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COREFLECTIVELY MODIFIED DUALITY

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ABSTRACT. This survey paper presents how several (old and new) results on stability of quotients (of various types) under product, on sequentiality of product of sequential spaces, on relationships between a topology and the upper Kuratowski convergence on its closed sets are derived from a general categorical mechanism of duality. Additional applications in the category of convergence-approach spaces are provided.

1. Introduction. In this paper **C** denotes a Cartesian-closed topological category, and I investigate two specific instances for **C**, namely, the category **Conv** of convergence spaces and the category **CAP** of convergence-approach spaces. All the considered categories are subcategories of **C**, and they are denoted by bold capitals. If a subcategory is (co)reflective, the associated (co)reflector will be denoted by the same (nonbold) capital letter. For example, if **J** is a reflective subcategory of **Conv**, the associated reflector is $J : \mathbf{Conv} \to \mathbf{J}$.

The equivalence between the exponentiality of an object X in a bireflective subcategory **L** of a Cartesian-closed topological category **C** [1] and the commutation problem (1.1) is known from Schwarz [**39**].

$$\forall Y \in \mathrm{Ob}\,(\mathbf{C}), \quad X \times LY \geq L(X \times Y). \tag{1.1}$$

Moreover, the link between exponentiality of X and quotientness (in **L**) of $\mathrm{Id}_X \times f$ for every quotient map f (in **L**) is well known (see for example [**35**]), so that (1.1) applies to problems of preservation of quotientness under product.

Let $X \in \mathrm{Ob}\,(\mathbf{C})$. The coarsest **C**-object W with the same underlying set as X for which

$$\forall Y \in \mathrm{Ob}\,(\mathbf{C}), \quad W \times Y \geq L(X \times Y), \tag{1.2}$$

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is the reflection of X in the Cartesian-closed topological hull of $\mathbf{L}$. A common generalization of (1.1) and (1.2) is

$$\forall Y \in \mathrm{Ob}\,(\mathbf{C}), \quad W \times JY \geq L(X \times Y), \tag{1.3}$$

where $\mathbf{J}$ is another bireflective subcategory of $\mathbf{C}$.

Applied in the category **Conv** of convergence spaces, this general scheme (1.3) allows a unified treatment of many problems concerning product of classically used types of quotient maps in general topology (quotient, hereditarily quotient, countably biquotient, biquotient, almost open, etc.). Relativizing (1.3) to Y in a bicoreflective subcategory of $\mathbf{C}$, several problems of preservation under product of many topological properties such as sequentiality, Fréchetness, strong Fréchetness, k-ness, quasi-k-ness and countably bi-k-ness, among others, can be handled simultaneously. In this way, classical results are unified and refined, new ones are obtained and some open problems are solved [**13, 29, 30**]. On the other hand, extending some of the techniques developed in **Conv** to **CAP**, the theory applies in convergence-approach spaces. In particular, it gives a new point of view on exponential objects in the category **PRAP** of pre-approach spaces [**31**]. The general categorical results are gathered in the next section. Section 3 illustrates how the method applies, with full proofs, in a concrete situation. Other applications are gathered, without proof, in the last section. Some variants of classical topological notions such as core-bi-k-ness or core-contour $(\mathcal{J})$-compactness are not explicitly defined in order to avoid nonessential technical aspects. The purpose of this survey paper is to give an account of applications of the new technique of coreflectively modified duality with the hope that the reader might apply it in other situations as well. I refer to [**1**] for categorical terminology, to [**14**] and [**27**] for undefined topological notions and to [**13**] and [**30**] for convergence theory.

2. General scheme.

2.1. Notations. In the Cartesian-closed concrete topological category $\mathbf{C}$, with $|\cdot|$ as forgetful functor over **Set**, there is a canonical Hom-structure $[X, Z]$ on every Hom-set $\mathrm{Hom}\,(X, Z)$. As usual, $X \geq Y$ signifies that the two $\mathbf{C}$-objects have the same underlying set $|X| = |Y|$

and that the identity carried map $\mathrm{Id}_{X,Y} : X \to Y$ is a morphism. Moreover, there always exists initial and final structures. In particular, given a *concrete*, i.e., $|DX| = |X|$ for every **C**-object X, functor D such that $DZ \geq Z$, the map $i : X \to [D[X,Z],Z]$ defined by $i(x)(f) = f(x)$ is a morphism and

$$\mathrm{Epi}\,{}^{Z}_{D}X,$$

denotes the associated initial **C**-object, with underlying set $|X|$. Let **L** be a bireflective subcategory of **C**. If $L \leq D$ then

$$\mathrm{Epi}^{L}_{D}X = \bigvee_{Z=LZ} \mathrm{Epi}\,{}^{Z}_{D}X.$$

If D is the identity functor, I simply write $\mathrm{Epi}^{\,L}$, respectively Epi^{Z}. This last case seems to be the only one classically used. Examples, within $\mathbf{C} = \mathbf{Conv}$, are the reflector on Antoine convergence spaces [**7**], when Z is the Sierpiński topology \$, and the reflector on c-embedded convergence spaces [**5, 8**], when Z is the real line $\mathbf{R}$ with its usual topology.

Given a bireflector J, a map $f : X \to Y$ is called J-*quotient* whenever Y is finer than the J-reflection of the final **C**-object, on $|Y|$, associated to X and f. If E is a bicoreflector and if J is a bireflector, a **C**-object X is said to be a JE-*structure* whenever the identity carried map $\mathrm{Id}_{EX,X} : EX \to X$ is J-quotient; in other words,

$$X \geq JEX.$$

The interest of these concepts will become clear from their use in the category **Conv** of convergence spaces in Section 3.1. The preservation result [**10**, Theorem 4.2] of JE-structures proved in **Conv** extends to topological categories.

Proposition 2.1. *Let* **C** *be a topological category. If* **J** *is a bireflective subcategory of* **C** *and* **E** *is a bicoreflective subcategory of* **C**, *then each J-quotient image of a JE-structure is a JE-structure.*

2.2. Main results. Recall that a subclass $\mathcal{D}$ of a category **L** is *initially dense* in **L** if for every **L**-object X there exists $\mathcal{D}_X \subset \mathcal{D}$ and an initial source $(f_D : X \to D)_{D \in \mathrm{Ob}(\mathcal{D}_X)}$. Dually $\mathcal{D}$ is *finally dense* in **L** if for every **L**-object Y there exists $\mathcal{D}_Y \subset \mathcal{D}$ and a final sink $(f_D : D \to Y)_{D\in \mathrm{Ob}\,(\mathcal{D}_Y)}$. If moreover the objects of $\mathcal{D}_Y$ can be chosen with the same underlying set as Y, $\mathcal{D}$ is called *rigidly finally dense in* **L**.

