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ON THE H-POLYNOMIAL OF CERTAIN MONOMIAL CURVES

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ABSTRACT. Let $n_1, \dots, n_e$ be an increasing sequence of positive integers with $\gcd(n_1, \dots, n_e) = 1$ and let A be the coordinate ring of the algebroid monomial curve in the affine algebroid e-space $\mathbf{A}_K^e$ over a field K, defined parametrically by $X_1 = t^{n_1}, \dots, X_e = t^{n_e}$. In this article assuming that some $e-1$ terms of $n_1, \dots, n_e$ form an arithmetic sequence, we compute (under some mild additional assumptions, see Theorem (2.7) for more precise assumptions) the h-polynomial (and hence the Hilbert function) of A explicitly in terms of the standard basis of the semi-group generated by $n_1, \dots, n_e$. Our special assumptions are satisfied in the case $e = 3$; in particular, for the class of algebroid monomial space curves, we can write down the h-polynomial and hence the Hilbert function explicitly.

1. Introduction. Let $(A, \mathfrak{m})$ be Noetherian local ring, and let $G := \mathrm{gr}_m(A) = \underset{i\geq 0}{\oplus} m^i/m^{i+1}$ be the associated graded ring of A. The Hilbert function $H_A : \mathbf{N} \to \mathbf{N}$ of A is the numerical function defined by $H_A(n) := \dim_{A/\mathfrak{m}}(\mathfrak{m}^n/\mathfrak{m}^{n+1})$. The Poincaré series of A is the series $P_A(Z) := \sum_{n\geq 0} H_A(n)Z^n$. By the Hilbert-Serre theorem, there exists a polynomial $h_A(Z) = \sum_{j=0}^{\deg h_A} h_j Z^j$ such that $P_A(Z) = h_A(Z)/(1-Z)^{\dim A}$. Then $h_0 = 1$, $h_1 = \mathrm{emdim}\,(A) := \dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$. The polynomial $h_A(Z)$ is called the h-polynomial of A and the vector $(h_0, h_1, \dots, h_{\deg h_A})$ is called the h-vector of A. It is clear that the h-vector of A and the Krull dimension of A determine the Hilbert function of A and conversely. Since the Hilbert function H_A of A is a good measure of singularity of the affine scheme Spec (A) at the closed point $\mathfrak{m}$, it is important to compute the Hilbert function, Poincaré series, h-vector, h-polynomial and its degree explicitly. These invariants are studied by many authors in the standard literature on local rings and still many interesting questions regarding these invariants are open in general (see, for example, [**1–3, 5, 6, 10–12**]).

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In this article we assume that A is the coordinate ring of an algebroid monomial curve in the affine e-space $\mathbf{A}_K^e$ over a field K, defined parametrically by $X_0 = T^{m_0}, \ldots, X_{e-1} = T^{m_{e-1}}$ with $0 < m_0 < m_1 < \cdots < m_{e-1}$, $\gcd(m_0, \ldots, m_{e-1}) = 1$ and the sequence $m_0, \ldots, m_{e-1}$ is an almost arithmetic sequence, that is, $m_0 < \cdots < m_p, \quad p := e-2$ is an arithmetic sequence and $n := m_{e-1}$ is arbitrary. In the case when the associated graded ring $\mathrm{gr}_m(A) := \underset{i\geq 0}{\oplus} m^i/m^{t+1}$ of A is not Cohen-Macaulay, in [12; Corollary 2.7], it is proved that the h-polynomial has non-negative coefficients. In this article, we assume $\mathrm{gr}_m(A)$ is Cohen-Macaulay and $m_0 < n$, $\mu \neq 0$ (see 2.3 for definition of μ) to give an algorithmic method to write down the h-polynomial of A explicitly. In the special case when the sequence $m_0, \cdots, m_{e-1}$ is an arithmetic sequence then $\mathrm{gr}_{\mathfrak{m}}(A)$ is always Cohen-Macaulay (see [**5**, Proposition (1.1)]) and the h-polynomial of A is written down explicitly (see [**5**, Corollary (1.10)]).

Our algorithmic method involves the nonnegative integers $\lambda, \mu, \nu, u, v, z, w$ which were defined in [**9**], using the explicit description of the standard basis S_{m_0} of the semi-group $\Gamma := \sum_{i=0}^{e-2} \mathbf{N}m_i + \mathbf{N}n$. Given integers $m_0, \ldots, m_p, n$, it is easy to find the nonnegative integers $\lambda, \mu, \nu, u, v, z, w$. The explicit description of S_{m_0} and the properties of the nonnegative integers $\lambda, \mu, \nu, u, v, z, w$ were used to find explicit minimal sets of generators for the relation ideal and the derivation module of A (see [**7**] and [**8**]). Further, in [**4**], the properties of the nonnegative integers $\lambda, \mu, \nu, u, v, z, w$ were used to give, in most cases, necessary and sufficient conditions for $\mathrm{gr}_{\mathfrak{m}}(A)$ to be Cohen-Macaulay (this condition is just one inequality which involves the nonnegative integers $\lambda, \mu, \nu, u, z, w$ and which is easy to check).

For the case of monomial space curves, we can write down, in most cases, the h-vector of A explicitly, since any three integers are in almost arithmetic sequence with $m_0 < m_1 < n$. We also give many examples to illustrate our algorithmic method.

2. Standard basis. In this section we recall the explicit description of the standard basis of a numerical semi-group generated by an almost arithmetic sequence given in [**9**, Section 3] (see also [**7**]). First we fix the following notations throughout this paper.

2.1 Notation. Let $\mathbf{Z}$, respectively $\mathbf{N}$, denote the set of all, respectively nonnegative, integers. For $a, b \in \mathbf{Z}$, let $[a, b] := \{i \in \mathbf{Z} \mid a \leq i \leq b\}$. Unless mentioned otherwise, the symbols $a, b, c, d, e, i, j, m, n, p, q, r, s, t, u, v, w, z$ denote integers.

