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THE ADMISSIBLE DISTURBANCE FOR DISCRETE NONLINEAR PERTURBED CONTROLLED SYSTEMS

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ABSTRACT. Consider the discrete perturbed controlled nonlinear system given by

$$\begin{cases} x^w(i+1) = Ax^w(i) + f(u_i+\alpha_i) \\ \qquad\qquad + g(v_i)\sum_{j=1}^{r}\beta_i^j h_j(x^w(i)) \quad i \geq 0, \\ x^w(0) = x_0 + \gamma \end{cases}$$

and the output function $y^w(i) = Cx^w(i)$, $i \geq 0$, where $w = (\gamma, (\alpha_i)_{i\geq 0}, (\beta_i)_{i\geq 0})$, is a disturbance which disturbs the system. The disturbance w is said to be ε-admissible if $\|y^w(i) - y^{(i)}\| \leq e$, for all $i \geq 0$, where $(y(i))_{i\geq 0}$ is the output signal corresponding to the uninfected controlled system. The set of all ε-admissible disturbances is the admissible set $\mathcal{E}(\varepsilon)$. The characterization of $\mathcal{E}(\varepsilon)$ is investigated and practical algorithms with numerical simulations are given. The admissible set $\overline{\mathcal{E}}(\varepsilon)$ for discrete delayed systems is also considered.

1. Introduction. The characterization of admissible sets have important application in the analysis and design of closed-loop systems with state and control constraints. During the control of a system we are always confronted with the presence of certain undesirable parameters that come from the natural relationship which exists between a system and its environment; let's mention as examples fires, transitory electric regimes, earthquakes, bacterial infecting, etc.

In order to face such problems, an important number of works have been developed, see [**1, 2, 4–10, 12**]. We contribute in this direction by exploring a technique which allows us to determine, among a class of disturbances which excite discrete nonlinear controlled systems, those which are ε-admissible.

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The linear uncontrolled perturbed systems have been dealt with by Rachik et al. in [**11**]. Bouyaghroumni et al. have also studied the bilinear case, see [**3**].

In this paper we consider the controlled perturbed nonlinear system defined by

(1)
$$\begin{cases} x^w(i+1) = Ax^w(i) + f(u_i + \alpha_i) + g(v_i)\sum_{j=1}^r \beta_i^j h_j(x^w(i)) & i \geq 0, \\ x^w(0) = x_0 + \gamma \end{cases}$$

the corresponding output signal is

$$y^w(i) = Cx^w(i), \quad i \geq 0 \tag{2}$$

where A and C are respectively $n \times n$ and $p \times n$ matrices; $x^w(i) \in \mathbf{R}^n$ is the state variable, $u_i \in \mathbf{R}^m$ and $v_i \in \mathbf{R}^q$ are the control variables, $f : \mathbf{R}^m \to \mathbf{R}^n$ and $h_j : \mathbf{R}^n \to \mathbf{R}^n$ are continuous functions and $g : \mathbf{R}^q \to \mathbf{R}$ is a given function. $w = (\gamma, (\alpha_i)_{i\geq 0}, (\beta_i)_{i\geq 0})$, where $\beta_i = (\beta_i^1, \beta_i^2, \dots, \beta_i^r)$, is an undesirable disturbance which infects the system because of its connections to the environment. The output signal corresponding to $\alpha = (\alpha_i)_{i\geq 0} = 0$, $\beta = (\beta_i)_{i\geq 0} = 0$ and $\gamma = 0$ is simply denoted by $(y(i))_{i\geq 0}$, i.e.,

$$y(i) = Cx(i), \quad i \geq 0 \tag{3}$$

where $(x(i))_{i\geq 0}$ is the uninfected state given by

$$\begin{cases} x(i+1) = Ax(i) + f(u_i) & i \geq 0, \\ x(0) = x_0 \in \mathbf{R}^n. \end{cases} \tag{4}$$

For physical considerations, we suppose that all the disturbances $\alpha = (\alpha_i)_{i\geq 0}$ and $\beta = (\beta_i)_{i\geq 0}$ susceptible of infecting our system have a finite life; consequently in all this work we suppose that

$$\alpha = (\alpha_i)_{i\geq 0} \in \mathcal{U}_m^I = \{(\gamma_i)_{i\geq 0}/\gamma_i \in \mathbf{R}^m \quad \text{and} \quad \gamma_i = 0,\ \forall\, i > I\}$$

and

$$\beta = (\beta_i)_{i\geq 0} \in \mathcal{U}_r^J = \{(\lambda_i)_{i\geq 0}/\lambda_i \in \mathbf{R}^r \quad \text{and} \quad \lambda_i = 0,\ \forall\, i > J\}$$

where I and J are, respectively, the life of disturbances $\alpha = (\alpha_i)_i$ and $\beta = (\beta_i)_i$. The ε-admissible set $\mathcal{E}(\varepsilon)$ defined by

$$\mathcal{E}(\varepsilon) = \{w = (\gamma, \alpha, \beta) \in \mathbf{R}^n \times \mathcal{U}_m^I \times \mathcal{U}_r^J / \|y^w(i) - y(i)\| \leq \varepsilon,\ \forall\, i \geq 0\}$$

The admissibility problem is also studied for discrete delayed systems. Simple examples and numerical simulations are presented.

Remark 1. As an example of disturbances with "finite life," we give the following motivation. Let's consider the temperature distributed in an industrial oven, see Figure I, whose simplified mathematical model is

$$\frac{\partial T}{\partial t}(x,t) = \alpha \frac{\partial^2 T}{\partial^2 x}(x,t) + \beta T(x,t) + \gamma u(t) T(x,t),\ \forall\, t \geq 0. \tag{5}$$

where $T(\cdot, t)$ is the temperature profile at time t. We suppose that the system is controlled via the flow of a liquid in an adequate metallic pipeline ($u(\cdot)$ is the variable control), $u(.)$ is applied to the system with the object to establish

$$y_u(t) = y_d(t), \quad \forall\, t \geq t_f$$

where $y_u(.)$ is the output function corresponding to the control $u(.)$ and $y_d(.)$ is the desired output.

The associated initial condition is supposed to be homogeneous

$$T(x, 0) = T_0(x), \quad \forall\, x \in [0, 1]$$

and the boundary condition is also homogeneous

$$T(0, t) = T(1, t) = 0, \quad \forall\, t \geq 0.$$

If we suppose that system (5) is thermically isolated, we should stop supervising the system as soon as we achieve our objective, i.e., at time t_f. Then, from instant t_f the evolution equation becomes

$$\frac{\partial T}{\partial t}(x,t) = \alpha \frac{\partial^2 T}{\partial^2 x}(x,t) + \beta T(x,t), \quad \forall\, t \geq t_f. \tag{6}$$

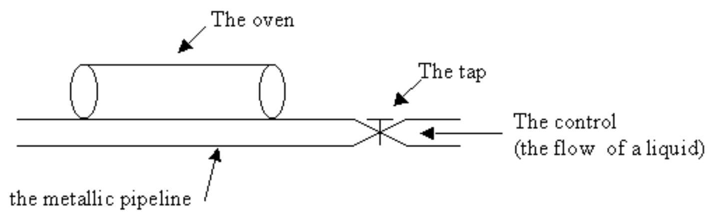

FIGURE I.

In reality we should by no means ignore that there are some disturbances "$e(t)$" which affect the system and which emanate essentially from

(i) The amount of heat preserved by the pipeline metal during a lap of the limited time $[t_f, t_1]$.