Theorem 2.2 [**31**, Theorem 3.1]. *Let* **C** *be a Cartesian-closed topological category.* **J** *and* **L** *are bireflective subcategories of* **C** *and* **E** *is a bicoreflective subcategory of* **C**. *For two* **C**-*objects* $X \geq W$, *the following are equivalent*:

1. *For every* $Y \geq JEY$ *in* $Ob\,(\mathbf{C})$

$$W \times JY \geq L(X \times Y); \tag{2.1}$$

2. *Equation* (2.1) *holds for every* $Y \in Ob\,(\mathbf{E})$;

3. $Id_{X,W} \times f$ *is* L-*quotient for every* J-*quotient map* f *with* JE-*domain*;

4. *For every* $Y \in Ob\,(\mathbf{E})$ *and every* $Z \in Ob\,(\mathbf{L})$,

$$\mathrm{Hom}\,(W \times JY, Z) = \mathrm{Hom}\,(X \times Y, Z); \tag{2.2}$$

5. *Equation* (2.2) *holds for every* $Y \in Ob\,(\mathbf{E})$ *and every* Z *in an initially dense subclass of the category* **L**;

6. *For every* $Z \in Ob\,(\mathbf{L})$,

$$JE[X, Z] \geq [W, Z]; \tag{2.3}$$

7. *Equation* (2.3) *holds for every* Z *in an initially dense subclass of* **L**;

8. $W \geq \mathrm{Epi}^L_{JE} X$.

In particular, if $\mathbf{E} = \mathbf{C}$, $\mathbf{J} = \mathbf{L}$ and $X = W \in \mathrm{Ob}\,(\mathbf{L})$, we get the following combination of [**35**, Theorem 3.1] and [**35**, Theorem 3.3] of Schwarz which is proved in the more general context of an epireflector in a monotopological Cartesian-closed category **C**.

Theorem 2.3. *Let* **C** *be a Cartesian-closed topological category, and let* **L** *be a finally dense epireflective subcategory. Let* $\mathcal{D}$ *be an initially dense subclass of* **L**. *If* $X \in Ob\,(\mathbf{L})$, *the following are equivalent*:

1. *X is exponential in* $\mathbf{L}$;
2. $X \times LY \geq L(X \times Y)$ *for every* $\mathbf{C}$*-object* Y;
3. $X\times$ *- preserves coproduct and quotient in* $\mathbf{L}$;
4. $[X, Z] = L[X, Z]$ *for every* $Z \in Ob\,(\mathbf{L})$;
5. $[X, Z] = L[X, Z]$ *for every* $Z \in \mathcal{D}$.

A $\mathbf{L}$-object X verifying one, hence all, of the conditions of Theorem 2.3 is said to be *exponential* in $\mathbf{L}$. In [**35**, p. 253], Schwarz addresses the problem of characterizing $\mathbf{C}$-objects X for which $[X, Z] = L[X, Z]$ for every $Z \in \mathrm{Ob}\,(\mathbf{L})$, dropping the condition that X is a $\mathbf{L}$-object and notices that little is known about this question. I call such objects $\mathbf{C}$-*quasi-exponential* in $\mathbf{L}$ and I usually drop $\mathbf{C}$ when it is clear from the context. Theorem 2.2 characterizes quasi-exponential objects when $\mathbf{E}$ is particularized to $\mathbf{C}$, the reflective subcategories $\mathbf{J}$ and $\mathbf{L}$ coincide and $W = X$. For instance, quasi-exponential objects in the category $\mathbf{T}$ of topological spaces were characterized in [**13**], while quasi-exponential objects in the category $\mathbf{P}$ of pretopological spaces and $\mathbf{P}_\omega$ were characterized in [**30**]. On the other hand, the use of two different reflective subcategories $\mathbf{J}$ and $\mathbf{L}$ and of a proper coreflective subcategory $\mathbf{E}$ appears to be of fundamental interest in applications. It is precisely this flexibility that allowed to apply the general scheme in [**13, 30**] and [**29**] to a large class of topological problems, including preservation under product of sequentiality, Fréchetness and k-ness, among others, a unified treatment of product theorems for quotient maps and new links between a topology and the upper Kuratowski convergence on its closed sets.

Proposition 2.4 [**31**, Proposition 4.2]. *Let* $\mathbf{E}$ *be a bicoreflective subcategory, and let* $\mathbf{L}$ *be a bireflective subcategory of a Cartesian-closed topological category* $\mathbf{C}$. *Then* Epi_E^L *is a* (*bi*)*reflector.*

Such reflectors play a key role in this method, which is not surprising because
$$L(X \times Y) = L(\mathrm{Epi}_E^L X \times Y),$$
for every $Y \in \mathrm{Ob}\,(\mathbf{E})$. One of their particular interests stems from their behavior under product.

Theorem 2.5 [**31**, Theorem 4.3]. *Let E and B be two finitely productive coreflectors in a Cartesian-closed topological category such that $E \geq B$. Then*

$$\mathrm{Epi}^L\ E\ \mathrm{Epi}^L_B X \times \mathrm{Epi}^L_E Y \geq \mathrm{Epi}^L_E(X \times Y),$$

for every $\mathbf{B}$*-object* Y.

The relevant point in this last result is that

$$EX \times \mathrm{Epi}^L_E Y \geq \mathrm{Epi}^L_E(X \times Y),$$

for every X and Y, provided that E is finitely productive.

Corollary 2.6 [**31**, Corollary 4.4]. *Let* $\mathbf{C}$ *be a Cartesian-closed topological category and let* $\mathbf{L}$ *be a bireflective subcategory. The Cartesian closed hull of the category* $\mathbf{L}$ *is the category* $\mathbf{Epi^L}$ *provided that* $\mathbf{L}$ *is finally dense in* $\mathbf{Epi^L}$.

Hence, Theorem 2.2 allows to describe both exponential objects in reflective subcategories of $\mathbf{C}$ and Cartesian closed hulls of such subcategories.

Theorem 2.7 [**31**, Theorem 4.5]. *Let E be a finitely productive coreflector and let L be a reflector in a Cartesian-closed topological category. The following are equivalent*:

1. $X \times Y$ *is an* LE*-structure for every* Y *in a rigidly finally dense subclass of the* $\mathbf{E}$;
2. $X \times Y$ *is an* LE*-structure for every* $\mathbf{E}$*-object* Y;
3. $X \times Y$ *is an* $\mathrm{Epi}^L_E E$*-structure for every* $Y \geq \mathrm{Epi}^L EY$;
4. $X \geq \mathrm{Epi}^L_E EX$.

Theorem 2.8 [**31**, Theorem 4.6]. *Let E be a finitely productive coreflector and let L be a reflector in a Cartesian-closed topological category. Let f be a surjective morphism. Then the following are equivalent*:

1. *f is* Epi_E^L*-quotient*;

2. *$f \times Id_Y$ is L-quotient for every Y in a rigidly finally dense subclass of* $\mathbf{E}$;

3. *$f \times Id_Y$ is L-quotient for every* $\mathbf{E}$*-object Y*;

4. *$f \times g$ is* Epi_E^L*-quotient for every* Epi^L*-quotient map g with* $\mathrm{Epi}^L E$*-range*^{2}.

Notice that in case $\mathbf{E} = \mathbf{C}$ and $\mathbf{L}$ is finally dense in $\mathbf{C}$, then, in view of Corollary 2.6, Theorem 2.8 states that a map is product-stable in $\mathbf{L}$ in the sense of Schwarz [**37**] if and only if it is quotient in the Cartesian-closed hull of $\mathbf{L}$. Hence it recovers [**37**, Theorem 3].