Let $m_0, \ldots, m_{e-1}$ be a sequence of positive integers with $m_0 < \cdots < m_{e-1}$ and $\gcd(m_0, \ldots, m_{e-1}) = 1$. We assume that $m_0, \ldots, m_{e-1}$ is a minimal set of generators for the semi-group $\Gamma := \sum_{i=0}^{e-1} \mathbf{N}_{m_i}$.

- Let $\mathcal{E} := (\mathbf{N})^e$ and for $i \in [0, e-1]$, we put $\mathbf{e}_i := (\delta_{ij})_{0 \leq j \leq e-1}$, where δ_{ij} denote the Kronecker delta.
- For $\alpha = \sum_{i=0}^{e-1} a_i \mathbf{e}_i$, let $\partial(\alpha) := \sum_{i=0}^{e-1} a_i m_i$.
- For $h \in \Gamma$, let $\mathcal{E}(h) := \{\alpha \in \mathcal{E} \mid \partial(\alpha) = h\}$.
- For $\alpha = \sum_{i=0}^{e-1} a_i \mathbf{e}_i \in \mathcal{E}(h)$, we put $\deg(\alpha) := \sum_{i=0}^{e-1} a_i$.
- For $\alpha, \beta \in \mathcal{E}(h)$, we write $\alpha \leq_{\deg} \beta$ if $\deg(\alpha) \leq \deg(\beta)$.

Then $\leq_{\deg}$ is an order on $\mathcal{E}(h)$ and since $\mathcal{E}(h)$ is a finite set, $\mathcal{E}(h)$ has maximal elements with respect to the order $\leq_{\deg}$. Let $\max(\mathcal{E}(h))$ denote the set of all maximal elements in $\mathcal{E}(h)$. Note that all elements of $\max(\mathcal{E}(h))$ have the same degree, therefore this degree we shall denote by $\max\deg(h)$.

2.2 Standard basis. Let Γ be a numerical semi-group generated by a sequence $m_0, m_1, \ldots, m_{e-1}$ of positive integers. Then the set $S_{m_0} := \{z \in \Gamma \mid z - m_0 \notin \Gamma\}$ is called the standard basis or the Apèry set of Γ with respect to m_0. It is clear that S_{m_0} depends on Γ and m_0, but for simplicity we write $S := S_{m_0}$. It is easy to see that $S = \{s_0 := 0, s_1, \ldots, s_{m_0-1}\}$, where $s_1, \ldots, s_{m_0-1} \in \Gamma$ are positive integers with the following properties:

(a) $s_i \equiv i \pmod{m_0}$ for every $i \in [0, m_0 - 1]$

(b) If $z \in \Gamma$ then $z \equiv i \pmod{m_0}$ for a unique $i \in [0, m_0 - 1]$ and $z \geq s_i$.

The following Key-Lemma from [**9**, Section 3] (see also [**7**, Section 1]) gives the explicit description of the standard basis of a semi-group generated by an almost arithmetic sequence.

2.3 Key-Lemma. *Let $p := e - 2$ and let d be a positive integer with $m_i = m_0 + id$ for all $0 \le i \le p$. Let n be an arbitrary positive integer with* $\gcd(m_0, d, n) = 1$*. Let $\Gamma' := \sum_{i=0}^{p} \mathbf{N}_{m_i}$ and $\Gamma = \Gamma' + \mathbf{N}n$. Let $S := S_{m_0}$ be the standard basis of Γ with respect to m_0. For $t \in \mathbf{N}$, let $q_t \in \mathbf{Z}$, $r_t \in [1, p]$ and $g_t \in \Gamma'$ be defined by $t = q_t p + r_t$ and $g_t = q_t m_p + m_{r_t}$.*

(1) *$g_s + g_t = \varepsilon m_0 + g_{s+t}$ with $\varepsilon = 1$ or 0 according to whether $r_s + r_t \le p$ or $r_s + r_t > p$.*

(2) *Let $u := \min\{t \in \mathbf{N} \mid g_t \notin S\}$ and $v := \min\{b \ge 1 \mid bn \in \Gamma'\}$. Then unique integers $w \in [0, v-1]$, $z \in [0, u-1]$, $\lambda \ge 1$, $\mu \ge 0$, exist such that*

(i) $g_u = \lambda m_0 + w_n$;

(ii) $vn = \mu m_0 + g_z$;

(iii) *$g_{u-z} + (v - w)n = \nu m_0$. Moreover, $\nu = \lambda + \mu + \varepsilon$ where $\varepsilon = 1$ or 0 according to whether $r_{u-z} < r_u$ or $r_{u-z} \ge r_u$.*

(3) *Let $V := [0, u-1] \times [0, v-1]$, $W := [u-z, u-1] \times [v-w, v-1]$ and $U := V \setminus W$. Then $S = \{g_s + bn \mid (s, b) \in U\}$. In particular, if $(s, b), (t, c) \in U$ with $g_s + bn \equiv g_t + cn \pmod{m_0}$, then $(s, b) = (t, c)$.*

(4) *Every element of Γ can be expressed uniquely in the form $am_0 + g_s + bn$ with $a \in \mathbf{N}$ and $(s, b) \in U$.*

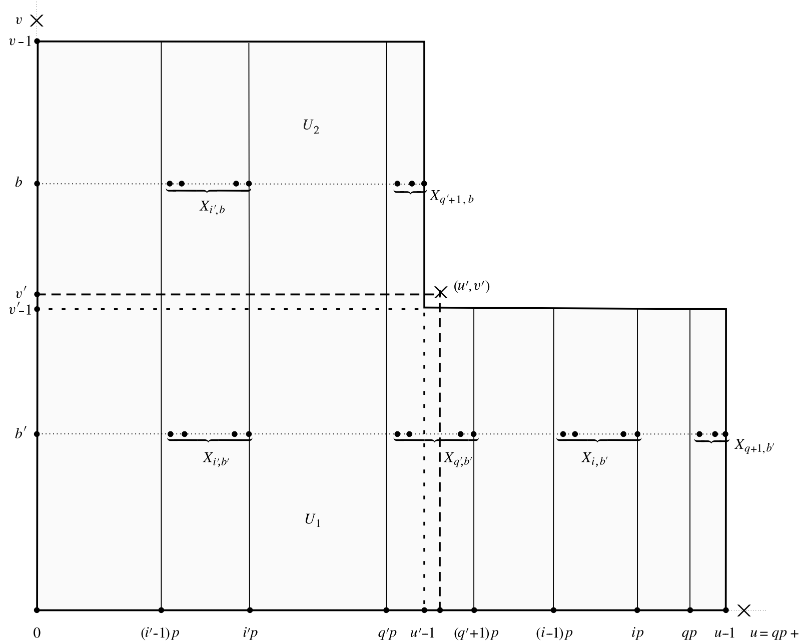
(5) *The map $(\mathbf{N})^{p+2} \to (\mathbf{N})^2$ defined by $\sum_{i=0}^{p+1} a_i \mathbf{e}_i \mapsto (\sum_{i=0}^{p} a_i m_i, a_{p+1} n)$ is a bijection between S and U.*