(ii) The delay "h existing between the stop control "$u(t)$" and its effects on the system.

So the evolution equation of the system can be written as follows

$$\frac{\partial T}{\partial t}(x,t) = \alpha \frac{\partial^2 T}{\partial^2 x}(x,t) + \beta T(x,t) + \gamma e(t) T(x,t), \quad \forall\, t \geq t_f \tag{7}$$

where

$$e(t) = 0 \quad \text{for} \quad t \geq \max(t_1, t_f + h).$$

A state space description. The equation (7) can be written

$$\frac{\partial T}{\partial t}(x,t) = AT(x,t) + \gamma e(t) T(x,t), \quad \forall\, t \geq t_f \tag{8}$$

where A is the operator $\alpha(\partial^2/\partial^2 x) + \beta$ whose domain $\mathcal{D}(A)$ and spectrum $\sigma(A)$ are respectively given by

$$\mathcal{D}(A) = \{f \in L^2(0,1)/f'' \in L^2(0,1) \text{ and } f(0) = f(1) = 0\}$$

$$\sigma(A) = \{\lambda_n = \beta - \alpha(\pi n)^2 / n \in \mathbf{N}^*\}$$

the associated eigenfunctions are

$$\varphi_n(x) = \sqrt{2}\sin(n\pi x), \quad n = 1, 2, \ldots .$$

Then we have

$$T(x,t) = \sum_{i=1}^{\infty} a_i(t)\varphi_i(x) \tag{9}$$

by replacing (9) in (8) and we obtain

$$\sum_{n=1}^{\infty} \dot{a}_n(t)\varphi_n(x) = \sum_{n=1}^{\infty} \lambda_n a_n(t)\varphi_n(x) + \sum_{n=1}^{\infty} \gamma e(t) a_n(t)\varphi_n(x)$$

which implies

$$\dot{a}_m(t) = \lambda_m a_m(t) + \gamma e(t) a_m(t); \quad m = 1, 2, \dots .$$

If we introduce the notations $\mathcal{A} = \mathrm{diag}(\lambda_1, \lambda_2, \dots)$, $a(t) = (a_1(t), a_2(t), \dots)^\top$ and $\mathcal{B} = \gamma\mathrm{Id}$, then equation (8) can be written as follows

$$\begin{cases} \dot{a}(t) = \mathcal{A}a(t) + e(t)\mathcal{B}a(t) & \text{for all } t > t_f, \\ a(0) = a_0 \end{cases} \tag{10}$$

where $a_0 = (a_1(0), a_2(0), \dots)^\top$ and $a_i(0) = \langle T(x,0), \varphi_i(x)\rangle_{L^2(0,1)}$.

Spatial approximation. If we project the system (10) on a finite dimensional subspace, we obtain

$$\begin{cases} \dot{a}^N(t) = \mathcal{A}_N a^N(t) + e(t)\mathcal{B}_N a^N(t) & \text{for all } t > t_f, \\ a^N(0) = (a_1(0), a_2(0), \dots, a_N(0))^\top \end{cases} \tag{11}$$

with $\mathcal{A}_N = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}$, $\mathcal{B}_N = \gamma I_N$ where I_N is the $N \times N$ matrix.

Sampling of time. In order to make the system abortable by a computer we proceed to a sampling of time; this means we put

$$[t_f, \infty[= \bigcup_{i=0}^{\infty} [t_i, t_{i+1}]$$

where

$$\begin{cases} t_0 = t_f \\ t_{i+1} = t_i + \Delta \quad \text{for all } i \geq 0 \end{cases}$$

with Δ sufficiently small. If we use the approximation

$$\dot{a}^N(t_i) = \frac{a^N(t_{i+1}) - a^N(t_i)}{\Delta}$$

we will have

$$\begin{cases} \dot{a}_{i+1}^N = (\Delta \mathcal{A}_N + I_N) a_i^N + e_i \mathcal{B}_N a_i^N \quad \text{for all } i \geq 0 \\ a_0^N \in \mathbf{R}^N \end{cases} \tag{12}$$

where $a_i^N = a^N(t_i)$ and $e_i = e(t_i)$. As $e(t) = 0$, for all $t > \max(t_1, t_f + h)$, then $(e_i)_{i \geq 0}$ is null from a certain integer I (I is called the age of disturbance).

2. Preliminary results. We define the operator $\mathcal{K}_i^\beta$ and $\phi^\beta(i,k)$ for every $\beta \in \mathcal{U}_r^J$ by

$$\mathcal{K}_i^\beta = A + g(v_i) \sum_{j=1}^{r} \beta_i^j h_j, \quad \forall\, i \geq 0$$

and

$$\phi^\beta(i,k) = \mathcal{K}_i^\beta \mathcal{K}_{i-1}^\beta \dots \mathcal{K}_k^\beta \quad \text{if } k \leq i.$$

By convention we have $\phi^\beta(i,k) = I_n$ if $k > i$ where I_n is an $n \times n$-unit matrix. Then we can easily show that the solution of systems (1) and (4) satisfies

$$x^w(i) = \phi^\beta(i-1,0) x^w(0) + \sum_{j=0}^{i-1} \phi^\beta(i-1, j+1) f(u_j + \alpha_j), \quad \forall\, i \geq 1$$

and

$$x(i) = A^i x(0) + \sum_{j=0}^{i-1} A^{i-1-j} f(u_j), \quad \forall\, i \geq 1,$$

then for $i = 0$ we have

$$y^w(0) - y(0) = C\gamma$$

and, for every $i \geq 1$, we have

$$\begin{aligned} y^w(i) - y(i) &= Cx^w(i) - Cx(i) \\ &= C\big[\phi^\beta(i-1,0)x^w(0) - A^i x(0)\big] \\ &\quad + \sum_{j=0}^{i-1} C\big[\phi^\beta(i-1,j+1)f(u_j+\alpha_j) - A^{i-1-j}f(u_j)\big]. \end{aligned}$$

Thus, the difference between the observation (3) and (2) can be equivalently rewritten in the form

$$y^w(i) - y(i) = \sum_{j=0}^{i} C\psi^\beta(i-1,j)\xi_j^w$$

with

$$\psi^\beta(i,j)(x,y) = \phi^\beta(i,j)x - A^{i-j+1}y \tag{13}$$

and

$$\begin{cases} \xi_j^w = (f(u_{j-1}+\alpha_{j-1}), f(u_{j-1})) & \text{for all } j > 0, \\ \xi_0^w = (x^w(0), x(0)). \end{cases} \tag{14}$$

Consequently, the set $\mathcal{E}(\varepsilon)$ can be written as follows

$$\mathcal{E}(\varepsilon) = \Big\{ w = (\gamma,\alpha,\beta) \in \mathcal{V} / \|C\gamma\| \leq \varepsilon; \Big\| \sum_{j=0}^{i} C\psi^\beta(i-1,j)\xi_j^w \Big\| \leq \varepsilon,\ \forall\, i \geq 1 \Big\}$$

where $\mathcal{V} = \mathbf{R}^n \times \mathcal{U}_m^I \times \mathcal{U}_r^J$.