3. A detailed example.

3.1. Convergence-theoretic characterization of topological notions. Recall that a *convergence* ξ on a set X is a relation between X and the set φX of filters on X

$$x \in \lim_\xi \mathcal{F}$$

that fulfills

(CONV1) $\forall x \in X,\ x \in \lim_\xi (x)$;

(CONV2) $\mathcal{G} \geq \mathcal{F} \Longrightarrow \lim_\xi \mathcal{G} \supset \lim_\xi \mathcal{F}$;

(CONV3) $\forall \mathcal{F}, \mathcal{G} \in \varphi X,\ \lim_\xi (\mathcal{F} \wedge \mathcal{G}) = \lim_\xi \mathcal{F} \bigcap \lim_\xi \mathcal{G}$;

where (x) denotes the principal ultrafilter generated by x. I denote the *convergence space* (X, ξ) simply by X when no confusion is possible and will consequently write $\lim_X$ instead of $\lim_\xi$. Let ξ and θ denote two convergences on the same set. The convergence ξ is *finer* than θ, $\xi \geq \theta$, whenever $\lim_\xi \mathcal{F} \subset \lim_\theta \mathcal{F}$ for every filter $\mathcal{F}$. If X denotes the convergence space associated to ξ and Y the convergence space associated to θ, I will also write $X \geq Y$. A map $f : X \to Y$ is *continuous* if $f(\lim_X \mathcal{F}) \subset \lim_Y f(\mathcal{F})$. The category **Conv** with convergence spaces as objects and continuous maps as morphisms is a Cartesian-closed topological category. Indeed, the canonical Hom-structure in **Conv** on the set $\mathrm{Hom}\,(X,Z)$ of continuous maps from the convergence space X to the convergence space Z, is the *continuous convergence* $[X,Z]$. A filter $\mathcal{F}$ converges to a continuous function $f : X \to Z$ for $[X,Z]$ if and only if $f(x) \in \lim_Z ev(\mathcal{F} \times \mathcal{G})$ for every $x \in |X|$ and every filter $\mathcal{G}$ such that $x \in \lim_X \mathcal{G}$. See [**5**] for other details.

Two families of subsets $\mathcal{A}$ and $\mathcal{B}$ *mesh* ($\mathcal{A}\#\mathcal{B}$) if $A \cap B \neq \varnothing$ for each $A \in \mathcal{A}$ and each $B \in \mathcal{B}$. If $\{A\}\#\mathcal{B}$, I denote $A\#\mathcal{B}$ or $A \in \mathcal{B}^{\#}$. The adherence of a filter is defined by

$$\mathrm{adh}_X \mathcal{F} = \bigcup_{\mathcal{H}\#\mathcal{F}} \lim{}_X \mathcal{H}. \tag{3.1}$$

The *adherence* $\mathrm{adh}_X A$ of a subset A of X is the adherence of the principal filter of A. A set V is a X-*vicinity* of x whenever $x \notin \mathrm{adh}_X V^c$. We denote $\mathcal{V}_X(x)$ the set of all the vicinities of x. A subset A of X is X-*closed* whenever for every filter $\mathcal{F}$ with $A \in \mathcal{F}$, one has $\lim_X \mathcal{F} \subset A$. A set is X-*open* if its complement is X-closed. The *closure* $\mathrm{cl}_X A$ is the least closed set that includes A. A set V is a *neighborhood* of x if and only if $x \notin \mathrm{cl}_X V^c$. The set of all the neighborhoods is denoted by $\mathcal{N}_\xi(x)$.

A convergence ξ on X is a *topology* if $x \in \lim_\xi \mathcal{F}$ amounts to $\mathcal{F} \geq \mathcal{N}_\xi(x)$; a *pretopology* if $x \in \lim_\xi \mathcal{F}$ amounts to $\mathcal{F} \geq \mathcal{V}_\xi(x)$; a *pseudotopology* if and only if

$$\lim{}_\xi \mathcal{F} = \bigcap_{\mathcal{U}\in\beta(\mathcal{F})} \lim{}_\xi \mathcal{U},$$

where $\beta(\mathcal{F})$ denotes the set of all the ultrafilters finer than $\mathcal{F}$. Analogously, $\beta(X)$ denotes the set of ultrafilters on X. The space X is then said to be *topological,* respectively *pretopological, pseudotopological.*

If a class of filters $\mathcal{J}$ fulfills $\mathcal{J}(X) \supset \mathcal{J}(Y)$ whenever $X \geq Y$, $\mathcal{J}(\mathrm{Adh}_{\mathcal{J}} X) = \mathcal{J}(X)$ and $f^- \mathcal{H} \in \mathcal{J}(X)$ [4] provided $\mathcal{H} \in \mathcal{J}(Y)$ and $f : X \to Y$, then the map $\mathrm{Adh}_{\mathcal{J}}$ defined by

$$\lim{}_{\mathrm{Adh}_{\mathcal{J}} X} \mathcal{F} = \bigcap_{\mathcal{J}\ni\mathcal{H}\#\mathcal{F}} \mathrm{adh}_X \mathcal{H} \tag{3.2}$$

is a bireflector in **Conv**. Such bireflectors constitute a very useful and fundamental tool in convergence theory. Indeed, if $\mathcal{J}$ stands respectively for the class of all filters, countably based filters, principal filters and principal filters of closed sets, $\mathrm{Adh}_{\mathcal{J}}$ is respectively the reflector S on pseudotopological spaces, P_ω on paratopological spaces, P on pretopological spaces and T on topological spaces [**10**].

Moreover, when J stands for I, S, P_ω, P and T, respectively, then J-quotientness extends to convergence spaces the topological notions of almost openess, biquotientness, countable biquotientness, hereditary quotientness and quotientness respectively [**10**].

A class $\mathcal{J}$ of filters also determines a bicoreflector $\mathrm{Base}_{\mathcal{J}}$ (under suitable assumptions on $\mathcal{J}$ [**10**]) defined by

$$\lim_{\mathrm{Base}_{\mathcal{J}} X} \mathcal{F} = \bigcup_{\mathcal{J} \ni \mathcal{G} \leq \mathcal{F}} \lim_X \mathcal{G}.$$

For instance, if $\mathcal{J}$ is the class of countably based filters, $\mathrm{Base}_{\mathcal{J}}$ is the coreflector on *first-countable convergence spaces* and is denoted by First. Analogously, if $\mathcal{J}$ is the class of principal filters, $\mathrm{Base}_{\mathcal{J}}$ is denoted Fin and is the coreflector on *finitely generated convergence spaces*. Notice that a pretopological space X is finitely generated in the sense of [**25**][5] if and only if $X = \mathrm{Fin}\, X$. As I intensively use those coreflectors, I write $\mathrm{Epi}_{\mathcal{J}}$ instead of $\mathrm{Epi}_{\mathrm{Base}_{\mathcal{J}}}$. Given a bireflector J and a bicoreflector E, X is a *JE-convergence space* provided

$$X \geq JEX. \tag{3.3}$$

The table below shows that many topological notions can be characterized as JE-properties. Moreover, Proposition 2.1 takes all its sense in view of the table below, from [**10**]: each property of a given row is preserved by the class of maps corresponding to this row (or to a higher row). Hence the **Conv** counterpart [**10**, Theorem 4.2] of Proposition 2.1 recovers many classical preservation theorems. See [**10**] and [**27**] for further details and precise definitions[6].

E coreflector J-quotient	First	K	First_K	$\mathrm{First}_{K_\omega}$
almost open I	first-countable First	locally compact K	point-countable type First_K	strict-q $\mathrm{First}_{K_\omega}$
biquotient S	bisequential S First	locally compact SK	bi-k S First_K	bi-quasi-k S $\mathrm{First}_{K_\omega}$
countably biquotient P_ω	strongly Fréchet P_ω First	strongly k' $P_\omega K$	countably bi$-k$ P_ω First_K	countably bi-quasi-k P_ω First K_ω
hereditarily quotient P	Fréchet P First	k' PK	singly bi-k P First_K	singly bi-quasi-k P $\mathrm{First}_{K_\omega}$
quotient T	sequential T First	k TK	k T First_K	quasi-k T $\mathrm{First}_{K_\omega}$

3.2. General scheme applied to $\mathrm{Adh}_{\mathcal{J}}$-reflectors. A convergence space X is called *atomic* if all its points, but one, are isolated. Notice that atomic topological spaces form a rigidly finally dense subclass of **Conv** while atomic metrizable topological spaces are rigidly finally dense in first-countable convergence spaces.