Proof. See [**9**, Section 3]. □

2.4 Notation. In addition to the notation in 2.1 and in Key-Lemma 2.3, we fix the following:

- $q := q_u, r := r_u$, that is, $u = qp + r$, $u' := u - z$, $q' := q_{u'}$, $r' := r_{u'}$, that is, $u' = q'p + r'$, and $v' := v - w$.
- $U_1 := \{(s, b) \in U \mid b \in [0, v' - 1]\}$ and $U_2 := \{(s, b) \in U \mid b \in [v', v-1]\}$.
- For $j \in \mathbf{N}$, let $Z_j := \{(s, b) \in U \mid q_s + 1 + b = j\}$ and let $Z_{1,j} := Z_j \cap U_1$, $Z_{2,j} := Z_j \cap U_2$,
- For $(i, b) \in \mathbf{Z} \times \mathbf{N}$, let $X_{i,b} := U \cap ([(i-1)p + 1, ip] \times \{b\})$.

2.5 The picture of U. The following picture of U (see 2.3) might be useful for computations or proofs:



With the notation in 2.3, 2.4 and using the above picture of U, the following two lemmas are immediate from the definitions.

2.6 Lemma. *Let* $(i,b) \in \mathbf{N} \times \mathbf{N}$. *Then*

(1) *If either* $b \geq v'$ *or* $i \geq q+2$, *then* $X_{i,b} \cap U_1 = \varnothing$.

(2) *If either* $b \geq v$ *or* $i \geq q'+2$, *then* $X_{i,b} \cap U_2 = \varnothing$.

(3) *If* $b \leq v'-1$, *then*

$$X_{i,b} \cap U_1 = \begin{cases} \{(0,b)\} & \text{if } i = 0, \\ [(i-1)p+1, ip] \times \{b\} & \text{if } 1 \leq i \leq q, \\ [qp+1, qp+r-1] \times \{b\} & \text{if } i = q+1, \\ \varnothing & \text{if } i \geq q+2. \end{cases}$$

In particular,

$$\operatorname{card}(X_{i,b}\cap U_1)=\begin{cases}1 & \text{if } i=0,\\ p & \text{if } 1\le i\le q,\\ r-1 & \text{if } i=q+1,\\ 0 & \text{if } i\ge q+2.\end{cases}$$

(4) *If* $v'\le b\le v-1$, *then*

$$X_{i,b}\cap U_2=\begin{cases}\{(0,b)\} & \text{if } i=0,\\ [(i-1)p+1,ip]\times\{b\} & \text{if } 1\le i\le q',\\ [q'p+1,q'p+r'-1]\times\{b\} & \text{if } i=q'+1,\\ \varnothing & \text{if } i\ge q'+2.\end{cases}$$

In particular,

$$\operatorname{card}(X_{i,b}\cap U_2)=\begin{cases}1 & \text{if } i=0,\\ p & \text{if } 1\le i\le q',\\ r'-1 & \text{if } i=q'+1,\\ 0 & \text{if } 1\ge q'+2.\end{cases}$$

2.7 *Lemma. Let* $j\in\mathbf{N}$. *Then*

(1) $Z_j=\cup_{b=0}^{v-1}X_{j-b,b}=\cup_{j-b=j-(v-1)}^{j}X_{j-b,b}$.

(2) $Z_{1,j}=\cup_{j-b=j-(v'-1)}^{j}X_{j-b,b}$ *and* $Z_{2,j}=\cup_{j-b=j-(v-1)}^{j-v'}X_{j-b,b}$.

(3) *If* $j\in[0,v-1]$, *then* $Z_{2,j}=\cup_{j-b=0}^{j-v'}X_{j-b,b}$.

(4) *If* $j\in[v,\infty)$, *then* $Z_{2,j}=\cup_{j-b=j-(v-1)\ge 1}^{j-v'}X_{j-b,b}$.

2.8 Theorem. *Let* K *be a field and let* $K[[T]]$ *be the power series ring. Let* $p,d,m\in\mathbf{N}^+$, $m_i=m+id$ *for* $i=0,\dots,p$ *and let* n *be an arbitrary positive integer with* $\gcd(m,d,n)=1$. *Let* $A:=K[[T^{m_0},\dots,T^{m_p},T^n]]\subseteq K[[T]]$, $\mathfrak{m}$ *the maximal ideal of* A *and let* $G:=gr_{\mathfrak{m}}(A)$ *be the associated graded ring of* A. *Let* $\tau_0,\dots,\tau_p$, τ *denote the images of* $T^{m_0},\dots,T^{m_p}$, T^n *in* G, *respectively, and let* $G':=G/(\tau_0)=\oplus_{j=0}^{t}G'_j$. *Suppose that* $m_0<n$, $\mu\neq 0$ (*see* 2.3),

and G is Cohen-Macaulay. Then $h_A(Z) = \sum_{j=0}^{t} \dim_K(G'_j)Z^j$ and $\dim_K(G'_j) = \text{card}\,(Z_j)$ for every $j \in \mathbf{N}$.