Since $\mathcal{U}_q^s$ can be identified to $\mathbf{R}^{q(s+1)}$ by the canonical isomorphism

$$\begin{aligned} \varphi : \mathcal{U}_q^s &\longrightarrow \mathbf{R}^{q(s+1)} \\ (z_i)_{i\geq 0} &\longrightarrow (z_i)_{i\leq s}^T \end{aligned}$$

where $(z_i)_{i\leq s}^T$ is the vector of $\mathbf{R}^{q(s+1)}$ given by

$$(z_i)_{i\leq s} = \begin{bmatrix} z_0 \\ \vdots \\ z_s \end{bmatrix}^\top \in \underbrace{\mathbf{R}^q \times \mathbf{R}^q \cdots \mathbf{R}^q}_{s+1\text{-times}}$$

then

$$\mathcal{E}(\varepsilon)=\Big\{w=(\gamma,(\alpha_i)_{i\geq 0},(\beta_i)_{i\geq 0})\in\mathcal{M}/\|C\gamma\|\leq\varepsilon;\ \Big\|\sum_{j=0}^{i}C\psi^{\beta}(i-1,j)\xi_j^w\Big\|\leq\varepsilon,\ \forall\, i\geq 1\Big\}$$

with

$$\mathcal{M}-\mathbf{R}^n\times\mathbf{R}^{m(I+1)}\times\mathbf{R}^r(J+1).$$

In order to characterize the set $\mathcal{E}(\varepsilon)$ by a finite number of functional inequalities, we rewrite $\mathcal{E}(\varepsilon)$ as follows

$$\mathcal{E}(\varepsilon)=R(\varepsilon)\cap\mathcal{S}(\varepsilon)$$

where

$$\mathbf{R}(\varepsilon)=\Big\{w\in\mathcal{M}/\|C\gamma\|\leq\varepsilon;\Big\|\sum_{j=0}^{i}C\psi^{\beta}(i–1,j)\xi_j^w\Big\|\leq\varepsilon,\ \forall\, i\in\{1,\dots,N\}\Big\}$$

and

$$\mathcal{S}(\varepsilon)=\Big\{w\in\mathcal{M}/\Big\|\sum_{j=0}^{i}C\psi^{\beta}(i-1,j)\xi_j^w\Big\|\leq\varepsilon,\ \forall\, i\geq N+1\Big\}$$

with $N=\max(I,J)+1$.

Let us define the functionals $(\mathcal{L}_i)_{0\leq i\leq N}$ by

$$\begin{aligned}\mathcal{L}_0:\mathcal{M}&\longrightarrow\mathbf{R}^n\\ w=(\gamma,\alpha,\beta)&\longmapsto\gamma\end{aligned}$$

and, for $i\in\{1,2,\dots,N\}$:

$$\begin{aligned}\mathcal{L}_i:\mathcal{M}&\longrightarrow\mathbf{R}^n\\ w=(\gamma,\alpha,\beta)&\longmapsto\sum_{j=0}^{i}\psi^{\beta}(i-1,j)\xi_j^w\end{aligned}$$

where $\psi^\beta(i,j)$ and ξ_j^w are defined respectively by equations (13) and (14). Hence

$$\mathbf{R}(\varepsilon) = \{w \in \mathcal{M}/\|C\mathcal{L}_i(w)\| \le \varepsilon;\ \forall\, i \in \{0,1,\dots,N\}\} \tag{15}$$

since the set $\mathbf{R}(\varepsilon)$ is characterized by a finite number of inequalities, our objective will be the characterization of the set $\mathcal{S}(\varepsilon)$, and consequently $\mathcal{E}(\varepsilon)$, by the finite number of inequalities.

Exploiting the fact that $\alpha_i = 0$, for all $i > I$ and $\beta_i = 0$, for all $i > J$, we have

$$\begin{aligned}
\mathcal{S}(\varepsilon) &= \Big\{w \in \mathcal{M}/\Big\|\sum_{j=0}^{N} C\psi^\beta(i-1,j)\xi_j^w\Big\| \le \varepsilon,\ \forall\, i \ge N+1\Big\}\\
&= \Big\{w \in \mathcal{M}/\Big\|C\Big[\sum_{j=0}^{N} \phi^\beta(i-1,N)\phi^\beta(N-1,j)f(u_j+\alpha_j)\\
&\qquad - A^{i-j-1}f(u_j)\Big]\Big\| \le \varepsilon,\ \forall\, i \ge N+1\Big\}\\
&= \Big\{w \in \mathcal{M}/\Big\|CA^{i-N}\Big[\sum_{j=0}^{N} \phi^\beta(N-1,j)f(u_j+\alpha_j)\\
&\qquad - A^{N-j-1}f(u_j)\Big]\Big\| \le \varepsilon,\ \forall\, i \ge N+1\Big\}\\
&= \Big\{w \in \mathcal{M}/\Big\|CA^{i-N}\sum_{j=0}^{N} C\psi^\beta(N-1,j)\xi_J^w\Big\| \le \varepsilon,\ \forall\, i \ge N+1\Big\}\\
&= \Big\{w \in \mathcal{M}/\Big\|CA^{k+1}\sum_{j=0}^{N} C\psi^\beta(N-1,j)\xi_j^w\Big\| \le \varepsilon,\ \forall\, k \ge 0\Big\}
\end{aligned}$$

then

$$\mathcal{S}(\varepsilon) = \{w \in \mathcal{M}/\|CA^{k+1}\mathcal{L}_N(w)\| \le \varepsilon,\ \forall\, k \ge 0\}. \tag{16}$$

Now we give some properties of the ε-admissible set.

Proposition 1.

i) *$\mathcal{E}(\varepsilon)$ is a closed set.*

ii) *If A is Lyapunov stable* (*the characteristic roots of A satisfy the following conditions*: $|\lambda| \leq 1$ *for every λ in the spectrum of A and* $|\lambda| = 1$ *implies λ is simple*), *then* $0 \in \operatorname{int}\mathcal{E}(\varepsilon)$.

Proof. i) Since $\mathcal{L}_0$ and $\mathcal{L}_i$ are continuous maps and the set $\{x \in \mathbf{R}^n/\|Cx\| \leq \varepsilon\}$ is closed, we deduce that $\mathbf{R}(\varepsilon)$, defined by equation (15), is closed.

On the other hand, we have $\mathcal{S}(\varepsilon) = \mathcal{L}_N^{-1}(\mathcal{T}(\varepsilon))$ where $\mathcal{T}(\varepsilon)$ is the closed set given by $\mathcal{T}(\varepsilon) = \{x \in \mathbf{R}^n/\|CA^{k+1}x\| \leq \varepsilon$, for all $k \geq 0\}$, hence $\mathcal{S}(\varepsilon)$ is closed (because $\mathcal{L}_N$ is a continuous map). So we conclude that $\mathcal{E}(\varepsilon) = \mathbf{R}(\varepsilon) \cap \mathcal{S}(\varepsilon)$ is a closed set.

ii) It is clear that $0 = \mathcal{L}_i(0) \in \operatorname{int}(\{x \in \mathbf{R}^n/\|Cx\| \leq \varepsilon\})$, moreover, $(\mathcal{L}_i)_{0\leq i\leq N-1}$ are continuous maps; hence for every integer $i \in \{0, \dots, N-1\}$, there exists an open set $\mathcal{O}_i = \mathcal{L}_i^{-1}(\{x \in \mathbf{R}^n/\|Cx\| < \varepsilon\})$ such that $0 \in \mathcal{O}_i$ and $\mathcal{L}_i(\mathcal{O}_i) \subset \{x \in \mathbf{R}^n/\|Cx\| \leq \varepsilon\}$. This implies that for every integer $i \in \{0, \dots, N-1\}$, we have $\mathcal{O}_i \subset \mathcal{L}_i^{-1}(\{x \in \mathbf{R}^n/\|Cx\| \leq \varepsilon\})$. Consequently, $0 \in \cap_{i=0}^{N-1}\mathcal{O}_i \subset \cap_{i=0}^{N-1}\mathcal{L}_i^{-1}(\{x \in \mathbf{R}^n/\|Cx\| \leq \varepsilon\}) = \mathbf{R}(\varepsilon)$, thus $0 \in \operatorname{int}\mathbf{R}(\varepsilon)$.