A class $\mathcal{J}$ of filters is said to be *composable* if it contains principal filters and if $\mathcal{HG}$, the filter generated by $\{HG : H \in \mathcal{H}, G \in \mathcal{G}\}$ [7] is a (possibly degenerate) $\mathcal{J}$-filter on Y whenever $\mathcal{H}$ is a $\mathcal{J}$-filter on $|X \times Y|$ and $\mathcal{G}$ a $\mathcal{J}$-filter on $|X|$. For example, the classes of principal filters and of countably based filters are composable, while that of filters generated by sequences and that of principal filters of closed sets are not. If $\mathcal{J}$ is composable, it contains principal filters, so that $\mathrm{Adh}_{\mathcal{J}} \geq P$. Moreover, if $\mathcal{J}$ is composable, then $\mathrm{Base}_{\mathcal{J}}$ is a finitely productive coreflector [**30**, Lemma 2.2].

Theorem 3.1 [**30**, Theorem 7.1]. *Let $\mathcal{J}$ be a composable class of filters. Then*

$$\mathrm{Epi}^P_{\mathcal{J}} = \mathrm{Adh}_{\mathcal{J}}.$$

Proof. Consider $x \in \lim_{\mathrm{Adh}_{\mathcal{J}} X} \mathcal{F}$, a $\mathcal{J}$-filter $\mathcal{G}$ such that $y \in \lim_Y \mathcal{G}$ and $A \# \mathcal{F} \times \mathcal{G}$. Since $\mathcal{J}$ is composable, $A\mathcal{G}$ is a $\mathcal{J}$-filter such that $A\mathcal{G} \# \mathcal{F}$. Thus, $x \in \mathrm{adh}_X A\mathcal{G}$. There exists an ultrafilter $\mathcal{U}$ of $A\mathcal{G}$ such that $(x,y) \in \lim_{X \times Y} \mathcal{U} \times \mathcal{G}$. Moreover $A \# \mathcal{U} \times \mathcal{G}$, so that $(x,y) \in \mathrm{adh}_{X \times Y} A$. Consequently, $\mathrm{Adh}_{\mathcal{J}} X \times Y \geq P(X \times Y)$ for every $\mathcal{J}$-based convergence space Y.

On the other hand, if $W \ngeq \mathrm{Adh}_{\mathcal{J}} X$, there is a filter $\mathcal{F}$ with $x_0 \in \lim_W \mathcal{F}$ and a $\mathcal{J}$-filter $\mathcal{H}$ such that $\mathcal{H} \# \mathcal{F}$ and $x_0 \notin \mathrm{adh}_X \mathcal{H}$. Let Y be the atomic $\mathcal{J}$-based topological space on $|X|$ defined by $\mathcal{N}_Y(x_0) = \mathcal{H} \wedge (x_0)$. Then $W \times Y \ngeq P(X \times Y)$. Indeed, $(x_0, x_0) \in \lim_{W \times Y} (\mathcal{F} \vee \mathcal{H} \times \mathcal{F} \vee \mathcal{H})$ but $(x_0, x_0) \notin \mathrm{adh}_{X \times Y} \{(x,x) : x \neq x_0\}$. Indeed $\mathcal{G} \geq \mathcal{H}$ whenever $(x_0, x_0) \in \lim_{X \times Y} \mathcal{G} \times \mathcal{G}$; a contradiction to $x_0 \notin \mathrm{adh}_X \mathcal{H}$. Consequently, Theorem 2.2 applies with $\mathbf{L} = \mathbf{P}$, $\mathbf{E} = \mathbf{Base}_{\mathcal{J}}$ and $\mathbf{J} = \mathbf{Conv}$, to the effect that $\mathrm{Epi}^P_{\mathcal{J}} = \mathrm{Adh}_{\mathcal{J}}$. $\square$

It is well known (e.g., [**7**, II.2]) that the pretopological space ¥ with underlying set $\{0,1,2\}$ and with vicinities $\mathcal{V}_{¥}(0) = \{\{0,1,2\}\}$, $\mathcal{V}_{¥}(1) = \{\{0,1,2\}\}$, $\mathcal{V}_{¥}(2) = \big\{\{0,1,2\},\ \{1,2\}\big\}$ is initially dense in the category $\mathbf{P}$ of pretopological spaces.

Theorem 3.2. *Let $\mathcal{D}$ and $\mathcal{J}$ be two composable classes of filters such that $\mathcal{D} \subset \mathcal{J}$. The following are equivalent*:

1. $W \times \mathrm{Adh}_{\mathcal{D}} Y \geq \mathrm{Adh}_{\mathcal{D}}(X \times Y)$, *for every* $Y \geq \mathrm{Adh}_{\mathcal{D}} \mathrm{Base}_{\mathcal{J}} Y$;
2. $W \times \mathrm{Adh}_{\mathcal{D}} Y \geq P(X \times Y)$, *for every $\mathcal{J}$-based convergence space* Y;
3. $\mathrm{Id}_{X,W} \times f$ *is* $\mathrm{Adh}_{\mathcal{D}}$*-quotient for every* $\mathrm{Adh}_{\mathcal{D}}$*-quotient map f with* $\mathrm{Adh}_{\mathcal{D}} \mathrm{Base}_{\mathcal{J}}$*-domain*;
4. $\mathrm{Adh}_{\mathcal{D}} \mathrm{Base}_{\mathcal{J}}[X,Z] \geq [W,Z]$ *for every* $Z = \mathrm{Adh}_{\mathcal{D}} Z$;
5. $\mathrm{Adh}_{\mathcal{D}} \mathrm{Base}_{\mathcal{J}}[X,¥] \geq [W,¥]$;
6. $W \geq S \mathrm{Base}_{\mathcal{D}} \mathrm{Adh}_{\mathcal{J}} X$.

