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EXTREMAL BOUNDED SLIT MAPPINGS FOR LINEAR FUNCTIONALS

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ABSTRACT. Let $S(M)$ be the class of holomorphic univalent functions $f(z) = z + a_2 z^2 + \dots$, $|f(z)| < M$, $|z| < 1$ and $L(f) = \sum_{k=2}^{n} \lambda_k a_k$, $(\lambda_2, \dots, \lambda_n) \in \mathbf{R}^{n-1}$. We prove that under some conditions among all bounded slit mappings only the Pick functions can be extremal for $\Re L(f)$ in $S(M)$ provided M is close to 1. In particular, if $\alpha > 0$, $(n-1)$ and $(m-1)$ are odd and relatively prime, then the Pick function maximizes $\Re(a_n + \alpha a_m)$ in $S(M)$ for M close to 1.

1. Introduction. Let $S(M)$, $M > 1$, be the class of holomorphic functions f in the unit disk $D = \{z : |z| < 1\}$,

$$f(z) = z + a_2 z^2 + \dots, \quad z \in D,$$

which are univalent and bounded by M in D, i.e., $|f(z)| < M$, $z \in D$.

Denote by $S^1(M)$ the class of functions $f \in S(M)$ which map D onto the disk D_M of radius M centered at the origin and slit along an analytic curve. An important member of $S^1(M)$ is the so-called Pick function $P_M(z)$ which maps D onto D_M slit along the segment $[-M, -M(2M - 1 - 2\sqrt{M(M-1)})]$.

Consider a linear continuous functional L on $S(M)$ given by

$$L(f) = \sum_{k=2}^{n} \bar{\lambda}_k a_k, \quad \lambda_k \in \mathbf{C}, \quad k = 2, \dots, n.$$

So L is determined by the vector $\boldsymbol{\lambda} = (\lambda_2, \dots, \lambda_n) \in \mathbf{C}^{n-1}$.

We will prove the following

Theorem 1. *Let* $\boldsymbol{\lambda} = (\lambda_2, \dots, \lambda_n) \in \mathbf{R}^{n-1}$ *and*

$$\max_{f \in S(M)} \Re L(f) = \Re L(f_0)$$

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for $f_0 \in S^1(M)$*. If*

$$\sum_{k=2}^{n}(k-1)\lambda_k \sin(k-1)u \tag{1}$$

has only single zeros on $[0, 2\pi]$*, then* f_0 *is either the Pick function* $P_M(z)$ *or* $-P_M(-z)$ *provided* $(M-1)$ *is small enough.*

Theorem 1 is applied to $L(f) = a_n + \alpha a_m$ where $\alpha > 0$, $(n-1)$ and $(m-1)$ are odd and relatively prime.

2. Loewner theory and optimization methods.

Theorem A (Loewner equation, see, e.g., [**1**]). *Let* $w = w(z,t)$ *be the solution of the Loewner equation*

$$\frac{dw}{dt} = -w\,\frac{e^{iu}+w}{e^{iu}-w}, \quad w|_{t=0} = z, \quad 0 \le t \le \log M, \tag{2}$$

with a piecewise continuous function $u = u(t)$*. Then*

$$w(z,t) = e^{-t}(z + a_2(t)z^2 + \cdots), \quad z \in D, \quad t \ge 0, \tag{3}$$

is holomorphic and univalent with respect to $z \in D$ *for every* $t \ge 0$*. Moreover, the functions given by the formula*

$$f(z) := Mw(z, \log M) \in S(M), \tag{4}$$

form a dense subclass of $S(M)$*.*

Remark 1. In the case $u(t) = \text{const}$, the functions $f(z)$ given by (4) are rotations of the Pick function $P_M(z)$. In particular, $u(t) = \pi$ corresponds to $P_M(z)$ while $u(t) = 0$ corresponds to $-P_M(-z)$. If $u(t)$ is analytic, then $f \in S^1(M)$.