On the other hand, the Lyapunov stability of A implies the existence of a constant $\rho > 0$ such that

$$\|CA^{k+1}x\| \leq \rho\|x\| \quad \text{for every } x \in \mathbf{R}^n \text{ and every } k \in \mathbf{N},$$

hence

$$\|CA^{k+1}\mathcal{L}_N(w)\| \leq \rho\|\mathcal{L}_N(w)\|, \quad \forall\, w \in M,\ \forall\, k \in \mathbf{N}.$$

Using the continuity of $\mathcal{L}_N$, we have

$$\forall\, \varepsilon > 0 \quad \exists \eta > 0, \quad \|w\| \leq \eta \Rightarrow \|\mathcal{L}_N(w)\| \leq \varepsilon/\rho$$

so for every $w \in B(0, \eta) = \{s \in M/\|s\| \leq \eta\}$ and every $k \in \mathbf{N}$ we have

$$\|CA^{k+1}\mathcal{L}_N(w)\| \leq \rho\|\mathcal{L}_N(w)\| \leq \varepsilon.$$

Hence by equation (16) we have $B(0, \eta) \subset \mathcal{S}(\varepsilon)$ thus $0 \in \operatorname{int}\mathcal{S}(\varepsilon)$. Consequently, $0 \in \operatorname{int}\mathbf{R}(\varepsilon) \cap \operatorname{int}\mathcal{S}(\varepsilon)$, i.e., $0 \in \operatorname{int}\mathcal{E}(\varepsilon)$. □

3. The characterization of $\mathcal{S}(\varepsilon)$. In order to characterize the set $\mathcal{S}(\varepsilon)$ by a finite number of inequalities, we rewrite it as follows

$$\mathcal{S}(\varepsilon) = \{w = (\gamma, (\alpha_i)_{i \le I}, (\beta_i)_{i \le J}) \in \mathcal{M} / \mathcal{L}_N(w) \in \mathcal{T}(\varepsilon)\}$$

where

$$\mathcal{T}(\varepsilon) = \{x \in \mathbf{R}^n / \|CA^{i+1}x\| \le \varepsilon,\ \forall\, i \ge 0\}.$$

For every $k \in \mathbf{N}$, we define the set $\mathcal{S}_k(\varepsilon)$ and $\mathcal{T}_k(\varepsilon)$ by

$$\begin{aligned} \mathcal{S}_k(\varepsilon) = \{w = (\gamma, (\alpha_i)_{i \le I}, (\beta_i)_{i \le J}) \in \mathcal{M} / \\ \|CA^{i+1}\mathcal{L}_N(w)\| \le \varepsilon,\ \forall\, i \in \{0, 1, \dots, k\}\} \\ \mathcal{T}_k(\varepsilon) = \{x \in \mathbf{R}^n / \|CA^{i+1}x\| \le \varepsilon,\ \forall\, i \in \{0, 1, \dots, k\}\}. \end{aligned}$$

$\mathcal{T}(\varepsilon)$ and $\mathcal{S}(\varepsilon)$ are said to be finitely accessible if there exists $k \in \mathbf{N}$ such that $\mathcal{T}(\varepsilon) = \mathcal{T}_k(\varepsilon)$ and $\mathcal{S}(\varepsilon) = \mathcal{S}_k(\varepsilon)$. We note k^* the smallest integer such that $\mathcal{T}(\varepsilon) = \mathcal{T}_{k^*}(\varepsilon)$ and $\mathcal{S}(\varepsilon) = \mathcal{S}_{k^*}(\varepsilon)$.

Remark 2. We have

$$\mathcal{T}(\varepsilon) \subset \mathcal{T}_{k_2}(\varepsilon) \subset \mathcal{T}_{k_1}(\varepsilon), \quad \forall\, k_1, k_2 \in \mathbf{N} \text{ such that } k_1 \le k_2.$$

Proposition 2. *$\mathcal{T}(\varepsilon)$ is finitely accessible if and only if $\mathcal{T}_{i+1}(\varepsilon) = \mathcal{T}_i(\varepsilon)$ for some $i \in \mathbf{N}$.*

Proof. If $\mathcal{T}(\varepsilon)$ is finitely accessible, then the equality $\mathcal{T}_{i+1}(\varepsilon) = \mathcal{T}_i(\varepsilon)$ holds for all $i \ge k^*$. Conversely, if $\mathcal{T}_{i+1}(\varepsilon) = \mathcal{T}_i(\varepsilon)$ for some $i \in \mathbf{N}$, we deduce that $\mathcal{T}_i(\varepsilon)$ is A-invariant (i.e., $A(\mathcal{T}_i(\varepsilon)) \subset \mathcal{T}_i(\varepsilon)$) which implies that $\mathcal{T}_i(\varepsilon)$ is A^k-invariant for every $k \in \mathbf{N}$, and so $\mathcal{T}_i(\varepsilon) \subset \mathcal{T}(\varepsilon)$. Finally we apply Remark 1 to end the proof. □

It is desirable to establish simple conditions which make the set $\mathcal{S}(\varepsilon)$, or $\mathcal{T}(\varepsilon)$, finitely accessible. Our main result in this direction is the following

Theorem 1. *Suppose the following assumptions hold*

i) *A is asymptotically stable ($|\lambda| < 1$ for every λ in spectrum of A).*

ii) *The pair* (C,A) *is observable* ($[C^\top | A^\top C^\top | \cdots | (A^\top)^{n-1} C^\top]$ *has rank* n).

Then $\mathcal{S}(\varepsilon)$ *is finitely accessible.*

Proof. Let $x \in \mathcal{T}_{n-1}(\varepsilon)$. Then $\|CA^{i+1}x\| \leq \varepsilon$ for all $i \in \{0, 1, \dots, n-1\}$ which implies that

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} Ax \in \overbrace{\mathcal{B}_p(0,\varepsilon) \times \cdots \times \mathcal{B}_p(0,\varepsilon)}^{n\text{-times}}$$

where

$$\mathcal{B}_p(0,\varepsilon) = \{x \in \mathbf{R}^p / \|x\| \leq \varepsilon\}.$$

Hence $\Lambda^\top \Lambda A c \in \Lambda^\top \overbrace{(\mathcal{B}_p(0,\varepsilon) \times \cdots \times \mathcal{B}_p(0,\varepsilon))}^{n\text{-times}}$ where Λ is the matrix is given by

$$\Lambda = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^{np});$$

consequently,

$$(\Lambda^T \Lambda A)(\mathcal{T}_{n-1}(\varepsilon)) \subset \Lambda^\top \overbrace{(\mathcal{B}_p(0,\varepsilon) \times \cdots \times \mathcal{B}_p(0,\varepsilon))}^{n\text{-times}},$$

so for every $x \in \mathcal{T}_{n-1}(\varepsilon)$ there exists $z \in \overbrace{(\mathcal{B}_p(0,\varepsilon) \times \cdots \times \mathcal{B}_p(0,\varepsilon))}^{n\text{-times}}$ such that $\Lambda^\top \Lambda A x = \Lambda^\top z$, which implies that

$$\langle \Lambda^\top \Lambda A x, Ax \rangle = \langle \Lambda^\top z, Ax \rangle, \quad \forall\, x \in \mathcal{T}_{n-1}(\varepsilon). \tag{17}$$