Proof. Since G is Cohen-Macaulay and $m_0 < n$ by assumption, by [**3**, Theorem 7], τ_0 is a nonzero divisor in G and hence G' is an Artinian reduction of G. In particular, $h_j = \dim_K(G'_j)$ for every $j \in \mathbf{N}$. Let $\bar{\tau}_1, \ldots, \bar{\tau}_p$, $\bar{\tau}$ denote the images of $\tau_1, \ldots, \tau_p$, τ in G'. Then for each $j \in \mathbf{N}$, G'_j is generated, as an $A/\mathfrak{m}$-vector space, by the set $\{\bar{\tau}_1^{a_1} \cdots \bar{\tau}_p^{a_p} \cdot \bar{\tau}^b \mid a_1m_1 + \cdots + a_pm_p + bn \in S$ and $T^{a_1m_1+\cdots+a_pm_p+b_n} \in \mathfrak{m}^j \setminus \mathfrak{m}^{j+1}\}$. In particular (see 2.3) $\dim_K(G'_j) = \text{card}\,(\{(s,b) \in U \mid \max\deg\,(g_s + bn) = j\})$ for every $j \in \mathbf{N}$. Now since $m_0 < n$, $\mu \neq 0$ and G is Cohen-Macaulay, by [**5**, Theorem (3.4)], we have $\lambda + w \geq q_u + 1$ and $v \leq \mu + q_z + 1$ (see 2.3 for definitions of $\lambda, \mu, u, v, w, z, q_u, q_z$) and so by [**4**, Proposition (3.2)] and the definition of Z_j (see 2.4), we have $\dim_K(G'_j) = \text{card}\,(Z_j)$ for every $j \in \mathbf{N}$. □

In the next section we shall give an algorithmic method to compute $\text{card}\,(Z_j)$, $j \in \mathbf{N}$, by using the nonnegative integers v, v', q and q'.

3. The h-polynomial. Let K be a field and let $K[[T]]$ be the power series ring. Let $p, d, m \in \mathbf{N}^+$, $m_i = m + id$ for $i = 0, \ldots, p$ and let n be an arbitrary positive integer with $m < n$ and $\gcd\,(m, d, n) = 1$. We assume that $m_0, \ldots, m_p, n$ is a minimal set of generators for the semi-group $\Gamma := \sum_{i=0}^{e-1} \mathbf{N}m_i$. We shall use the explicit description of the standard basis S_{m_0} of the semi-group $\Gamma := \sum_{i=0}^{p} \mathbf{N}m_i + \mathbf{N}n$ given in the Key-Lemma 2.3, particularly, the definitions (see 2.3) of the nonnegative integers $\lambda, \mu, \nu, u, v, z, w$.

Let $A := K[[T^{m_0}, \ldots, T^{m_p}, T^n]] \subseteq K[[T]]$, $\mathfrak{m}$ be the maximal ideal of A and let $G := \text{gr}_{\mathfrak{m}}(A)$ be the associated graded ring of A. Suppose that $m_0 < n$, $\mu \neq 0$ (see 2.3) and G is Cohen-Macaulay.

With all the above assumptions, in this section we shall compute $\deg h_A$ of the h-polynomial and its coefficients h_j, $0 \leq j \leq \deg h_A$, explicitly.

For convenience we shall subdivide $\mathbf{N}$ into the two intervals $J_1 := [0, v-1]$ and $J_2 := [v, \infty)$. In the proposition below, we shall compute h_j for $j \in J_1$. For this subdivide the interval J_1 into the following six disjoint subsets:

- $J_{11} := [0, q'] \cap J_1$.
- $J_{12} := [q'+1, q] \cap [0, v'+q'] \cap J_1$.
- $J_{13} := [q'+1, q] \cap [v'+q'+1, v-1] \cap J_1$.
- $J_{14} := [q+1, v-1] \cap [0, v'+q'] \cap J_1$.
- $J_{15} := [q+1, v-1] \cap [v'+q'+1, v'+q] \cap J_1$.
- $J_{16} := [q+1, v-1] \cap [v'+q+1, v-1] \cap J_1$.

With this we have:

3.1 Proposition. *Suppose that $j \in J_1$. Then*

(1) *If $j \in J_{11}$, then $h_j = jp+1$.*

(2) *If $j \in J_{12}$, then $h_j = jp+1$.*

(3) *If $j \in J_{13}$, then $h_j = (v'+q')q+r'$.*

(4) *If $j \in J_{14}$, then $h_j = qp+r$.*

(5) *If $j \in J_{15}$, then $h_j = (v'+q-j+q')p+r+r'-1$.*

(6) *If $j \in J_{16}$, then $h_j = q'p+r'$.*

Proof. (1) Since $0 \leq j \leq q' \leq q$, by 2.7 (1) we have $h_j = \text{card}\,(Z_j) = jp+1$.

(2) We consider the two cases $j \in [0, v'-1]$ and $j \in [v', v'+q']$ separately.

Case 1. $j \in [0, v'-1]$. In this case, since $q'+1 \leq j \leq q$ and $j-v' < 0$, by 2.7 (2) and (3) we have $\text{card}\,(Z_{1,j}) = jp+1$ and $\text{card}\,(Z_{2,j}) = 0$. Therefore, $h_j = \text{card}\,(Z_j) = jp+1$.

Case 2. $j \in [v', v'+q']$. In this case, since $v' \leq j \leq q$ and $j-v' \leq q'$, by 2.7 (2) and (3) we have $\text{card}\,(Z_{1,j}) = v'p$ and $\text{card}\,(Z_{2,j}) = (j-v')p+1$. Therefore, $h_j = \text{card}\,(Z_j) = jp+1$.

(3) Since $j \leq q$ and $0 \leq q'+1 \leq j-v'$, by 2.7 (2) and (3) we have $\text{card}\,(Z_{1,j}) = v'p$ and $\text{card}\,(Z_{2,j}) = q'p+r'$. Therefore, $h_j = \text{card}\,(Z_j) = (v'+q')p+r'$.

(4) We consider the two cases $j \in [0, v'-1]$ and $j \in [v', v'+q']$ separately.