Proof. In view of Theorem 2.5 combined with Theorem 3.1, $S\mathrm{Base}_{\mathcal{D}} \times \mathrm{Adh}_{\mathcal{J}} X \times \mathrm{Adh}_{\mathcal{D}} Y \geq \mathrm{Adh}_{\mathcal{D}}(X \times Y)$, for every $Y \geq \mathrm{Adh}_{\mathcal{D}}\mathrm{Base}_{\mathcal{J}} Y$, so that 6 implies 1, which obviously implies 2. Assume that $W \ngeq S\mathrm{Base}_{\mathcal{D}}\mathrm{Adh}_{\mathcal{J}} X$. Thus, there exists an ultrafilter $\mathcal{U}$ such that $x_0 \in \lim_W \mathcal{U} \backslash \lim_{\mathrm{Base}_{\mathcal{D}}\mathrm{Adh}_{\mathcal{J}} X} \mathcal{U}$. Hence, for every $\mathcal{D}$-filter $\mathcal{H}$ that meshes with $\mathcal{U}$, there exists a $\mathcal{J}$-filter $\mathcal{L}_{\mathcal{H}}$, such that $\mathcal{L}_{\mathcal{H}} \# \mathcal{H}$ but $x_0 \notin \mathrm{adh}_X \mathcal{L}_{\mathcal{H}}$. Let Y denote the atomic convergence space on $|X|$ in which $x_0 \in \lim_Y \mathcal{F}$ if and only if there exists a $\mathcal{D}$-filter $\mathcal{H} \# \mathcal{U}$ such that $\mathcal{F} \geq \mathcal{L}_{\mathcal{H}} \wedge (x_0)$. The convergence space Y is $\mathcal{J}$-based. On the other hand, $(x_0, x_0) \in \lim_{W \times \mathrm{Adh}_{\mathcal{D}} Y}(\mathcal{U} \times \mathcal{U})$ but $(x_0, x_0) \notin \lim_{P(X \times Y)}(\mathcal{U} \times \mathcal{U})$. Indeed, if $(x_0, x_0) \in \lim_{X \times Y}(\mathcal{G} \times \mathcal{G})$ for $\mathcal{G} \neq (x_0)$, then there exists a $\mathcal{D}$-filter $\mathcal{H} \# \mathcal{U}$ such that $\mathcal{G} \geq \mathcal{L}_{\mathcal{H}}$. Then $x_0 \notin \lim_X \mathcal{G}$ because $x_0 \notin \mathrm{adh}_X \mathcal{L}_{\mathcal{H}}$. Hence $W \times \mathrm{Adh}_{\mathcal{D}} Y \ngeq P(X \times Y)$. Consequently, 2 implies 6. The other equivalences follow immediately from Theorem 2.2. ◻

Notice that Theorem 3.2 combined with Corollary 2.6 applies with the class $\mathcal{D} = \mathcal{J}$ of all filters to the effect that

Corollary 3.3 [**7**, Theorem II.4.1]. *The Cartesian-closed hull of the category* $\mathbf{P}$ *of pretopological spaces is the category* $\mathbf{S}$ *of pseudotopological spaces.*

On the other hand, if $\mathcal{J}$ is the class of all filters while $\mathcal{D}$ stands for the class of principal filters, Theorem 3.2 applies to the effect that

Corollary 3.4. *X is quasi-exponential in* $\mathbf{P}$ *if and only if* $X \geq S\,\mathit{Fin}\,SX$. *In particular, a pretopological space X is exponential in* $\mathbf{P}$ *if and only if it is finitely generated.*

Hence [**25**, Theorem 3.4] is refined. The following table gathers the topological corollaries of Theorem 3.2 (sometimes via Theorem 2.8) in terms of product of quotient maps (see [**30**] for details). The parentheses mark an equivalent condition.

for every g	$f \times g$ **is**	**iff** f **is**
hereditarily quotient	hereditarily quotient	biquotient with finitely generated range
biquotient with finitely generated range		hereditarily quotient
hereditarily quotient with Fréchet domain		countably biquotient with finitely generated range
countably biquotient	hereditarily quotient (countably biquotient)	biquotient with bisequential range
countably biquotient with strongly Fréchet domain		countably biquotient with bisequential range
biquotient with bisequential range (identity of metrizable topology)	countably biquotient	countably biquotient
biquotient (identity)	hereditarily quotient (biquotient)	biquotient

Moreover, some of the above results can be stated in a more general form when f is an identity map. For instance, Theorem 3.2 applies with the class $\mathcal{D}$ of principal filters and the class $\mathcal{J}$ of countably based filters to the effect that

Corollary 3.5 [**30**, Corollary 7.11]. *Let X be a paratopological space. The following are equivalent*:

1. $Id_X \times f$ *is hereditarily quotient for every hereditarily quotient map* f (*equivalently with Fréchet domain*);
2. $X \times Y$ *is Fréchet for every Fréchet* Y (*equivalently every atomic Fréchet topological space*);
3. $X \geq S\mathrm{Fin}\, X$;
4. $[X, Z] = P[X, Z]$ *for every* $Z = PZ$;
5. $P\mathrm{First}\,([X, \yen]) \geq [X, \yen]$.

Analogously, Theorem 3.2 applies with the class $\mathcal{D} = \mathcal{J}$ of countably based filters to the effect that

Corollary 3.6 [**30**, Theorem 7.7, Corollary 7.19]. *Exponential objects in the category of paratopological spaces are bisequential paratopological spaces. If X is a paratopological space, the following are equivalent*:

1. $Id_X \times f$ *is countably biquotient for every countably biquotient map* f;
2. $Id_X \times f$ *is countably biquotient for every countably biquotient map* f *with strongly Fréchet domain*;
3. $[X, Z]$ *is a paratopological space for every paratopological space* Z;
4. $[X, ¥]$ *is a paratopological space*;
5. $P_\omega \mathrm{First}\,[X, ¥] \geq [X, ¥]$;
6. X *is bisequential*.

On the other hand, Theorem 2.7 together with Theorem 3.2 in case $\mathcal{D} = \mathcal{J}$ is the class of countably based filters leads to the following extension to convergence spaces of a combination of well-known results of E. Michael: [**27**, Proposition 4.D.4] and [**27**, Proposition 4.D.5].

Theorem 3.7 [**30**, Theorem 7.10]. *The following are equivalent*:

1. X *is strongly Fréchet*;
2. $\mathrm{adh}_X \mathcal{H} \subset \mathrm{adh}_{\mathrm{First}\, X} \mathcal{H}$ *for each countably based* $\mathcal{H}$;
3. $X \times Y$ *is Fréchet for every first-countable* Y;
4. $X \times Y$ *is Fréchet for every atomic metrizable topological space* Y;
5. $X \times Y$ *is strongly Fréchet for every bisequential* Y.

3.3. Extension to CAP. Following [**20**] and [**21**], I call *convergence-approach limit* on X a map $\lambda : \varphi X \to [0, \infty]^X$ which fulfills the properties:

$$\begin{array}{ll} (CAL1) & \forall\, x \in X,\ \lambda(x)(x) = 0; \\ (CAL2) & \mathcal{G} \geq \mathcal{F} \Longrightarrow \lambda(\mathcal{F}) \geq \lambda(\mathcal{G}); \\ (CAL3) & \forall\, \mathcal{F}, \mathcal{G} \in \varphi X,\ \lambda(\mathcal{F} \wedge \mathcal{G}) = \lambda(\mathcal{F}) \vee \lambda(\mathcal{G}). \end{array}$$

(X, λ), shortly X, is called a *convergence-approach space*. A map $f : X \to Y$ between two convergence-approach spaces is a *contraction* if

$$\lambda_Y(f(\mathcal{F}))\ (f(\cdot)) \le \lambda_X(\mathcal{F})(\cdot),$$

for every $\mathcal{F} \in \varphi X$. The category with convergence-approach spaces as objects and contractions as morphisms is a Cartesian-closed topological category denoted **CAP** [**20**]. Each convergence space X can be considered as a convergence-approach space by stating

$$\lambda_X(\mathcal{F})(x) = \begin{cases} 0 & \text{if } x \in \lim_X \mathcal{F} \\ \infty & \text{otherwise.} \end{cases}$$

Moreover, **Conv**, together with continuous maps, is included both reflectively and coreflectively in **CAP**. Indeed, if λ is a convergence-approach, then its **Conv**-coreflection is $c(\lambda)$ defined by $x \in \lim_{c(\lambda)} \mathcal{F}$ if and only if $\lambda(\mathcal{F})(x) = 0$, while its **Conv**-reflection is $r(\lambda)$ defined by $x \in \lim_{r(\lambda)} \mathcal{F}$ if and only if $\lambda(\mathcal{F})(x) < \infty$. In **CAP** the canonical Hom-structure is described for example in [**21**]. If X and Z are now two convergence-approach spaces, the limit λ on the set $\mathrm{Hom}\,(X, Z)$ of contractions from X to Z is given by

$$\lambda(\mathcal{F})(f) = \bigwedge \Big\{ \alpha : \underset{\mathcal{G} \in \varphi(|X|)}{\forall} \lambda_Z(ev(\mathcal{G} \times \mathcal{F})) \circ f(\cdot) \le \lambda_X(\mathcal{G})(\cdot) \bigvee \alpha \Big\},$$

and is called *continuous convergence-approach*. Since λ coincides with the continuous convergence in case X and Z are convergences, I extend to **CAP** the notation of **Conv** and use $[X, Z]$ instead of λ.