On the other hand, the observability of (C,A) implies that $\Lambda^\top \Lambda$ is coercive, i.e.,

$$\exists \alpha > 0 / \langle \Lambda^\top \Lambda x, x \rangle \geq \alpha \|x\|^2, \quad \forall\, x \in \mathbf{R}^n.$$

Then it follows from (17) that

$$\alpha\|Ax\|^2 \leq (cste)\|Ax\| \times \|z\|, \quad \forall\, x \in \mathcal{T}_{n-1}(\varepsilon),$$

and consequently

$$\|Ax\| \leq (cste)\|z\|, \quad \forall\, x \in \mathcal{T}_{n-1}(\varepsilon).$$

Then since $\overbrace{(\mathcal{B}_p(0,\varepsilon) \times \cdots \times \mathcal{B}_p(0,\varepsilon))}^{n\text{-times}}$ is a bounded set, we deduce the existence of a constant $r > 0$ such that

$$A\mathcal{T}_k(\varepsilon) \subset \mathcal{B}_n(0,r) = \{x \in \mathbf{R}^n / \|x\| \leq r\}, \quad \forall\, k \geq n-1. \tag{18}$$

Using the asymptotic stability of A, it follows that there exists $k_0 \geq n-1$ such that $\|CA^{k_0+1}\| \leq \varepsilon/r$, hence

$$CA^{k_0+1}(\mathcal{B}_n(0,r)) \subset \mathcal{B}_p(0,\varepsilon) \tag{19.}$$

Then we use (18) and (19) to deduce that

$$\|CA^{k_0+2}x\| \leq \varepsilon, \quad \forall\, x \in \mathcal{T}_{k_0}(\varepsilon)$$

which implies that

$$x \in \mathcal{T}_{k_0+1}(\varepsilon);$$

consequently

$$\mathcal{T}_{k_0}(\varepsilon) \subset \mathcal{T}_{k_0+1}(\varepsilon).$$

Finally we use Proposition 2 to end the proof. □

Using Proposition 2 we can establish a first formal algorithm to determine the smallest integer k^* such that $\mathcal{T}_{k^*}(\varepsilon) = \mathcal{T}(\varepsilon)$ and consequently to characterize the set $\mathcal{S}(\varepsilon)$ by

$$\mathcal{S}(\varepsilon) = \mathcal{S}_{k^*}(\varepsilon) = \mathcal{L}_N^{-1}(\mathcal{T}_{k^*}(\varepsilon)).$$

Algorithm I.

Step 1: Set $k = 0$

Step 2: If $\mathcal{T}_{k+1}(\varepsilon) = \mathcal{T}_k(\varepsilon)$ then set $k^* = k$ and stop, else continue.

Step 3: Replace k by $k+1$ and return to step 2.

It is obvious that Algorithm 1 is not practical because it does not describe how the test $\mathcal{T}_{k+1}(\varepsilon) = \mathcal{T}_k(\varepsilon)$ is implemented; moreover, it produces k^* if and only if $\mathcal{T}(\varepsilon)$ is finitely accessible. In order to overcome this difficulty, let $\mathbf{R}^p$ be endowed with the following norm

$$\|x\| = \max_{1 \le i \le p} |x_i|, \quad \forall\, x = (x_1, \dots, x_p) \in \mathbf{R}^p.$$

The set $\mathcal{T}_k(\varepsilon)$ is then described as follows

$$\mathcal{T}_k(\varepsilon) = \{x \in \mathbf{R}^n / D_j(CA^{i+1}x) \le 0 \\ \text{for } j = 1, 2, \dots, 2p \text{ and } i = 0, 1, \dots, k\}$$

where $D_j : \mathbf{R}^p \to \mathbf{R}$ are defined for every $x = (x_1, \dots, x_p) \in \mathbf{R}^p$ by

$$D_{2m-1}(x) = x_m - \varepsilon \quad \text{for } m \in \{1, 2, \dots, p\}$$
$$D_{2m}(x) = -x_m - \varepsilon \quad \text{for } m \in \{1, 2, \dots, p\}.$$

It follows from Remark 1 that

$$\mathcal{T}_{k+1}(\varepsilon) = \mathcal{T}_k(\varepsilon) \Longleftrightarrow \mathcal{T}_k(\varepsilon) \subset \mathcal{T}_{k+1}(\varepsilon)$$

so

$$\mathcal{T}_{k+1}(\varepsilon) = \mathcal{T}_k(\varepsilon) \Longleftrightarrow [\forall\, x \in \mathcal{T}_k(\varepsilon), \\ \forall\, j \in \{1, 2, \dots, 2p\}\ D_j(CA^{k+2}x) \le 0]$$

or equivalently

$$\sup_{x \in \mathcal{T}_k(\varepsilon)} D_j(CA^{k+2}x) \le 0 \quad \forall\, j \in \{1, 2, \dots 2p\},$$

hence Algorithm I can be rewritten as follows.

Algorithm II.

Step 1: Let $k = 0$;

Step 2: For $i = 1, \dots, 2p$, do:

Maximize $J_i(x) = D_i(CA^{k+2}x)$

$\begin{cases} D_i(CA^l x) \leq 0, \\ i = 1, \ldots, 2p, \; l = 1, \ldots, k+1. \end{cases}$

Let J_i^* be the maximum value of $J_i(x)$.

If $J_i^* \leq 0$, for $i = 1, \ldots, 2p$, then set $k^* := k$ and stop.

Else continue.

Step 3: Replace k by $k+1$ and return to Step 2.

Remark 3. The optimization problem cited in Step 2 is a mathematical programming problem and can be solved by standard methods.

4. Examples. In this section we give two simple examples where we present the set $\mathcal{E}(\varepsilon)$.

Example 1. Let A, C and ε be given by

$$A = \begin{pmatrix} 0.6 & 0 \\ 1 & 0.2 \end{pmatrix}, \quad C = (1, 1) \quad \text{and} \quad \varepsilon = 0,3$$

then we use Algorithm II to establish that $k^* = 1$.