Remark 2. If a control function $u(t)$ generates a function $f(z)$ by (2)–(4), then $-u(t)$ generates $\overline{f(\bar{z})}$.

Let $a_j(t)$ be given by (3), $a_j(t) = x_{2j-1}(t) + ix_{2j}(t)$, $j = 2, \dots, n$, and $a(t) = (x_3(t), \dots, x_{2n}(t))$. Comparing Taylor coefficients in both sides of the Loewner equation (2) we obtain the system of differential equations

$$\begin{aligned} \frac{dx_k}{dt} = g_k(t, \boldsymbol{a}, u), \quad x_k(0) = 0, \quad k = 3, \dots, 2n \\ x_{2j-1}(\log M) + ix_{2j}(\log M) = a_j, \quad j = 2, \dots, n. \end{aligned} \tag{5}$$

The explicit formulas for g_k are given in [**5**]. Note that

$$g_{2j-1}(0, \mathbf{0}, u) + ig_{2j}(0, \mathbf{0}, u) = -2e^{-i(j-1)u}, \quad j \geq 2.$$

The coefficient region

$$V_n^M = \{\boldsymbol{a} = (a_2, \dots, a_n) : f \in S(M)\}$$

is the closure of the attainable set for the system (5). Let $f^* \in S^1(M)$ be extremal for $\Re L$ in $S(M)$, i.e., $\max_{f \in S(M)} \Re L(f) = \Re L(f^*)$ and correspond to a boundary point $\boldsymbol{a}^* \in \partial V_n^M$. Then f is represented by (2)–(4) with $u^*(t)$ satisfying certain optimization conditions.

In fact, consider the Hamilton function

$$H(t, \boldsymbol{a}, \boldsymbol{\psi}, u) = \sum_{k=3}^{2n} g_k(t, \boldsymbol{a}, u)\psi_k, \tag{6}$$

where $\boldsymbol{\psi} = (\psi_3(t), \dots, \psi_{2n}(t))$ is the nonzero conjugate vector which satisfies the conjugate Hamiltonian system

$$\frac{d\psi_k}{dt} = -\frac{\partial H}{\partial x_k}, \quad \psi_k(0) = \xi_k, \quad k = 3, \dots, 2n. \tag{7}$$

The following theorem is a version of Pontryagin's maximum principle together with the corresponding transversality conditions both of which are necessary conditions for extremal trajectories in an optimal control problem (see, e.g., [**4**, pp. 254, 319] for the autonomous case).

Theorem B. *Let $\boldsymbol{a}^*(t)$ be a solution of the system* (5) *with a continuous control function $u^*(t)$. If $\boldsymbol{a}^* = \boldsymbol{a}^*(\log M)$ is a boundary point* *of V_n^M which gives $\max_{f\in S(M)} \Re L(f)$, then there exists the solution $\boldsymbol{\psi}^* = \boldsymbol{\psi}^*(t)$ of the system (7) exists with the same control function $u^*(t)$ such that*

(8)

$$\max_u H(t,\boldsymbol{a}^*(t),\boldsymbol{\psi}^*(t),u) = H(t,\boldsymbol{a}^*(t),\boldsymbol{\psi}^*(t),u^*(t)), \quad t\in[0,\log M],$$

and

$$\boldsymbol{\psi}^*(\log M) = (\Re\lambda_2, \Im\lambda_2, \dots, \Re\lambda_n, \Im\lambda_n). \tag{9}$$

The condition (8) is called the *Pontryagin maximum principle* and (9) is called the *transversality condition* at $\boldsymbol{a}^*$. Evidently $u^*(t)$ is a root of the equation

$$H_u(t,\boldsymbol{a},\boldsymbol{\psi},u) = 0 \tag{10}$$

with $\boldsymbol{a} = \boldsymbol{a}^*$ and $\boldsymbol{\psi} = \boldsymbol{\psi}^*$.