Case 1. $j \in [0, v'-1]$. In this case since $q+1 \leq j$ and $j - v' < 0$, by 2.7 (2) and (3) we have $\text{card}\,(Z_{1,j}) = qp + r$ and $\text{card}\,(Z_{2,j}) = 0$. Therefore, $h_j = \text{card}\,(Z_j) = qp + r$.

Case 2. $j \in [v', v'+q']$. In this case, since $q+1 \leq j$ and $0 \leq j - v' \leq q'$, by 2.7 (2) and (3) we have $\text{card}\,(Z_{1,j}) = (q - j + v')p + r - 1$ and $\text{card}\,(Z_{2,j}) = (j - v')p + 1$. Therefore $h_j = \text{card}\,(Z_j) = qp + r$.

(5) Since $q + 1 \leq j$ and $0 \leq q' + 1 \leq j - v' \leq q$, by 2.7 (2) and (3) we have $\text{card}\,(Z_{1,j}) = (q - j + v')p + r - 1$ and $\text{card}\,(Z_{2,j}) = q'p + r'$. Therefore $h_j = \text{card}\,(Z_j) = (v' + q - j + q')p + r + r' - 1$.

(6) Since $0 \leq q' + 1 \leq q + 1 \leq j - v'$, by 2.7 (2) and (3) we have $\text{card}\,(Z_{1,j}) = 0$ and $\text{card}\,(Z_{2,j}) = q'p + r'$. Therefore $j_j = \text{card}\,(Z_j) = q'p + r'$. $\square$

Now to compute the coefficients h_j for $j \in J_2$, we subdivide J_2 into the two disjoint subsets $J_{21} := [v, \infty) \cap [0, q]$ and $J_{22} := [v, \infty) \cap [q+1, \infty)$. Further, for convenience we shall subdivide the set J_{21} into the following four disjoint subsets:

- $J_{211} := [v, q'] \cap J_{21}$.
- $J_{212} := (q', q] \cap [v, v' + q'] \cap J_{21}$.
- $J_{213} := (q', q] \cap [v' + q' + 1, v + q'] \cap J_{21}$.
- $J_{214} := (q', q] \cap [v + q' + 1, \infty) \cap J_{21}$.

With this we can now write down h_j, $j \in J_{21}$ in the proposition below

3.2 Proposition. *Suppose that* $j \in J_{21}$. *Then*

(1) *If* $j \in J_{211}$, *then* $h_j = vp$.

(2) *If* $j \in J_{212}$, *then* $h_j = vp$.

(3) *If* $j \in J_{213}$, *then* $h_j = (v + v' + q' - j)p + r' - 1$.

(4) *If* $j \in J_{214}$, *then* $h_j = v'p$.

Proof. Since $v' \le v \le j \le q$, by 2.7 (2) we have

$$\text{card}\,(Z_{1,j}) = v'p. \tag{3.2a}$$

(1) and (2) Since $j - v' \le q'$, $\text{card}\,(Z_{2,j}) = wp$ by 2.7 (4) and so $h_j = \text{card}\,(Z_j) = (v' + w)p = vp$ by (3.2a).

(3) Since $1 \le q' + 1 \le j - v'$, by 2.7 (4) we have $\text{card}\,(\mathbf{Z}_{2,j}) = (q' - j + v)p + r' - 1$ and so $h_j = \text{card}\,(Z_j) = (v + v' + q' - j)p + r' - 1$ by (3.2a).

(4) Since $j - (v - 1) \ge q' + 2$, $\text{card}\,(Z_{2,j}) = 0$ by 2.7 (4) and so $h_j = \text{card}\,(Z_j) = v'p$ by (3.2a). □

Now to compute the coefficients h_j for $j \in J_{22}$, we subdivide J_{22} into the two disjoint subsets $J_{221} := [v, v + q] \cap [q + 1, \infty)$ and $J_{222} := [v + q + 1, \infty) \cap [q + 1, \infty)$. Further, for convenience we shall subdivide the set J_{221} into the following five disjoint subsets:

- $J_{2211} := [v, v' + q'] \cap J_{221}$.
- $J_{2212} := [v' + q' + 1, v' + q] \cap [v' + q' + 1, v + q'] \cap J_{221}$.
- $J_{2213} := [v' + q' + 1, v' + q] \cap [v + q' + 1, v + q] \cap J_{221}$.
- $J_{2214} := [v' + q + 1, v + q] \cap [v' + q + 1, v + q'] \cap J_{221}$.
- $J_{2215} := [v' + q + 1, v + q]] \cap [v + q' + 1, v + q] \cap J_{221}$.

With this we have:

3.3 Proposition. *Suppose that $j \in J_{221}$. Then*

(1) *If $j \in J_{2211}$, then $h_j = (q - j + v)p + r - 1$.*

(2) *If $j \in J_{2212}$, then $h_j = (q + q' - 2j + v + v')p + r + r' - 2$.*

(3) *If $j \in J_{2213}$, then $h_j = (q - j + v')p + r - 1$.*

(4) *If $j \in J_{2214}$, then $h_j = (q' - j + v)p + r' - 1$.*

(5) *If $j \in J_{2215}$, then $h_j = 0$.*

Proof. (1) Since $1 \le j-(v'-1) \le q'+1 \le q+1 \le j$ and $1 \le j-v' \le q'$, by 2.7 (2) and (4) we have $\text{card}\,(Z_{1,j}) = (q - j + v')p + r - 1$ and $\text{card}\,(Z_{2,j}) = wp$. Therefore, $h_j = (q - j + v)p + r - 1$.

(2) Since $1 \leq j-(v'-1) \leq q+1 \leq j$ and $q \leq q'+1 \leq j-v'$, by 2.7 (2) and (4) we have card $(Z_{1,j}) = (q-j+v')p+r-1$ and card $(Z_{2,j}) = (q'-j+v)p+r'-1$. Therefore $h_j = (q+q'-2j+v+v')p+r+r'-2$.

(3) Since $1 \leq j-(v'-1) \leq q+1 \leq j$ and $j-(v-1) \geq q'+2$, by 2.7 (2) and (4) we have card $(Z_{1,j}) = (q-j+v')p+r-1$ and card $(Z_{2,j}) = 0$. Therefore $h_j = (q-j+v')p+r-1$.