We suppose that $f = 0$, $r = 1$, $h_1 : \mathbf{R}^2 \to \mathbf{R}^2 : \begin{pmatrix} x \\ y \end{pmatrix} \hookrightarrow \begin{pmatrix} x^2+y^2 \\ xy \end{pmatrix}$, $g(x) = \sin(x)$ and $J = 0$. Then, for $v_0 = \pi/2$ and $x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we have

$$\mathcal{E}(\varepsilon) = \mathbf{R}(\varepsilon) \cap \mathcal{S}(\varepsilon)$$

where

$$\begin{aligned} \mathbf{R}(\varepsilon) = \big\{ &(\gamma_1, \gamma_2, \beta_0) \in \mathbf{R}^3 / \big|\gamma_1 + \gamma_2\big| \leq 0.3, \\ &\big|1.6\gamma_1 + 0.2\gamma_2 + \beta_0[(\gamma_1+1)^2 + \gamma_2^2 + \gamma_2(\gamma_1+1)]\big| \leq 0.3\big\} \\ \mathcal{S}(\varepsilon) = \big\{ &(\gamma_1, \gamma_2, \beta_0) \in \mathbf{R}^3 / \\ &\big|1.16\gamma_1 + 0.04\gamma_2 + \beta_0[1.6((\gamma_1+1)^2 + \gamma_2^2) + 0.2\gamma_2(\gamma_1+1)]\big| \\ &\leq 0.3, \\ &\big|0.736\gamma_1 + 0.008\gamma_2 + \beta_0[1.16((\gamma_1+1)^2 + \gamma_2^2) + 0.04\gamma_2(\gamma_1+1)]\big| \\ &\leq 0.3\big\}. \end{aligned}$$

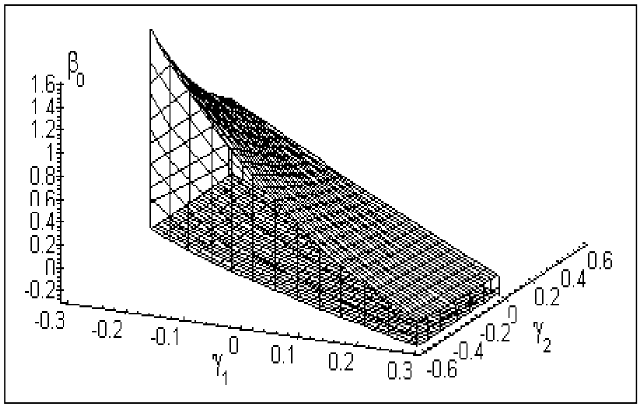

FIGURE 1. The set $\mathcal{E}(\varepsilon)$ corresponding to Example 1.

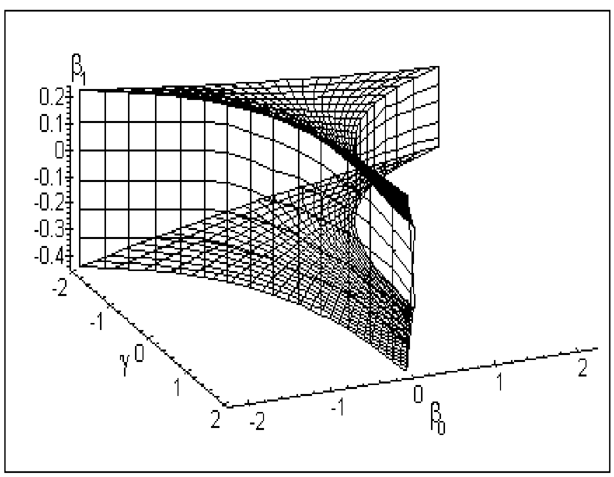

FIGURE 2. The set $\mathcal{E}(\varepsilon)$ corresponding to Example 2.

Example 2. For $A = 0.8$, $C = 5$ and $\varepsilon = 2$, we obtain $k^* = 0$. If we take $f = 0$, $r = 1$, $h_1(x) = e^x$, $g(x) = 1/x$ and $J = 1$. Then, for $v_0 = 2$, $v_1 = 1$ and $x_0 = 1$, we have

$$\mathcal{E}(\varepsilon) = \mathbf{R}(\varepsilon) \cap \mathcal{S}(\varepsilon)$$

where

$$\begin{aligned}\mathbf{R}(\varepsilon) = \Big\{&(\gamma, \beta_0, \beta_1) \in \mathbf{R}^3 / |\gamma| \leq 2, \Big|\frac{5}{2}\beta_0 e^{\gamma+1}\Big| \leq 2, \\ &|0.8 + 5\beta_1 e^{0.8+(\beta_0/2)e^{\gamma+1}}| \leq 2\Big\} \\ \mathcal{S}(\varepsilon) = \Big\{&(\gamma, \beta_0, \beta_1) \in \mathbf{R}^3 / \big|0.61 + 4\beta_1 e^{0.8+(\beta_0/2)e^{\gamma+1}}\big| \leq 2\Big\}.\end{aligned}$$

5. Admissible disturbances for discrete delayed nonlinear controlled systems. This section is devoted to the characterization of admissible disturbances for the discrete infected controlled nonlinear delayed system given by

$$(20)\qquad \begin{cases} x^w(i+1) = \sum_{j=0}^{d} A_j x^w(i-j) + f(u_i + \alpha_i) \\ \qquad\qquad + g(v_i + \beta_i) \sum_{j=0}^{r} \beta_i^j h_j(x^w(i), x^w(i-1), \dots, \\ x^w(i-d)) \quad i \geq 0, \\ x^w(0) = x_0 + \gamma_0, \\ x^w(k) = \theta_k + \gamma_k \quad \text{for } k \in \{-d, -d+1, \dots, -1\}. \end{cases}$$

The corresponding delayed output function is

$$(21)\qquad y^w(i) = \sum_{j=0}^{t} C_j x^w(i-j), \quad i \geq 0,$$

where $A_j \in \mathcal{L}(\mathbf{R}^n)$, $C_j \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^p)$, r, d and t are integers such that $t \leq d$, $h_j : \mathbf{R}^{n(d+1)} \to \mathbf{R}^n$ and $f : \mathbf{R}^m \to \mathbf{R}^n$ are a continuous function and $g : \mathbf{R}^q \to \mathbf{R}$ is a given function. $(\gamma_{-d}, \gamma_{-d+1}, \dots, \gamma_{-1}, \gamma_0) \in (\mathbf{R}^n)^{d+1}$ is a perturbation which infects the initial state $(\theta_{-d}, \theta_{-d+1}, \dots, \theta_{-1}, x_0)$.

As before, we suppose that I and J are respectively the life of $(\alpha_i)_{i\geq 0}$ and $(\beta_i)_{i\geq 0}$, where $\beta_i = (\beta_i^1, \beta_i^2, \dots, \beta_i^r)$ and we investigate the set $\overline{\mathcal{E}}(\varepsilon)$ of all ε-admissible disturbances

$$w = ((\gamma_k)_{-d\leq k\leq 0}, (\alpha_i)_{i\geq 0}, (\beta_i)_{i\geq 0}) \in (\mathbf{R}^n)^{d+1} \times \mathcal{U}^I(\mathbf{N}, \mathbf{R}^m) \times \mathcal{U}^J(\mathbf{N}, \mathbf{R}^r),$$

i.e.,

$$\overline{\mathcal{E}}(\varepsilon) = \{w \in \mathbf{R}^{n(d+1)} \times \mathcal{U}_m^I \times \mathcal{U}_r^J / \|y^w * (i) - y(i)\| \leq \varepsilon, \ \forall\, i \geq 0\},$$

where $(y(i))_{i\geq 0}$ is the output function corresponding to the uninfected controlled system, namely,

$$y(i) = \sum_{j=0}^{t} C_j x(i-j), \quad i \geq 0 \tag{22}$$

where

$$\begin{cases} x(i+1) = \sum_{j=0}^{d} A_j x(i-j) + f(u_i) & i \geq 0, \\ x(0) \in \mathbf{R}^n \\ x(k) = \theta_k \quad \text{for } k \in \{-d, -d+1, \dots, -1\}. \end{cases} \tag{23}$$

Consider the new state variables X_i^w and X_i defined in $\mathbf{R}^{n(d+1)}$ by

$$X^w(i) = (x^w(i), x^w(i-1), \dots, x^w(i-d))^\top, \quad i \geq 0$$
$$X(i) = (x(i), x(i-1), \dots, x(i-d))^\top, \quad i \geq 0.$$