A convergence-approach λ is a *pseudo-approach space* [**21**] if

$$(PSAP) \quad \forall\, \mathcal{F} \in \varphi X,\ \lambda(\mathcal{F}) = \bigvee_{\mathcal{U} \in \beta(\mathcal{F})} \lambda(\mathcal{U});$$

and it is a *pre-approach space* [**20**] if $(CAL3)$ is strengthened to

$$(PRAP) \quad \lambda\Big(\bigwedge_{j \in J} \mathcal{F}_j\Big) = \bigvee_{j \in J} \lambda(\mathcal{F}_j), \qquad \text{for any family } (\mathcal{F}_j)_{j \in J} \text{ of filters.}$$

The category **PSAP** of pseudo-approach spaces contains the category **S** of pseudotopological spaces and the category **PRAP** of pre-approach spaces contains the category **P** of pretopological spaces both reflectively and coreflectively (via the restrictions of c and r).

An *approach space* is a pre-approach space fulfilling

$$(AP) \quad \text{for any } \mathcal{F} \in \varphi X \text{ and any selection } (\mathcal{F}_x)_{x\in X} \text{ of filters,}$$
$$\lambda\Big(\bigvee_{F\in\mathcal{F}}\bigwedge_{x\in F}\mathcal{F}_x\Big)(\cdot) \leq \lambda(\mathcal{F})(\cdot) + \bigvee_{x\in X}\lambda(\mathcal{F}_x)(x).$$

The category **T** of topological spaces (with continuous maps) is a reflective and coreflective (via the restrictions of r and c) subcategory of the category **AP** of approach spaces [**23**]. There are several other equivalent descriptions of **AP** and **PRAP** (see [**24**] and [**22**] for details). There exists an approach space M which is initially dense in **AP** [**21**, Theorem 3.7]. The associated #-object (in the sense of [**15**]) $M^{\triangle}$ is initially dense in **PRAP** [**21**, Theorem 4.1].

As observed in [**31**], the definitions of the reflectors $\mathrm{Adh}_{\mathcal{J}}$ and of the coreflectors $\mathrm{Base}_{\mathcal{J}}$ extend to **CAP** via

$$\mathrm{Adh}_{\mathcal{J}}\lambda(\mathcal{F})(x) = \bigvee_{\mathcal{J}\ni\mathcal{H}\#\mathcal{F}} \mathrm{adh}_{\lambda}\mathcal{H}(x) = \bigvee_{\mathcal{J}\ni\mathcal{H}\#\mathcal{F}}\ \bigwedge_{\mathcal{U}\in\beta(\mathcal{H})}\lambda(\mathcal{U})(x); \tag{3.4}$$

and

$$(\mathrm{Base}_{\mathcal{J}}\lambda)(\mathcal{F})(\cdot) = \bigwedge_{\mathcal{J}\ni\mathcal{G}\leq\mathcal{F}}\lambda(\mathcal{G})(\cdot). \tag{3.5}$$

When $\mathcal{J}$ is respectively the class of all filters and of principal filters, $\mathrm{Adh}_{\mathcal{J}}$ is respectively the reflector on **PSAP** and on **PRAP**. This last fact [**31**, Theorem 5.7] is in no way obvious and gives a new explicit description of the reflector on **PRAP**. Moreover, the category of para-approach spaces is introduced as the category of fixed points for $\mathrm{Adh}_{\mathcal{J}}$ with the class $\mathcal{J}$ of countably based filters.

The main achievement of [**31**] is that Theorems 3.1 and 3.2 both extend to **CAP**.[9] Hence all their corollaries also extend to **CAP** with the convention that the terminology used in **Conv** for $\mathrm{Adh}_{\mathcal{D}}\mathrm{Base}_{\mathcal{J}}$ properties also extends to **CAP**. In particular, all the product results for quotient maps in the table above remain true in **CAP**. Analogously, Corollaries 3.5, 3.6 and Theorem 3.7 remain true, just replacing the initially dense pretopological space ¥ by the intially dense pre-approach space $M^{\triangle}$ and the inclusion $\subset$ by $\geq$. In particular, the **CAP** extension of Corollary 3.3 characterizes the Cartesian-closed hull of **PRAP** as **PSAP** [**21**, Corollary 5.10]. Analogously the extension of Corollary 3.4 characterizes exponential objects in **PRAP**. The characterization of exponential objects in **PRAP** obtained by E. Lowen, R. Lowen and Verbeeck [**22**, Theorem 3.7] was given in a different language: exponential objects in **PRAP** are the pre-metric pre-approach spaces, i.e., the pre-approach structure λ is determined by a map $d : |\lambda| \times |\lambda| \to [0, \infty]$ which is zero on the diagonal in the following way

$$\lambda(\mathcal{F})(x) = \bigwedge_{F \in \mathcal{F}} \bigvee_{y \in F} d(x, y).$$

To each object (X, λ) in **CAP** we can associate a pre-metric d_λ defined by $d_\lambda(x, y) = \lambda(y)(x)$. Conversely, a pre-approach λ_d can be associated to each pre-metric d via $\lambda_d(\mathcal{F})(x) = \bigwedge_{F \in \mathcal{F}} \bigvee_{y \in F} d(x, y)$. If $\lambda = \operatorname{Fin} \lambda$ then $\lambda(\mathcal{F})(x) = \bigwedge_{F \in \mathcal{F}} \lambda(F)(x) = \bigwedge_{F \in \mathcal{F}} \lambda(\bigwedge_{y \in F} y)(x)$. If moreover $\lambda = P\lambda$ then $\lambda(\bigwedge_{y \in F} y)(x) = \bigvee_{y \in F} \lambda(y)(x)$ so that λ is a pre-metric. Conversely, a pre-metric convergence-approach space is obviously a pre-approach space fixed by Fin. This new viewpoint allows to see the characterization [**25**, Theorem 3.4] of exponential objects in **P** as a corollary of the characterization of exponential objects in **PRAP**. Analogously, exponential objects in para-approach spaces are the bisequential ones [**31**, Corollary 7.2].