(4) Since $j-(v'-1) \geq q+2$ and $1 \leq q'+1 \leq q+1 \leq j-v'$, by 2.7 (2) and (4) we have card $(Z_{1,j}) = 0$ and card $(Z_{2,j}) = (q'-j+v)p+r'-1$. Therefore $h_j = (q'-j+v)p+r'-1$.

(5) Since $j-(v'-1) \geq q+2$ and $j-(v-1) \geq q'+2$, by 2.7 (2) and (4) we have card $(Z_{1,j}) = 0$ and $(Z_{2,j}) = 0$. Therefore, $h_j = 0$. □

3.4 Proposition. *Suppose that $j \in J_{222}$. Then $h_j = 0$.*

Proof. Since $j-(v'-1) \geq j-v+1 \geq q+2 \geq q'+2$ by 2.7 (2) and (4) we have card $(Z_{1,j}) = 0$ and card $(Z_{2,j}) = 0$. Therefore, $h_j = 0$. □

3.5 Proposition. *Let $j \in \mathbf{N}$.*

(1) *$h_j \neq 0$ for all $j \in J_1 \cup J_{21}$.*

(2) *Suppose that $v' < v$.*

(a) *If $v+q' < v'+q$, then*

$$h_j = \begin{cases} 0 & \text{if } j > v'+q, \\ r-1 & \text{if } j = v'+q, \\ p+r+r'-2 & \text{if } j = v'+q-1 \text{ and} \\ & v+q' = v'+q-1, \\ p+r-1 & \text{if } j = v'+q-1 \text{ and} \\ & v+q' < v'+q-1. \end{cases}$$

(b) *If $v+q' \geq v'+q$, then*

$$h_j = \begin{cases} 0 & \textit{if } j > v+q', \\ r+r'-2 & \textit{if } j = v+q' \textit{ and} \\ & v+q' = v'+q, \\ r'-1 & \textit{if } j = v+q' \textit{ and} \\ & v+q' > v'+q, \\ (q-q'-w+2)p & \\ \ +r+r'-2 & \textit{if } j = v+q'-1 \textit{ and} \\ & v'+q'+1 \leq v+q'-1 \leq v'+q, \\ p+r'-1 & \textit{if } j = v+q'-1 \textit{ and} \\ & v'+q+1 \leq v+q'-1. \end{cases}$$

(3) *Suppose that $v' = v$. Then*

$$h_j = \begin{cases} 0 & \textit{if } j > v+q, \\ r-1 & \textit{if } j = v+q, \\ p+r-1 & \textit{if } j = v+q-1. \end{cases}$$

Proof. (1) is immediate from 3.1 and 3.2.

(2)(a) For $j \in \mathbf{N}$, $j \geq v'+q-1$, we have

$$j \in \begin{cases} J_{2215} & \text{if } j > v'+q, \\ J_{2213} & \text{if } j = v'+q, \\ J_{2212} & \text{if } j = v'+q-1 \text{ and} \\ & v+q' = v'+q-1, \\ J_{2213} & \text{if } j = v'+q-1 \text{ and} \\ & v+q' < v'+q-1. \end{cases}$$

Therefore the assertion follows from 3.3.

(b) For $j \in \mathbf{N}$, $j \geq v + q' - 1$, we have

$$j \in \begin{cases} J_{2215} & \text{if } j > v + q', \\ J_{2212} & \text{if } j = v + q' \text{ and } v + q' = v' + q, \\ J_{2214} & \text{if } j = v + q' \text{ and } v + q' > v' + q, \\ J_{2212} & \text{if } j = v + q' - 1 \text{ and} \\ & v' + q' + 1 \leq v + q' - 1 \leq v' + q, \\ J_{2214} & \text{if } j = v + q' - 1 \text{ and} \\ & v' + q + 1 \leq v + q' - 1. \end{cases}$$

Therefore the assertion follows from 3.3.

(3) Since $v' = v$, $w = 0$ and for $j \in \mathbf{N}$, $j \geq v + q - 1$, we have

$$j \in \begin{cases} J_{2215} & \text{if } j > v + q, \\ J_{2211} & \text{if } j = v + q \text{ and } q' = q, \\ J_{2213} & \text{if } j = v + q \text{ and } q' < q, \\ J_{2211} & \text{if } j = v + q - 1 \text{ and } q' \geq q - 1, \\ J_{2213} & \text{if } j = v + q - 1 \text{ and } q' < q - 1. \end{cases}$$

Therefore the assertion follows from 3.3. □

3.6 Corollary. *Let $p, d, m \in \mathbf{N}^+$, $m_i = m + id$ for $i = 0, \ldots, p$ and n any positive integer with* $\gcd(m, d, n) = 1$. *Let K be a field, $A := K[[T^{m_0}, \ldots, T^{m_p}, T^n]] \subseteq K[[T]]$ and let $\mathfrak{m}$ be the maximal ideal of A. Suppose that $m_0 < n$, $\mu \neq 0$ (see 2.3) and $\operatorname{gr}_{\mathfrak{m}}(A)$ is Cohen-Macaulay. Then the degree $\deg h_A$ of the h-polynomial is*

$$\deg_{h_A} = \begin{cases} v' + q - 1 & \text{if } v' < v,\ v + q' < v' + q \text{ and } r = 1, \\ v' + q & \text{if } v' < v,\ v + q' < v' + q \text{ and } r \neq 1, \\ \max\{q, v + q' - 1\} & \text{if } v' < v,\ v + q' = v' + q,\ r = 1 \text{ and } r' = 1, \\ \max\{q, v + q'\} & \text{if } v' < v,\ v + q' = v' + q \text{ and} \\ & \text{either } r \neq 1 \text{ or } r' \neq 1, \\ \max\{q, v + q' - 1\} & \text{if } v' < v,\ v + q' > v' + q \text{ and } r' = 1, \\ \max\{q, v + q'\} & \text{if } v' < v,\ v + q' > v' + q \text{ and } r' \neq 1, \\ v + q - 1 & \text{if } v' = v \text{ and } r = 1, \\ v + q & \text{if } v' = v \text{ and } r \neq 1. \end{cases}$$

Now we give examples to illustrate the use of 3.1, 3.2, 3.3, 3.4 and 3.6 to compute the h-polynomial and its degree. Note that in each of the following examples $\mathrm{gr}_{\mathfrak{m}}(A)$ is the Cohen-Macaulay (since in each of them, we have $m_0 < n$ and $\mu \neq 0$, we can use [**4**, Theorem (3.4)] and just need to verify the inequalities $\lambda + w \geq q + 1$ and $v \leq \mu + q_z + 1$).