Let's define the matrices $\tilde{A} \in \mathcal{L}(\mathbf{R}^{n(d+1)})$ and F by

$$\tilde{A} = \begin{pmatrix} A_0 & A_1 & \cdots & \cdots & A_d \\ I_n & 0_n & \cdots & \cdots & 0_n \\ 0_n & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_n & \cdots & 0_n & I_n & 0_n \end{pmatrix}, \tag{24}$$

and

$$F : \mathbf{R}^q \longrightarrow \mathbf{R}^{n(d+1)}$$
$$x \longrightarrow (f(x), 0_{n\times m}, \dots, 0_{n\times m})^\top$$

where I_n is the $n \times n$-unit matrix, O_n is the $n \times n$-zero matrix and $0_{n\times m}$ is the $n\times n$-zero matrix. Then it is easy to deduce from (20) and (23) that

$$\begin{cases} X^w(i+1) = \tilde{A}X^w(i) + F(u_i + \alpha_i) \\ \qquad\qquad + g(v_i + \beta_i) \sum_{j=0}^{r} \beta_i^j h_j(X^w(i)) \quad \text{for all } i \geq 0, \\ X^w(0) = \theta_0 = W \end{cases} \tag{25}$$

and

$$\text{(26)} \qquad \begin{cases} X(i+1) = \tilde{A}X(i) + F(u_i) & \text{for all } i \geq 0, \\ X(0) = \theta_0 \end{cases}$$

where

$$\theta_0 = (x_0 \theta_{-1}, \dots, \theta_{-d})^\top, \quad W = (\omega_0, \omega_{-1}, \dots, \omega_{-d})^\top.$$

Moreover, if we consider the matrix

$$\text{(27)} \qquad \overline{C} = (C_0|C_1|\cdots|C_t \underbrace{|O_{p\times n}|\cdots|O_{p\times n}}_{d-t\text{-times}}) \in \mathcal{L}(\mathbf{R}^{n(d+1)}, \mathbf{R}^p)$$

then, by (21) and (22) are given in terms of the new state variables $X^w(i)$ and $X(i)$ by

$$\begin{aligned} y^w(i) &= \overline{C}X^w(i), \quad \forall\, i \geq 0 \\ y(i) &= \overline{C}X(i), \quad \forall\, i \geq 0. \end{aligned}$$

As above, define the set $\tilde{\mathcal{D}}(\varepsilon)$ by

$$\overline{\mathcal{E}}(\varepsilon) = \overline{\mathbf{R}}(\varepsilon) \cap \overline{\mathcal{S}}(\varepsilon)$$

where

$$\overline{\mathbf{R}}(\varepsilon) = \{w \in \overline{\mathcal{M}} / \|y_i^w - y_i\| \leq \varepsilon,\ \forall\, i \in \{0, \dots, N\}\}$$

and

$$\overline{\mathcal{S}}(\varepsilon) = \{w \in \mathcal{M} / \|y_i^w - y_i\| \leq \varepsilon,\ \forall\, i \geq N+1\}$$

with $\overline{\mathcal{M}} = \mathbf{R}^{n(d+1)} \times \mathbf{R}^{m(I+1)+q(J+1)}$ and $N = \max(I, J) + 1$.

The set $\overline{\mathbf{R}}(\varepsilon)$ is described by a finite number of equations while $\overline{\mathcal{S}}(\varepsilon)$ is not. So in the sequel, we will investigate the characterization of $\overline{\mathcal{S}}(\varepsilon)$ (and consequently $\overline{\mathcal{E}}(\varepsilon)$) by a finite number of equations (finite accessibility), i.e., the existence of an integer k such that $\overline{\mathcal{S}}(\varepsilon) = \overline{\mathcal{S}}_k(\varepsilon)$ where

$$\overline{\mathcal{S}}_k(e) = \{w \in \overline{\mathcal{M}} / \|y^w(i) - y(i)\| \leq \varepsilon,\ \forall\, i \in \{0, \dots, k\}\}.$$

Since the systems (20) and (25) are infected by perturbation $(\gamma, (\alpha_i)_{i\geq 0}, (\beta_i)_{i\geq 0})$ and have the same output functions $y^w(i)$, then we can use results of Section 4 to characterize the set $\overline{\mathcal{E}}(\varepsilon)$.

Now we develop conditions which are adequate to the delayed case, under which the set $\overline{\mathcal{S}}(\varepsilon)$ is finitely accessible. To realize this we consider the two following situations

a) First case, $p = n$ (i.e., the observation space and the state space have the same dimension).

b) Second case, $p < n$ (which is the usual one).

First case, $p = n$. In this case every C_i is an $n \times n$ matrix.

Proposition 3. *Suppose the following assumptions hold*

i) *C_i commutes with A_j for all i and j such that $0 \leq i \leq t$, $0 \leq j \leq d$.*

ii) *$\| \sum_{i=0}^{d} A_i z_i \| \leq \varepsilon$ for every $(z_0, \ldots, z_d) \in \underbrace{\mathcal{B}_n(0,\varepsilon) \times \cdots \times B_n(0,\varepsilon)}_{(d+1)\text{-times}}$*

where $B_n(0,\varepsilon) = \{x \in \mathbf{R}^n / \|x\| \leq \varepsilon\}$, then $\overline{\mathcal{S}}(\varepsilon) = \overline{\mathcal{S}}_d(\varepsilon)$ where d is the number of delays in the state variable of system (20).

Proof. Applying techniques in Sections 3 and 4, we have

$$\begin{aligned}\overline{\mathcal{S}}(\varepsilon) &= \{w \in \overline{\mathcal{M}} / \|\overline{C}\tilde{A}^{i+1}\overline{\mathcal{L}}_N(w)\| \leq \varepsilon,\ \forall\, i \geq 0\} \\ &= \{w \in \overline{\mathcal{M}} / \overline{\mathcal{L}}_N(w) \in \overline{\mathcal{T}}(\varepsilon)\}\end{aligned}$$

where

$$\overline{\mathcal{T}}(\varepsilon) = \{z \in \mathbf{R}^{n(d+1)} / \|\overline{C}\tilde{A}^{i+1} z\| \leq \varepsilon,\ \forall\, i \geq 0\}$$

$\tilde{A}, \overline{C}$ are given respectively by (24) and (27), operator $\overline{\mathcal{L}}_N$ is defined in a similar way, with appropriate changes, than the operator $\mathcal{L}_N$ defined in Section 3. We have

$$\begin{aligned}\overline{\mathcal{S}}_k(\varepsilon) &= \{w \in \overline{\mathcal{M}} / \|\overline{C}\tilde{A}^{i+1}\overline{\mathcal{L}}_N(w)\| \leq \varepsilon,\ 0 \leq i \leq k\} \\ &= \{w \in \overline{\mathcal{M}} / \overline{\mathcal{L}}_N(w) \in \overline{\mathcal{T}}_k(\varepsilon)\}\end{aligned}$$

where

$$\tilde{\mathcal{T}}_k(\varepsilon) = \{z \in \mathbf{R}^{n(d+1)} / \|\overline{C}\tilde{A}^{i+1} z\| \leq \varepsilon,\ 0 \leq i \leq k\}.$$

To prove the proposition it suffices to prove that $\tilde{\mathcal{T}}_d(\varepsilon) = \tilde{\mathcal{T}}_{d+1}(\varepsilon)$.