Note that $g_3,\dots,g_{2n}$ in (5) do not depend on x_{2n-1} and x_{2n}. Hence

$$\frac{d\psi_{2n-1}}{dt} = \frac{d\psi_{2n}}{dt} = 0$$

and taking into account the transversality conditions (9) we assume that

$$\psi_{2n-1}(t) + i\psi_{2n}(t) = \xi_{2n-1} + i\xi_{2n} = \lambda_n.$$

Denote $\boldsymbol{\xi} = (\xi_3,\dots,\xi_{2n-2})$. In particular, at $t=0$ we have

$$H(0,\mathbf{0},\boldsymbol{\xi},u) = -2\sum_{k=2}^{n}(\xi_{2k-1}\cos(k-1)u - \xi_{2k}\sin(k-1)u).$$

Let $\boldsymbol{\lambda} = (\lambda_2,\dots,\lambda_n) \in \mathbf{R}^{n-1}$. Since M is close to 1 and the functions $-(\partial H/\partial x_k)$ in the righthand side of (7) are bounded for $0 \le t \le \log M$, $\boldsymbol{\psi}^*(0)$ is close to $\boldsymbol{\psi}^*(\log M)$ and $H(t,\boldsymbol{a}^*,\boldsymbol{\psi}^*,u)$ is close to $H(0,\mathbf{0},\boldsymbol{\xi},u) = -2\sum_{k=2}^{n}\lambda_k\cos(k-1)u$. According to (1) $H_u(0,\mathbf{0},\boldsymbol{\xi},u)$ has only single zeros on $[0,2\pi]$ and this property is preserved for $H_u(t,\boldsymbol{a}^*,\boldsymbol{\psi}^*,u)$ which means that $H_{uu}(t,\boldsymbol{a}^*,\boldsymbol{\psi}^*,u^*) < 0$.

The last assertion guarantees that the control function u in the righthand side of (5) and (7) is the analytic branch of the implicit function $u = u(t, \boldsymbol{a}, \boldsymbol{\psi})$ determined by the equation (10) with the initial value $u(0, \mathbf{0}, \boldsymbol{\xi}^*) = u^*(0)$, $\boldsymbol{\xi}^* = \boldsymbol{\psi}^*(0)$. Indeed, this follows from the analytical properties of the Hamilton function H and the inequality $H_{uu}(t, \boldsymbol{a}, \boldsymbol{\psi}, u) \neq 0$ which holds in a neighborhood of $(t, \boldsymbol{a}, \boldsymbol{\psi}, u) = (0, \mathbf{0}, \boldsymbol{\xi}^*, u^*(0))$.

Vectors $\boldsymbol{a}$ and $\boldsymbol{\psi}$, being the solution of the systems (5) and (7) with $u = u(t, \boldsymbol{a}, \boldsymbol{\psi})$ in their righthand sides, depend only on t and $\boldsymbol{\xi}$, i.e., $\boldsymbol{a} = \boldsymbol{a}(t, \boldsymbol{\xi})$ and $\boldsymbol{\psi} = \boldsymbol{\psi}(t, \boldsymbol{\xi})$.

Denote

$$u(t, \boldsymbol{\xi}) = u(t, \boldsymbol{a}(t, \boldsymbol{\xi}), \boldsymbol{\psi}(t, \boldsymbol{\xi})).$$

Lemma A. *Let $u = u(t, \boldsymbol{\xi})$ and $H(t, \boldsymbol{a}, \boldsymbol{\psi}, u)$ be the Hamilton function (6). Then $|H_{uu}(0, \mathbf{0}, \boldsymbol{\xi}, u)| \geq \delta > 0$ in a neighborhood of $\boldsymbol{\xi}^0 = (\lambda_2, 0, \dots, \lambda_{n-1}, 0)$.*

Lemma A was proved in [**3**] for a partial case corresponding to the nonlinear functional $I(f) = \Re(a_2 a_n)$. Therefore we will give here only a sketch of the proof which is very close to that of [**3**].