4. Other applications in Conv.

4.1. Strong sequentiality versus strong Fréchetness. Theorem 3.7 characterizes topological spaces, and more generally convergence spaces, whose product with every first-countable, equivalently metrizable, topological space is Fréchet. In [**40**], Y. Tanaka adresses the similar problem of charaterizing topological spaces whose product with every first-countable, equivalently metrizable, topological space is sequential. Contrary to the situation of Theorem 3.7, the problem had been solved by Tanaka in [**40**] only for regular Hausdorff Fréchet spaces. Under these assumptions, the spaces that answer the problem are once again the strongly Fréchet ones. Thanks to the method of coreflectively modified duality, I was able to solve the general problem in

[**29**]. Indeed, in view of the characterization of sequential topologies as TFirst-structures (see Table, page 742) the problem extended to **Conv** is to find convergence spaces X for which $X \times Y \geq T\mathrm{First}\,(X \times Y)$, for every $Y = \mathrm{First}\,Y$. In other words, the extended problem of Tanaka is to characterize the convergence spaces X for which

$$X \times Y \geq T(\mathrm{First}\,X \times Y),$$

for every $Y = \mathrm{First}\,Y$. In view of Theorem 2.2, X answers the problem if and only if

$$X \geq \mathrm{Epi}^{T}_{\mathrm{First}}\mathrm{First}\,X. \tag{4.1}$$

I call such a convergence space *strongly sequential*. The remaining problem is an explicit description of $\mathrm{Epi}^{T}_{\mathrm{First}}$. The reflector $\mathrm{Epi}^{T}_{\mathcal{J}}$ is denoted $A_{\mathcal{J}}$, as a coreflectively modified version of the reflector A on Antoine convergence spaces, which is obtained when $\mathcal{J}$ is the class of all filters. The closure of points plays a key role in the classical characterization [**7**] by Bourdaud of Antoine convergences. More generally, to describe the reflectors $A_{\mathcal{J}}$ we need:

$$\mathrm{ad}_X A = \bigcup_{a \in A} \lim\nolimits_X(a), \quad \mathrm{ad}_{TX} A = \bigcup_{a \in A} \mathrm{cl}_X a.$$

Let $\mathrm{ad}^{\natural}_{TX}\mathcal{H}$ denote the filter generated by $\{\mathrm{ad}_{TX} H : H \in \mathcal{H}\}$ and let $(\mathcal{J})_{\mathrm{ad}_{TX}}$ denote the class of $\mathcal{J}$-filters $\mathcal{H}$ for which $\mathcal{H} = \mathrm{ad}^{\natural}_{TX}\mathcal{H}$.

Theorem 4.1 [**29**, Theorem 2.3]. *If $\mathcal{J}$ is a composable class of filters, then the reflector $A_{\mathcal{J}}$ is given by*

$$\lim\nolimits_{A_{\mathcal{J}}X}\mathcal{F} = \bigcap_{(\mathcal{J})_{\mathrm{ad}_{TX}} \ni \mathcal{H} \# \mathcal{F}} \mathrm{cl}_X(\mathrm{adh}_X \mathcal{H}). \tag{4.2}$$

Notice that Theorem 4.1 gives in particular (when $\mathcal{J}$ is the class of all filters) an explicit description of the reflector on Antoine convergence spaces. Thus it refines the classical characterization [**7**, Theorem I.4.4] of Antoine spaces by Bourdaud.

Finally, in view of Theorem 2.7, the following answers Tanaka's problem.

Theorem 4.2 [**29**, Theorem 3.1]. *The following are equivalent*:

1. X *is strongly sequential*;

2. $\mathrm{adh}_X\mathcal{H} \subset \mathrm{cl}_{\mathrm{First}X}(\mathrm{adh}_{\mathrm{First}X}\mathcal{H})$ *for each countably based* $\mathcal{H}$ *such that* $\mathcal{H} = \mathrm{ad}^{\natural}_{TX}\mathcal{H}$;

3. $X \times Y$ *is sequential for each first-countable convergence space* Y;

4. $X \times Y$ *is sequential for each metrizable atomic topological space* Y;

5. $X \times Y$ *is strongly sequential for each quasi-bisequential convergence space* Y.

A convergence space X is *quasi-bisequential* whenever $X \geq A\mathrm{First}\, X$. Since $AX = SX$ for each Hausdorff convergence space X (see for example [**7**]) quasi-bisequentiality and bisequentiality coincide for Hausdorff convergences. Strongly sequential spaces appear naturally in other product problems for sequential spaces.

Theorem 4.3 [**29**, Theorem 5.1]. *Let* Y *be a first-countable regular* T_1 *topological space. Then* $X \times Y$ *is sequential if and only if* X *is strongly sequential or* Y *is locally countably compact.*

Theorem 4.4 [**29**, Theorem 5.3]. *If* X^{ω} *is sequential, then* X *is strongly sequential.*

Theorem 4.5 [**13**, Theorem 12.1]. *A topology* X *is core-contour* (*First*)*-compact and strongly sequential if and only if* $X \times Y$ *is sequential for each sequential convergence* (*equivalently, topological*) *space* Y.

4.2. Products of quotient maps. Since the general scheme allows to derive product theorems for quotient maps from every commutation result as (1.3), the following corollaries are obtained as byproducts of the study of the commutation of the topologizer with product in [**13**] and of the following commutation problem in [**30**, Theorem 6.3]

$$\forall Y = \mathrm{Base}_{\mathcal{J}} Y,\ W \times \mathrm{Adh}_{\mathcal{D}} Y \geq T(X \times Y). \tag{4.3}$$

See [**13**] and [**30**] for details and precise definitions.

<table>
<tr><th>for every g</th><th>$f \times g$ is</th><th>iff f is</th></tr>
<tr><td>quotient</td><td rowspan="6">quotient</td><td>A-quotient with
core-compact topological range</td></tr>
<tr><td>quotient with
sequential domain</td><td>A_ω-quotient with
core-contour (First)-compact
topological range</td></tr>
<tr><td>hereditarily quotient</td><td>A-quotient with
T-core-compact range</td></tr>
<tr><td>hereditarily quotient
with Fréchet domain</td><td>A_ω-quotient with
T-core-countably compact
range</td></tr>
<tr><td>countably biquotient</td><td>A-quotient with
core-bi-k range</td></tr>
<tr><td>countably biquotient with
strongly Fréchet domain</td><td>A_ω-quotient
with core-bi-quasi-k range</td></tr>
<tr><td>A-quotient
(identity)</td><td>quotient
(A-quotient)</td><td>A-quotient</td></tr>
<tr><td>A-quotient with
quasi-bisequential domain
(identity of
metrizable topology)</td><td>quotient
(A_ω-quotient)</td><td>A_ω-quotient</td></tr>
</table>

The classes of A-quotient (Antoine-quotient) and A_ω-quotient maps (countably Antoine quotient) maps are characterized as follows.[10]

Theorem 4.6 [**13**, Theorem 11.3]. *Let $f : X \to Y$ be a quotient map. Then the following are equivalent*:

1. *f is A-quotient* (*respectively A_ω-quotient*);

2. *$f \times \mathrm{Id}_Y$ is quotient for every convergence space Y* (*respectively every first-countable convergence space Y*) *equivalently for every atomic* (*respectively atomic metrizable*) *topological space*;

3. *$f \times g$ is A-quotient* (*respectively A_ω-quotient*) *for every A-quotient map g* (*respectively A-quotient map with bisequential domain*);

4. *If $y \in \lim_Y \mathcal{F}$, then $\mathcal{F}$ is Y-compactoid* (*respectively Y-countably compactoid*) *in $\mathcal{N}_Y(y)$* [11];

5. *If $\mathcal{H} = \mathrm{ad}^\natural_Y \mathcal{H}$* (*respectively and is countably based*) *and $y \in \mathrm{adh}_Y \mathcal{H}$, then*

$$f^-(y) \bigcap \mathrm{cl}_W(\mathrm{adh}_X f^- \mathcal{H}) \neq \varnothing;$$

where W is the initial object (in **Conv**) *with respect to f and Y.*

6. *If $y \in \lim_Y \mathcal{F}$, V is a Y-open set containing y, and $\mathcal{S}$ is a X-cover* (*respectively a countable X-cover*) *of $f^- V$, there exists a finite subfamily $\mathcal{P} \subset \mathcal{S}$ such that the intersection of all Y-open sets containing $\underset{P \in \mathcal{P}}{\cup} f(P)$ is an element of $\mathcal{F}$.*

This result generalizes in particular [**9**, Theorem 2] of Day and Kelly.