For this, let $z = (z_0, z_1, \dots, z_d) \in \mathbf{R}^{n(d+1)}$, then we have

$$\overline{C}\tilde{A}^i z = \bar{y}(i), \quad \forall\, i \geq 0$$

where $(\bar{y}(i))_{i\geq 0}$ is the output function

$$\bar{y}(i) = \overline{C}Z(i), \quad i \geq 0$$

and $(Z(i))_{i\geq 0}$ is the solution of the system

$$\begin{cases} Z(i+1) = \tilde{A}Z(i) & i \geq 0, \\ Z(0) = z. \end{cases}$$

We have also

$$\bar{y}(i) = \sum_{j=0}^{t} C_j \xi(i-j)$$

where $(\xi_i)_{i\geq 0}$ is the solution of the following system

$$\begin{cases} \xi(i+1) = \sum_{j=0}^{d} A_j \xi(i-j) & i \geq 0, \\ \xi(-k) = z_k & \text{for all } k \in \{0, 1, \dots, d\}. \end{cases}$$

Let $z = (z_0, z_1, \dots, z_d) \in \tilde{\mathcal{T}}_d(\varepsilon)$, then

$$\overline{C}\tilde{A}^{i+1} z = \bar{y}(i+1) \in B_n(0, \varepsilon), \quad \forall\, i \in \{0, \dots, d\}. \tag{28}$$

On the other hand,

$$\begin{aligned} \bar{y}(d+2) &= \sum_{j=0}^{t} C_j \xi(d+2-j) \\ &= \sum_{j=0}^{t} C_j \sum_{k=0}^{d} A_k \xi(d+1-k-j) \\ &= \sum_{k=0}^{d} A_k \sum_{j=0}^{t} C_j \xi(d+1-k-j) \\ &\quad \text{[using hypothesis i) cited in Proposition 4]} \\ &= \sum_{k=0}^{d} A_i \bar{y}(d+1-k); \end{aligned}$$

then it follows from hypotheses ii) and (28) that

$$\|\bar{y}(d+2)\| \leq \varepsilon,$$

and hence

$$\overline{\mathcal{T}}_d(\varepsilon) \subset \overline{\mathcal{T}}_{d+1}(\varepsilon)$$

or equivalently

$$\overline{\mathcal{T}}_d(\varepsilon) = \overline{\mathcal{T}}_{d+1}(\varepsilon). \qquad \square$$

Remark 4. After the statement of the proposition one could observe that, in particular, in the case where the state-space dimension n and the output-space dimension p are the same, and no delay appears, $d = t = 0$, if the matrices A, C commute and A is Lyapunov stable, then $\overline{\mathcal{S}}(\varepsilon)$ is finite accessible and one has $\overline{\mathcal{S}}(\varepsilon) = \overline{\mathcal{S}}_0(\varepsilon)$.

Second case $p < n$. Since every C_i is a $p \times n$ matrix, we define the matrix $\hat{C}_i = \begin{pmatrix} C_i \\ 0 \end{pmatrix}$ as an $n \times n$-matrix. If we introduce the new observation variables $\tilde{y}^w(i)$ and $\hat{y}(i)$ by

$$\hat{y}^w(i) = \begin{pmatrix} y^w(i) \\ 0_{\mathbf{R}^{n-p}} \end{pmatrix} \in \mathbf{R}^n, \quad \hat{y}(i) = \begin{pmatrix} y(i) \\ 0_{\mathbf{R}^{n-p}} \end{pmatrix} \in \mathbf{R}^n,$$

then clearly we have

$$\hat{y}^w(i) = \sum_{j=0}^{t} \hat{C}_j x^w(i-j), \quad \hat{y}(i) = \sum_{j=0}^{t} \hat{C}_j x(i-j).$$

Consequently, the set $\overline{\mathcal{T}}(\varepsilon)$ is given by

$$\begin{aligned}\overline{\mathcal{S}}(\varepsilon) &= \{w \in \overline{\mathcal{M}} / \|y^w(i) - y(i)\| \leq \varepsilon, \quad \forall\, i \geq 0\} \\ &= \{w \in \overline{\mathcal{M}} / \|\hat{y}^w(i) - \hat{y}(i)\| \leq \varepsilon, \quad \forall\, i \geq 0\}.\end{aligned}$$

Since $\hat{C}_i$ are $n \times n$-matrices, we apply the results established in the first case, $p = n$, to deduce the following proposition:

Proposition 4. *Suppose the following hypotheses hold*

i) $\hat{C}_i$ *commutes with* A_j *for all* i *and* j *such that* $0 \leq i \leq t$, $0 \leq j \leq d$.

ii) $\| \sum_{i=0}^{d} A_i z_i \| \leq \varepsilon$ *for all* $(z_0, \dots, z_d) \in \underbrace{B_n(0,\varepsilon), \dots, B_n(0,\varepsilon)}_{(d+1)\text{-times}}$.

Then $\overline{\mathcal{S}}(\varepsilon)$ *is finitely accessible, moreover* $\overline{\mathcal{S}}(\varepsilon) = \overline{\mathcal{S}}_d(\varepsilon)$.

6. Conclusion. In this paper the problem of the characterization of the admissible disturbances set for discrete-time controlled systems with some state and input constraints is considered. An efficient algorithm for constructing the admissible set is given and numerical simulation have been done for some examples. The case of controlled discrete-time delayed systems have also been investigated.

REFERENCES

1. A. Bensoussan, *On some singular perturbation problems avising in stochastic control*, Stochastic Anal. Appl. (1984), 13–53.

2. F. Blanchini, *Feedback control for linear time-invariant systems with state and control bounds in the presence of disturbances*, IEEE Trans. Automat. Control **35** (1990), 1231–1234.

3. J. Bouyaghroumni, A. El Jai and M. Rachik, *Admissible disturbances set for discrete perturbed systems*, Internat. J. Appl. Math. Computer Sci. **10** no. 3, (2001).

4. R.F. Curtain and K. Glover, *Robust stabilisation of infinite dimensional systems by finite dimensional controllers*, Systems Control Lett. **7** (1986), 41–47.

5. E.G. Gilbert and K. Tin Tan, *Linear systems with state and control constraints*: *The theory and application of maximal output admissible sets*, IEEE Trans. Automat. Control **36** (1991), 1008–1020.

6. P.O. Gutman and M. Cwikel, *An algorithm to find maximal state constraint sets for discrete-time linear dynamical systems with bounded controls and states*, IEEE Trans. Automat. Control **32** (1987), 251–254.

7. J.L. Lions, *Sentinelles pour les systèmes distribués à données incomplète*, Collection R.M.A., Masson, Paris, 1992.

8. M. Malabre and R. Rabah, *Structure at infinity model matching and disturbance rejection for linear systems with delays*, Kibernetika **29** (1993), no. 5, 485–498,

9. L. Pandolfi, *Disturbance decoupling and invariant subspaces for delay systems*, Appl. Math. Optim. **14** (1986), 55–72.

10. A.M. Perdon and G. Conte, *The disturbance decoupling problem for systems over a principal ideal domain*, In Proc. New Trends in Systems and Control Theory, vol. 7, 1991, Birkhauser, Berlin, pp. 583–592.

11. M. Rachik, E. Labriji, A. Abkari and J. Bouyaghroumni, *Infected discrete linear systems*: *On the admissible sources*, Optimization **48** (2000), 271–289.

12. G. Zames, *Feedback and optional sensitivity*: *Model reference transformations, multiplicative semi-norms and approximate inverse*, IEEE Trans. Automat. Control **26** (1981), 301–320.

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