Since $H_u(0, \mathbf{0}, \boldsymbol{\xi}, u(0, \boldsymbol{\xi}))$ has a single zero at $u(0, \boldsymbol{\xi}^0)$,

$$r(\boldsymbol{\xi}) = H_{uu}(0, \mathbf{0}, \boldsymbol{\xi}, u(0, \boldsymbol{\xi})) < 0$$

in a neighborhood of $\boldsymbol{\xi}^0$. After differentiating $r(\boldsymbol{\xi})$ and the equation (10) at $t = 0$, we obtain

$$r'(\boldsymbol{\xi}) = H_{uu\boldsymbol{\psi}}(0, \mathbf{0}, \boldsymbol{\xi}, u) - H_{u\boldsymbol{\psi}}(0, \mathbf{0}, \boldsymbol{\xi}, u) H_{uuu}(0, \mathbf{0}, \boldsymbol{\xi}, u)/r(\boldsymbol{\xi}).$$

Due to boundedness of partial derivatives of $H(0, \mathbf{0}, \boldsymbol{\xi}, u)$ in a neighborhood of $\boldsymbol{\xi}^0$, the derivative $r'_{\mathbf{e}}(\boldsymbol{\xi})$ for any direction $\mathbf{e}$ satisfies

$$|r'_{\mathbf{e}}(\boldsymbol{\xi})| \leq \frac{A}{|r(\boldsymbol{\xi})|} + B \tag{11}$$

for some positive numbers A and B. Let l be the smallest number such that $r(\boldsymbol{\xi}) = r(\boldsymbol{\xi}^0)/2$ for a certain $\boldsymbol{\xi}$, $\|\boldsymbol{\xi} - \boldsymbol{\xi}^0\| = l$, i.e., $|r(\boldsymbol{\xi})| \geq$

$|r(\boldsymbol{\xi}^0)|/2 = \delta$, if $\|\boldsymbol{\xi} - \boldsymbol{\xi}^0\| \le l$. Integrating the differential inequality (11) from $\boldsymbol{\xi}^0$ to $\boldsymbol{\xi}$ in the direction $\mathbf{e} = \boldsymbol{\xi} - \boldsymbol{\xi}^0$, we obtain

$$|r(\boldsymbol{\xi}) - r(\boldsymbol{\xi}^0)| = |r(\boldsymbol{\xi}^0)|/2 \le \left(\frac{2A}{|r(\boldsymbol{\xi}^0)|} + B\right) l,$$

which gives a lower bound for l and completes the proof of Lemma A.

Lemma B [**3**]. *Let* $|H_{uu}(0, \mathbf{0}, \boldsymbol{\xi}, u(0, \boldsymbol{\xi}))| \ge \delta > 0$ *for all* $\boldsymbol{\xi}$, $\|\boldsymbol{\xi} - \boldsymbol{\xi}^0\| \le l$. *Then there exists* $M > 1$ *such that the inequality*

$$|H_{uu}(t, \boldsymbol{a}(t, \boldsymbol{\xi}), \boldsymbol{\psi}(t, \boldsymbol{\xi}), u(t, \boldsymbol{\xi}))| \ge \delta/2$$

holds for all $t \in [0, \log M]$.

Lemma C [**3**]. *Let* $u = u(t, \boldsymbol{\xi})$. *The partial derivatives* u_t *and* $u_{\boldsymbol{\xi}}$ *are bounded if* $\boldsymbol{\xi}$ *is close to* $\boldsymbol{\xi}^0$ *and* t *is close to* 0.

3. Proof of Theorem 1.

Proof of Theorem 1. First we show that there exists a unique point $\boldsymbol{\xi}$ in a neighborhood of $\boldsymbol{\xi}^0$ for which the solution of the systems (5) and (7) satisfies the maximum principle (8) and the transversality condition (9).

