpha d) + T_{m+2}(1-|m+2|^\alpha d) + O\left(d^2\right)\\&= d\left(\left(|m+1|^\alpha - |m|^\alpha\right)T_m - \left(|m+2|^\alpha - |m+1|^\alpha\right)T_{m+2}\right) + O\left(d^2\right)\\&= d\left(f_{m,m+2}\right).\end{aligned}$$

Then we have

$$\Delta_4^{(\alpha)} = \begin{vmatrix} f_{0,2} & f_{-1,1} & f_{-2,0} & f_{-3,-1} & f_{-4,-2} \\ f_{1,3} & f_{0,2} & f_{-1,1} & f_{-2,0} & f_{-3,-1} \\ f_{2,4} & f_{1,3} & f_{0,2} & f_{-1,1} & f_{-2,0} \\ \mu_3^{(\alpha)} & \mu_2^{(\alpha)} & \mu_1^{(\alpha)} & \mu_0^{(\alpha)} & \mu_{-1}^{(\alpha)} \\ \mu_4^{(\alpha)} & \mu_3^{(\alpha)} & \mu_2^{(\alpha)} & \mu_1^{(\alpha)} & \mu_0^{(\alpha)} \end{vmatrix} d^3 + O\left(d^4\right) \tag{4.3.5}$$

We make a *second* simplification of the determinant. This time we make *column* operations. From the left to the right we use the following column-numbers: $c = \{1, 2, \cdots, n+1\}$. Notice that the indices of the "moments" are decreasing in the rows.

Replace column c by

$$\text{column}\,(c)\text{-2x column}\,(c+1)\,\text{+column}\,(c+2)\,.$$

In our situation we obtain

$$\begin{aligned}(4.3.5)\quad & f_{m,m+2} - 2xf_{m-1,m+1} + f_{m-2,m}\\&= \left(\left(|m+1|^\alpha - |m|^\alpha\right)T_m - \left(|m+2|^\alpha - |m+1|^\alpha\right)T_{m+2}\right)\\&\quad - 2x\left(\left(|m|^\alpha - |m-1|^\alpha\right)T_{m-1} - \left(|m+1|^\alpha - |m|^\alpha\right)T_{m+1}\right)\\&\quad + \left(\left(|m-1|^\alpha - |m-2|^\alpha\right)T_{m-2} - \left(|m|^\alpha - |m-1|^\alpha\right)T_m\right)\end{aligned}$$

We rewrite the expression

$$\begin{aligned}&= -\left(|m|^\alpha - 2\,|m-1|^\alpha + |m-2|^\alpha\right)T_{m-2}\\&\quad + 2\left(|m+1|^\alpha - 2\,|m|^\alpha + |m-1|^\alpha\right)T_m\\&\quad - \left(|m+2|^\alpha - 2\,|m+1|^\alpha + |m|^\alpha\right)T_{m+2}\end{aligned} \tag{4.3.6}$$

For the determinant in question we have three expressions. The first one P is for $m = 0$:

$$\begin{aligned} f_{0,2} - 2xf_{-1,1} + f_{-2,0} &= -\left(2^\alpha - 2\right) T_{-2} + 4T_0 - \left(2^\alpha - 2\right) T_2 \\ &= \left(4T_0 - \left(2^{\alpha+1} - 4\right) T_2\right) =: P \end{aligned}$$

The second one Q is for $m = -1$:

$$\begin{aligned} &f_{-1,1} - 2xf_{-2,0} + f_{-3,-1} \\ &\qquad = -\left(1 - 2^{\alpha+1} + 3^\alpha\right) T_{-3} + \left(-4 + 2^{\alpha+1}\right) T_{-1} - (1+1)\, T_1 \\ &\qquad = \left(2^{\alpha+1} - 6\right) T_1 - \left(3^\alpha - 2^{\alpha+1} + 1\right) T_3 =: Q \end{aligned}$$

and the third one R is for $m = -2$:

$$\begin{aligned} f_{-2,0} - 2xf_{-3,-1} + f_{-4,-2} &= -\left(2^\alpha - 2\right) T_0 + 2\left(\left(3^\alpha\right) - 2^{\alpha+1} + 1\right) T_2 \\ &\quad - \left(4^\alpha - 2\left(3^\alpha\right) + 2^\alpha\right) T_4 =: R. \end{aligned}$$

The determinant (4.3.5) now looks like

$$\Delta_4^{(\alpha)} = \begin{vmatrix} P & Q & R & f_{-3,-1} & f_{-4,-2} \\ Q & P & Q & f_{-2,0} & f_{-3,-1} \\ R & Q & P & f_{-1,1} & f_{-2,0} \\ df_{3,1} & df_{2,0} & f_{1,-1} & \mu_0^{(\alpha)} & \mu_1^{(\alpha)} \\ df_{4,2} & df_{3,1} & df_{2,0} & \mu_1^{(\alpha)} & \mu_0^{(\alpha)} \end{vmatrix} d^3 + O\left(d^4\right) \tag{4.3.7}$$