3.7 Example (see [**5**, Corollary (1.10)]). Let a be an integer ≥ 2, $p, b \in \mathbf{N}$, $p \geq 1$, $b \in [0, p]$, $m := a(p+1) + b$, d an integer ≥ 1 with $\gcd(m, d) = 1$ and let $m_i := m + id$ for $i = 0, 1, \ldots, p+1$. Note that we are taking $n := m_{p=1}$. Then

$$u = p + 1, \quad \lambda = 1, \quad w = 1, \quad q = 1, \quad r = 1,$$

$$v = \begin{cases} a & \text{if } b = 0, \\ a + 1 & \text{if } b \neq 0, \end{cases}$$

$$\mu = d + a, \quad z = \begin{cases} 0 & \text{if } b = 0, \\ p + 1 - b & \text{if } b \neq 0, \end{cases},$$

$$v' = v - w = v - 1 = \begin{cases} a - 1 & \text{if } b = 0, \\ a & \text{if } b \neq 0, \end{cases}$$

$$u' = u - z = \begin{cases} u & \text{if } b = 0, \\ b & \text{if } b \neq 0, \end{cases}$$

$$q' = \begin{cases} 1 & \text{if } b = 0, \\ 0 & \text{if } b \neq 0, \end{cases} \qquad r' = \begin{cases} 1 & \text{if } b = 0, \\ b & \text{if } b \neq 0. \end{cases}$$

3.7.1 *Case*: $b = 0$

$J_1 = [0, a-1]$	$J_{11} = [0, 1]$	$J_{12} = \varnothing$	$J_{13} = \varnothing$	$J_{14} = [2, a-1]$	$J_{15} = \varnothing$	$J_{16} = \varnothing$
	$h_0 = 1$, $h_1 = p+1$	——	——	$h_j = p+1$, $2 \leq j \leq a-1$	——	——
$J_2 = [a, \infty)$						
$J_{21} = \varnothing$	$J_{211} = \varnothing$	$J_{212} = \varnothing$	$J_{213} = \varnothing$	$J_{214} = \varnothing$		
	——	——	——	——		
$J_{22} = [a, \infty)$	$J_{221} = [a, a+1]$					$J_{222} = [a+2, \infty)$
	$J_{2211} = \{a\}$	$J_{2212} = \varnothing$	$J_{2213} = \varnothing$	$J_{2214} = \{a+1\}$	$J_{2215} = \varnothing$	
	$h_a = p$	——	——	$h_{a+1} = 0$	——	$h_j = 0$, $j \geq a+2$
$h_A =$			$\deg h_A =$			
$1 + \sum_{j=1}^{a-1}(p+1)Z^j + pZ^a$			$a = \max\{q, v + q' - 1\}$			

3.7.2 *Case*: $b \neq 0$

$J_1 = [0, a]$	$J_{11} = \{0\}$	$J_{12} = \{1\}$	$J_{13} = \varnothing$	$J_{14} = [2, a]$	$J_{15} = \varnothing$	$J_{16} = \varnothing$
	$h_0 = 1$,	$h_1 = p+1$	——	$h_j = p+1$, $2 \leq j \leq a$	——	——
$J_2 = [a+1, \infty)$						
$J_{21} = \varnothing$	$J_{211} = \varnothing$	$J_{212} = \varnothing$	$J_{213} = \varnothing$	$J_{214} = \varnothing$		
	——	——	——	——		
$J_{22} = [a+1, \infty)$	$J_{221} = [a+1, a+2]$					$J_{222} = [a+3, \infty)$
	$J_{2211} = \varnothing$	$J_{2212} = \{a+1\}$	$J_{2213} = \varnothing$	$J_{2214} = \varnothing$	$J_{2215} = \{a+2\}$	
	——	$h_{a+1} = b-1$	——	——	$h_{a+2} = 0$	$h_j = 0$, $j \geq a+3$
$h_A =$			$\deg h_A =$			
$1 + \sum_{j=1}^{a}(p+1)Z^j + (b-1)Z^{a+1}$			$\begin{cases} a = \max\{q, v + q' - 1\}, & \text{if } b = 1, \\ a+1 = \max\{q, v + q'\}, & \text{if } b \geq 2 \end{cases}$.			

3.8 Example. Let a be an integer ≥ 2, $p \in \mathbf{N}$, $p \geq 1$, $m_i := 2a(2p{+}1)-p{+}i$ for $i = 0,1,\ldots,p$, and let $n := m_0{+}2p{+}1$. Then $u = 2p{+}1$, $\lambda = 2$, $w = 1$, $q = 2$, $r = 1$, $v = 2a$, $\mu = 2a$, $z = p$, $v' = v{-}w = v{-}1 = 2a{-}1$, $u' = u{-}z = p{+}1$, $q' = 1$, $r' = 1$.