Let us consider the mapping

$$\mathbf{F} : \boldsymbol{\xi} \longrightarrow (\psi_3(\log M, \boldsymbol{\xi}), \dots, \psi_{2n-2}(\log M, \boldsymbol{\xi})), \quad \|\boldsymbol{\xi} - \boldsymbol{\xi}^0\| \le l.$$

The function $\mathbf{F}(\boldsymbol{\xi})$ maps the initial data $\boldsymbol{\xi}$ onto the solution of the Cauchy problem (7) for $t = \log M$. Hence $\mathbf{F}$ is an analytic function and its derivative $\mathbf{F}_{\boldsymbol{\xi}}$ is the Jacobi matrix $A(t, \boldsymbol{\xi})$ with the elements

$$a_{jk} = \frac{\partial \psi_j(\log M, \boldsymbol{\xi})}{\partial \psi_k}, \quad j, k = 3, \dots, 2n-2.$$

Clearly, $A(0, \boldsymbol{\xi}^0)$ is the unit matrix. Hence $\det A(\log M, \boldsymbol{\xi}^0) > 0$ if $(M-1)$ is small enough. This means that the matrix $A(\log M, \boldsymbol{\xi}^0) =$

$\mathbf{F}_{\boldsymbol{\xi}}(\boldsymbol{\xi}^0)$ is invertible and $\mathbf{F}$ maps a neighborhood $U_\varepsilon(\boldsymbol{\xi}^0) = \{\boldsymbol{\xi} : \|\boldsymbol{\xi} - \boldsymbol{\xi}^0\| < \varepsilon\}$, $\varepsilon > 0$, of $\boldsymbol{\xi}^0$ one-to-one onto a neighborhood of $\mathbf{F}(\boldsymbol{\xi}^0)$. Therefore there exists a unique $\boldsymbol{\xi} \in U_\varepsilon(\boldsymbol{\xi}^0)$ for which the maximum principle (8) and the transversality condition (9) are satisfied.

Second, suppose to the contrary that the extremal function $f^*(z) \in S^1(M)$ for the functional L is different from $P_M(z)$ and $-P_M(-z)$. This means that f^* maps D onto D_M slit along an analytic curve which is nonsymmetrical with respect to the real axis. Hence $f^{**}(z) = \overline{f^*(\bar{z})}$ is different from $f^*(z)$. As soon as real parts of coefficients of f^* and f^{**} are equal, both of them are extremal for L.

Let, according to Theorem A, the functions f^* and f^{**} be represented as

$$f^*(z) = Mw^*(z, \log M), \quad f^{**}(z) = Mw^{**}(z, \log M),$$

where $w^*(z,t)$ and $w^{**}(z,t)$ are the solutions of the Loewner differential equation (2) with $u = u^*(t)$ and $u = u^{**}(t)$ respectively.

Let $w^*(z,t)$ correspond to $\boldsymbol{a} = \boldsymbol{a}^*(t) = (x_3^*(t), \dots, x_{2n}^*(t))$, $u = u^*(t)$ and $\boldsymbol{\psi} = \boldsymbol{\psi}^*(t) = (\psi_3^*(t), \dots, \psi_{2n}^*(t))$ in (5), (7), $0 \le t \le \log M$. Then $w^{**}(z,t)$ corresponds to $\boldsymbol{a} = \boldsymbol{a}^{**}(t) = (x_3^*(t), -x_4^*(t), \dots, x_{2n-1}^*(t), -x_{2n}^*(t))$, $u = u^{**}(t) = -u^*(t)$ and $\boldsymbol{\psi} = \boldsymbol{\psi}^{**}(t) = (\psi_3^*(t), -\psi_4^*(t), \dots, \psi_{2n-1}^*(t), -\psi_{2n}^*(t))$ which implies that f^* and f^{**} correspond to the distinct data values $\boldsymbol{\xi}^* = (\xi_3^*, \xi_4^*, \dots, \xi_{2n-3}^*, \xi_{2n-2}^*)$ and $\boldsymbol{\xi}^{**} = (\xi_3^*, -\xi_4^*, \dots, \xi_{2n-3}^*, -\xi_{2n-2}^*)$ respectively.