Now we have the cases:

(i)

$$\begin{vmatrix} P & Q & R \\ Q & P & Q \\ R & Q & P \end{vmatrix} = 0 \tag{4.3.8}$$

(ii)

$$\begin{vmatrix} P & Q & R \\ Q & P & Q \\ R & Q & P \end{vmatrix} \neq 0 \tag{4.3.9}$$

Let us consider the "border" case $\alpha = 2$ We have

$$\begin{aligned} P &= 4T_0 - 4T_2 = 8\left(1-x^2\right) = 8\left(1-x^2\right)T_0 \\ Q &= 2T_1 - 2T_3 = 8x\left(1-x^2\right) = 8\left(1-x^2\right)T_1 \\ R &= -2T_0 + 4T_2 - 2T_4 = 8\left(1-x^2\right)T_2. \end{aligned}$$

Let $C = 8\left(1-x^2\right)$. From (4.3.7) we find

$$\Delta_4^{(2)} = \begin{vmatrix} C\ T_0 & C\ T_1 & C\ T_2 & f_{-3,-1} & f_{-4,-2} \\ C\ T_1 & C\ T_0 & C\ T_1 & f_{-2,0} & f_{-3,-1} \\ C\ T_2 & C\ T_1 & C\ T_0 & f_{-1,1} & f_{-2,0} \\ df_{3,1} & df_{2,0} & f_{1,-1} & \mu_0^{(2)} & \mu_1^{(2)} \\ df_{4,2} & df_{3,1} & df_{2,0} & \mu_1^{(2)} & \mu_0^{(2)} \end{vmatrix} d^3 + O\left(d^4\right) \tag{4.3.10}$$

A second row operation leads (in a similar way as the first one) to

$$\Delta_4^{(2)} = \begin{vmatrix} d\,(*) & d\,(*) & d\,(*) & C\ T_1 & C\ T_2 \\ C\ T_1 & C\ T_0 & C\ T_1 & f_{-2,0} & f_{-3,-1} \\ C\ T_2 & C\ T_1 & C\ T_0 & f_{-1,1} & f_{-2,0} \\ df_{3,1} & df_{2,0} & f_{1,-1} & \mu_0^{(2)} & \mu_1^{(2)} \\ df_{4,2} & df_{3,1} & df_{2,0} & \mu_1^{(2)} & \mu_0^{(2)} \end{vmatrix} d^3 + O\left(d^4\right) \tag{4.3.11}$$

where $(*)$ is an expression in x, d.

A second column operation leads, in a similar way as the first one, to

$$\Delta_4^{(2)} = \begin{vmatrix} d\,(\#) & d\,(*) & d\,(*) & C\ T_1 & C\ T_2 \\ d\,(*) & C\ T_0 & C\ T_1 & f_{-2,0} & f_{-3,-1} \\ d\,(*) & C\ T_1 & C\ T_0 & f_{-1,1} & f_{-2,0} \\ dT_1 & df_{2,0} & f_{1,-1} & \mu_0^{(2)} & \mu_1^{(2)} \\ dT_2 & df_{3,1} & df_{2,0} & \mu_1^{(2)} & \mu_0^{(2)} \end{vmatrix} d^3 + O\left(d^4\right) \tag{4.3.12}$$

where $(*)$ and $(\#)$ are expressions in x, d.

The important fact is that we can pull out d from the first column in (4.3.12), and hence the power of d is d^4 (4.3.2). For more details, see [**10**]. The value $\alpha = 2$ is the only value where we can pull out more d's from the determinant.

If $\alpha \neq 2$ we have no possibility to pull out more d's in the determinant, so we have the situation (4.3.9). In that situation the determinant (4.3.7) may be written

$$\Delta_4^{(2)} = \begin{vmatrix} P & Q & R \\ Q & P & Q \\ R & Q & P \end{vmatrix} \begin{vmatrix} \mu_0^{(2)} & \mu_{-1}^{(2)} \\ \mu_1^{(2)} & \mu_0^{(2)} \end{vmatrix} d^3 + O\left(d^4\right).$$

For all $\alpha \in (0, 2)$ we thus have the situation stated in Proposition 4.

Proposition 4 holds for a signal with an arbitrary number of frequencies and an arbitrary degree $n \geq n_0$. For more details, see [**11**]. If, in addition, Conjecture 3 holds, a result of type (1.4) holds, but not necessarily (1.5).

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Sør-Trøndelag University College, N-7004 Trondheim, Norway
E-mail address: vigdis.petersen@alt.hist.no