4.3. Continuous convergence and upper Kuratowski convergence. Analogously, equivalences between 6, 7 and 8 in Theorem 2.2 lead to new relations between a convergence space and its continuous duals. In particular, as the Sierpiński (topological) space \$ is initially dense in **T**, the results of [**13**] on the commutation of the topologizer with product and [**30**, Theorem 6.3] on (4.3) apply to the effect that

$[X, \$]$ **verifies**	**iff** X **is**
$[X, \$] = P([X, \$])$	T-core-compact
$[X, \$] = T([X, \$])$	core-compact (if topological)
$[X, \$] = P_\omega([X, \$])$	core-bi-k
$P\mathrm{First}\, [X, \$] \geq [X, \$]$	T-core-countably compact
$P_\omega\mathrm{First}\, [X, \$] \geq [X, \$]$	core-bi-quasi-k

See [**13**] and [**30**] for details and precise definitions.

Notice that $[X, \$]$ is homeomorphic to the upper Kuratowski convergence on the set of X-closed sets and, transposed on open sets, it is homeomorphic to the Scott convergence (e.g., [**12**]). Among the above results, the first one was only known for a topological space X. The second one is [**18**, Proposition 4.2] and the others are new.

On the other hand, variants of the results of [**13**] contained in [**32**] allows to extend [**19**, Theorem 3.2] from the usual topology of the real line to every regular topological space[12]:

Corollary 4.7 [**32**]. *If X is a locally compact convergence space and if Z is a regular topological space, then the continuous convergence $[X, Z]$ is a topology.*

Moreover, the following new variant is obtained.

Corollary 4.8 [**32**]. *If X is a locally countably compact convergence space and if Z is a regular topological space, then the continuous convergence $[X, Z]$ fulfills*:

$$T\mathrm{First}\,[X, Z] \geq [X, Z].$$

4.4. When does compactness, countable compactness, Lindelöfness imply topologicity? Let Ω denote the reflector on completely regular topological spaces. Notice that $\mathrm{Epi}^{\mathbf{R}}_{\mathrm{Fin}} = \Omega$ while $\mathrm{Epi}^{\mathbf{R}}$ is the reflector on (non-necessarily Hausdorff) c-embedded convergence spaces in the sense of Binz. In [**6**], Binz proved

Theorem 4.9. *A compact c-embedded convergence is a topology.*

Recall that a convergence is *compact* provided that every ultrafilter converges. More generally, I call $\mathcal{J}$-*compact* a convergence for which $\mathrm{adh}\,\mathcal{H} \neq \varnothing$ for every $\mathcal{J}$-filter $\mathcal{H}$. Notice that $\mathcal{J}$-compactness extends to convergences the usual notions of compactness, countable compactness and Lindelöfness respectively, provided $\mathcal{J}$ is the class of all, of countably based and of countably deep[13] filters respectively.

In [**33**], it is proved that for $W \geq \Omega X$,

$$W \times PY \geq \Omega(X \times Y),$$

for every $Y = \mathrm{Base}_{\mathcal{J}} Y$ provided that W is locally X-$\mathcal{J}$-compact, i.e., every W-convergent filter contains a X-$\mathcal{J}$-compact set. Of course, if X is $\mathcal{J}$-compact, then ΩX is locally X-$\mathcal{J}$-compact. Thus

$$\Omega X \times Y \geq \Omega(X \times Y),$$

for every $Y = \mathrm{Base}_{\mathcal{J}} Y$. In view of Theorem 2.2, $\Omega X = \mathrm{Epi}^{\mathbf{R}}_{\mathcal{J}}$. Hence Theorem 4.9 of Binz is generalized to the following.

Theorem 4.10 [**33**]. *Let $\mathcal{J}$ be a composable class of filters. If a convergence is $\mathcal{J}$-compact, then its* $\mathrm{Epi}^{\mathbf{R}}_{\mathcal{J}}$*-reflection is a* (*not-necessarily Hausdorff*) *completely regular topology.*

An explicit description of convergence spaces fixed by $\mathrm{Epi}^{\mathbf{R}}_{\mathcal{J}}$ that generalizes the classical characterization of c-embedded convergences [**8**, Theorem 4.6] by Bourdaud is given in [**33**].

ENDNOTES

1. That is, in this context, the existence of canonical Hom-structures on $\mathrm{Hom}\,(X,Y)$ for every $Y \in \mathrm{Ob}\,(\mathbf{C})$.

2. Notice that Epi^L_E being a reflector while E is a coreflector, the range of a Epi^L_E-quotient map is a $\mathrm{Epi}^L_E E$-structure whenever the domain is a $\mathrm{Epi}^L_E E$-structure by Proposition 2.1.

3. That is, $f \times \mathrm{Id}_Y$ is L-quotient for every $\mathbf{L}$-object Y.

4. f^- denotes the inverse relation of f and $f^-\mathcal{H}$ denotes the filter generated by $\{f^- H : H \in \mathcal{H}\}$.

5. Every point has a smallest vicinity.

6. $x \in \lim_{KX} \mathcal{F}$ if $x \in \lim_X \mathcal{F}$ and there exists $K \in \mathcal{F}$ such that $\lim_X \mathcal{U} \cap K \neq \varnothing$ for every ultrafilter $\mathcal{U}$ on K. Analogously, $x \in \lim_{\mathrm{First}_K X} \mathcal{F}$, respectively $x \in \lim_{\mathrm{First}_{K_\omega} X} \mathcal{F}$, if $x \in \lim_X \mathcal{F}$ and there exists a countably based filter $\mathcal{H} \leq \mathcal{F}$ such that $\mathrm{adh}_X \mathcal{G} \cap \mathcal{H} \neq \varnothing$ for every $\mathcal{G} \# \mathcal{H}$, respectively for every countably based $\mathcal{G} \# \mathcal{H}$.

7. $HG = \{y : \exists_{x \in G} (x,y) \in H\}$.

8. In [**27**], Michael uses the term *countably bisequential* for strongly Fréchet.

9. Moreover, their counterparts in **Conv** described above are corollaries of the general versions in **CAP**.

10. Theorem 4.6 characterizes A-quotient and A_ω-quotient maps among quotient maps. They can also be characterized among continuous surjections. See [**13**].

11. A filter $\mathcal{F}$ is (*countably*) *X-compactoid* in $\mathcal{A}$ if $\mathrm{adh}_X \mathcal{H} \# \mathcal{A}$ for every (countably based) filter $\mathcal{H} \# \mathcal{F}$.

12. Beattie and Butzmann give an alternative proof of this result [**3**, Theorem 1.4.17] for Hausdorff convergences.

13. A filter $\mathcal{F}$ is *countably deep* if $\cap\mathcal{A} \in \mathcal{F}$ for every countable family $\mathcal{A} \subset \mathcal{F}$.

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