But the transversality condition (9) means that

$$\begin{aligned}\boldsymbol{\psi}(\log M, \boldsymbol{\xi}^*) &= \boldsymbol{\psi}^*(\log M) = \boldsymbol{\psi}^{**}(\log M) \\ &= \boldsymbol{\psi}(\log M, \boldsymbol{\xi}^{**}) = (\lambda_2, 0, \dots, \lambda_n, 0).\end{aligned}$$

If $(M-1)$ is small enough, then $\boldsymbol{\xi}^*$ and $\boldsymbol{\xi}^{**}$ are close to $\boldsymbol{\xi}^0$ and belong to a neighborhood $U_\varepsilon(\boldsymbol{\xi}^0)$ of $\boldsymbol{\xi}^0$ where $\mathbf{F}$ has an inverse mapping $\mathbf{F}^{-1}$. This contradicts the statement that $\mathbf{F}(\boldsymbol{\xi}^*) = \mathbf{F}(\boldsymbol{\xi}^{**})$ and ends the proof of Theorem 1.

Remark 3. Requirement of Theorem 1 that the trigonometrical polynomial $\sum_{k=2}^n (k-1)\lambda_k \sin(k-1)u$ has only single zeros on $[0, 2\pi]$ can be weakened. Indeed, we need only the singleness of zeros which are maximum points of $H(0, \mathbf{0}, \boldsymbol{\xi}^0, u)$.

4. Application to estimates for $\Re(a_n+\alpha a_m)$. Schiffer and Tammi [**7**] and Siewierski [**6**] showed that the Pick functions are not extremal for $\max_{f\in S(M)} \Re a_n$ if $n > 2$ and $(M-1)$ is small. More precisely, they showed that there exists $M_n > 1$ such that $\Re a_n$ is maximized in $S(M)$ by the function

$$P_{M,n}(z) = [P_{M^{n-1}}(z^{n-1})]^{1/(n-1)} \in S(M)$$

for all $M \in (1, M_n)$.

Given $\alpha > 0$ and even m and n such that $(m-1)$ and $(n-1)$ are relatively prime, $n > m \geq 2$, consider the linear functional

$$L(f) = a_n + \alpha a_m$$

and the extremal problem

$$\Re L(f) \longrightarrow \max, \quad f \in S(M). \tag{12}$$

According to the Pontryagin maximum principle (8) and the transversality condition (9), the Hamilton function $H(t, \boldsymbol{a}(t,\boldsymbol{\xi}), \boldsymbol{\psi}(t,\boldsymbol{\xi}), u)$ is close to

$$H(0, \mathbf{0}, \boldsymbol{\xi}^0, u) = q(u) = -2(\cos(n-1)u + \alpha\cos(m-1)u)$$

if $\boldsymbol{\xi}$ is close to $\boldsymbol{\xi}^0 = (0, \dots, 0, \alpha, 0, \dots, 0, 1, 0)$ and $(M-1)$ is small.

Since $q(u)$ has only one absolute maximum in $[0, 2\pi]$ at $u = \pi$ and $q''(\pi) < 0$, the Hamilton function also has only one absolute maximum in $[0, 2\pi]$ at $u(t, \boldsymbol{\xi})$. This means that an extremal function f^* of the problem (12) belongs to $S^1(M)$ and all the conditions of Theorem 1 are satisfied.

So we proved

Theorem 2. *Given $\alpha > 0$ and even m and n such that $(m-1)$ and $(n-1)$ are relatively prime, there exists $M(m,n,\alpha) > 1$ such that the Pick function $P_M(z)$ is extremal for the problem* (12) *for all $M \in (1, M(m,n,\alpha))$.*

It is proved in [**2**] that the Pick function $-P_M(-z)$ is extremal for the nonlinear problem $\Re(a_m a_n) \to \max$ in $S(M)$ if $(m-1)$ and $(n-1)$ are relatively prime and $(M-1)$ is small enough.

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