<table>
<tr><td rowspan="2">$J_1 = [0, 2a{-}1]$</td><td>$J_{11} = [0,1]$</td><td>$J_{12} = \{2\}$</td><td>$J_{13} = \varnothing$</td><td>$J_{14} = [3, 2a{-}1]$</td><td>$J_{15} = \varnothing$</td><td>$J_{16} = \varnothing$</td></tr>
<tr><td>$h_0 = 1$,
$h_1 = p+1$</td><td>$h_2 = 2p+1$</td><td>——</td><td>$h_j = 2p+1$,
$3 \leq j \leq 2a-1$</td><td>——</td><td>——</td></tr>
<tr><td>$J_2 = [2a, \infty)$</td><td colspan="6"></td></tr>
<tr><td rowspan="2">$J_{21} = \varnothing$</td><td>$J_{211} = \varnothing$</td><td>$J_{212} = \varnothing$</td><td>$J_{213} = \varnothing$</td><td colspan="3">$J_{214} = \varnothing$</td></tr>
<tr><td>——</td><td>——</td><td>——</td><td colspan="3">——</td></tr>
<tr><td>$J_{22} = [2a, \infty)$</td><td colspan="5">$J_{221} = [2a, 2a+2]$</td><td>$J_{222} = [2a{+}3, \infty)$</td></tr>
<tr><td rowspan="2"></td><td>$J_{2211} = \{2a\}$</td><td>$J_{2212} = \{2a{+}1\}$</td><td>$J_{2213} = \varnothing$</td><td>$J_{2214} = \varnothing$</td><td>$J_{2215} = \{2a{+}2\}$</td><td></td></tr>
<tr><td>$h_a = 2p$</td><td>$h_{2a+1} = 0$</td><td>——</td><td>——</td><td>$h_{2a+2} = 0$</td><td>$h_j = 0$,
$j \geq 2a+3$</td></tr>
<tr><td colspan="3">$h_A =$</td><td colspan="4">$\deg h_A =$</td></tr>
<tr><td colspan="3">$1 + (p+1)Z + \sum_{j=2}^{2a-1}(2p+1)Z^j + 2pZ^{2a}$</td><td colspan="4">$2a = \max\{q, v+q'-1\}$</td></tr>
</table>

3.9 Example. Let $p, r, q, v \in \mathbf{N}$, $p \geq r \geq 1$, $v > q \geq 2$, $m_i := (v-1)(qp+r)+i+1$ for $i = 0, 1, \ldots, p$, and let $n := v(qp+r)+1$. Then $u = qp+r$, $\lambda = q$, $w = 1$, $\mu = v-q$, $z = qp+r-1$, $v' = v-w = v-1$, $u' = 1$, $q' = 0$, $r' = 1$.

$J_1=[0,v-1]$	$J_{11}=\{0\}$	$J_{12}=[1,q]$	$J_{13}=\varnothing$	$J_{14}=[q+1,v-1]$	$J_{15}=\varnothing$	$J_{16}=\varnothing$
	$h_0=1$	$h_j=jp+1$, $1\leq j\leq q$	——	$h_j=qp+r$, $q+1\leq j\leq v-1$	——	——
$J_2=[v,\infty)$						
$J_{21}=\varnothing$	$J_{211}=\varnothing$	$J_{212}=\varnothing$	$J_{213}=\varnothing$	$J_{214}=\varnothing$		
	——	——	——	——		
$J_{22}=[v,\infty)$	$J_{221}=[v,v+q]$					$J_{222}=[v+q+1,\infty)$
	$J_{2211}=\varnothing$	$J_{2212}=\{v\}$	$J_{2213}=[v+1,v'+q]$	$J_{2214}=\varnothing$	$J_{2215}=\{v+q\}$	
	——	$h_v=(q-1)p+r-1$	$h_j=(q-j+v')p+r-1$, $v+1\leq j\leq v'+q$	——	$h_{v+q}=0$	$h_j=0$, $j\geq q+3$
$h_A =$			$\deg h_A =$			
$1+\sum_{j=1}^{q}(jp+1)Z^j+\sum_{j=q+1}^{v-1}(qp+r)Z^j$ $+[(q-1)p+r-1]Z^v$ $+\sum_{j=v+1}^{v+q-1}[(q-j+v')p+r-1]Z^j$			$\begin{cases} v+q-2=v'+q-1, & \text{if } r=1, \\ v+q-1=v'+q, & \text{if } r\neq 1 \end{cases}$.			

3.10 Example. Let p, r, q, d, q'', r'' be positive integers with $p \geq r \geq r'' \geq 1$, $d := q''p{+}r''$ and $q > 3d$. Let $m := m_0 := (qp{+}r)d$ and $n := m{+}d{+}1$. Then $u = qp{+}r$, $\lambda = q{+}2$, $w = 0$, $v = d$, $\mu = d{-}q''{-}1$, $z = d{+}1$, $v' = v = d$, $u' = u{-}z = (q{-}q'')p{+}(r{-}r'')+1$, $q' = q-q''$, $r' = r-r''+1$.

$J_1=[0,d-1]$	$J_{11}=[0,d-1]$	$J_{12}=\varnothing$	$J_{13}=\varnothing$	$J_{14}=\varnothing$	$J_{15}=\varnothing$	$J_{16}=\varnothing$
	$h_j=jp+1$ $0\le j\le d-1$	——	——	——	——	——
$J_2=[d,\infty)$						
$J_{21}=[d,q]$	$J_{211}=[d,q']$	$J_{212}=(q',q]$	$J_{213}=\varnothing$	$J_{214}=\varnothing$		
	$hj=dp$, $d\le j\le q'$	$hj=dp$, $q'<j\le q$	——	——		
$J_{22}=$ $[q+1,\infty)$	$J_{221}=[q+1,d+q]$					$J_{222}=$ $[d+q+1,\infty)$
	$J_{2211}=[q+1,d+q']$	$J_{2212}=\varnothing$	$J_{2213}=[d+q'+1,d+q]$	$J_{2214}=\varnothing$	$J_{2215}=\varnothing$	
	$h_j=(q-j+d)p+r-1$ $q+1\le j\le d+q'$	——	$h_j=(q-j+d)p+r-1$, $d+q'+1\le j\le d+q$	——	——	$h_j=0$, $j\ge d+q+1$
$h_A=$			$\deg h_A=$			
$\sum_{j=0}^{d-1}(jp+1)Z^j+\sum_{j=d}^{q}(dp)Z^j$ $+\sum_{j=q+1}^{d+q'}[(q-j+d)p+(r-1)]Z^j$ $+\sum_{j=d+q'+1}^{d+q}[(q-j+d)p+(r-1)]Z^j$			$\begin{cases} d+q-1=v+q-1, & \text{if } r=1,\\ d+q=v+q, & \text{if } r\neq 1\end{cases}$.			

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