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# TWO THETA FUNCTION IDENTITIES AND SOME EISENSTEIN SERIES IDENTITIES OF RAMANUJAN

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ABSTRACT. In this paper we prove two general theta function identities by using the complex variable theory of elliptic functions. As applications, we provide completely new proofs of some Eisenstein series identities of Ramanujan by these two theta function identities and one famous identity of Ramanujan for the Rogers-Ramanujan continued fraction. We also derive two remarkable theta function identities relating to the modular equations of degree 5.

**1. Introduction.** We suppose throughout that $q = e^{2\pi i\tau}$, $\operatorname{Im}\tau > 0$; this condition ensures that all the sums and products that appear here converge. The Dedekind eta-function is defined by

$$\eta(\tau) = q^{(1/24)}(q;q)_\infty = e^{(\pi i\tau)/12}\prod_{n=1}^{\infty}(1-e^{2\pi in\tau}), \tag{1.1}$$

where and throughout the paper

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \quad |q|<1. \tag{1.2}$$

The Rogers-Ramanujan continued fraction $R(q)$ is defined by

$$R(q) = q^{1/5}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{1.3}$$

To prove several of Ramanujan's claims on $R(q)$ that were made in his first two letters to Hardy, Watson [**16**] first proved the following

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two identities about $R(q)$ that can be found in Ramanujan's second notebook [**15**, pp. 265–267].

**Theorem 1.** *Let* $R(q)$ *be defined by* (1.3). *Then we have*

$$\frac{1}{R(q^5)} - 1 - R(q^5) = \frac{(q;q)_\infty}{q(q^{25};q^{25})_\infty} = \frac{\eta(\tau)}{\eta(25\tau)}, \tag{1.4}$$

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{(q;q)_\infty^6}{q(q^5;q^5)_\infty^6} = \frac{\eta^6(\tau)}{\eta^6(5\tau)}. \tag{1.5}$$

Recently, in [**10**], we derived a very general theta function identity, which includes (1.4) and (1.5) as special cases. Our method is quite different from Watson's method.

Define the Eisenstein series $L(\tau), M(\tau)$, and $N(\tau)$ by

$$L(\tau) = 1 - 24\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}, \tag{1.6}$$

$$M(\tau) = 1 + 240\sum_{n=1}^{\infty} \frac{n^3q^n}{1-q^n}, \tag{1.7}$$

$$N(\tau) = 1 - 504\sum_{n=1}^{\infty} \frac{n^5q^n}{1-q^n}. \tag{1.8}$$

In [**5**], by the logarithmic differentiation of (1.4) and (1.5), the authors gave short elegant proofs of the following two beautiful Eisenstein series identities of Ramanujan.

**Theorem 2.** *If we define*

$$A(\tau) = 25L(25\tau) - L(\tau) \quad \textit{and} \quad B(\tau) = 5L(5\tau) - L(\tau), \tag{1.9}$$

*then we have*

$$A(\tau) = 24\frac{\eta^5(5\tau)}{\eta(\tau)}\left\{\frac{\eta^2(\tau)}{\eta^2(25\tau)} + 2\frac{\eta(\tau)}{\eta(25\tau)} + 5\right\}^{1/2} \tag{1.10}$$

*and*

$$B(\tau) = 4\left\{\frac{\eta^{10}(\tau)}{\eta^2(5\tau)} + 22\eta^4(\tau)\eta^4(5\tau) + 125\frac{\eta^{10}(5\tau)}{\eta^2(\tau)}\right\}^{1/2}. \tag{1.11}$$

Equations (1.9) and (1.10) are entries 4(i) and 7(iii) in Chapter 21 of Ramanujan's second notebook [**2**, pp. 463, 475]. In his lost notebook [**14**], Ramanujan recorded formulas for $L(n\tau)$, $M(n\tau)$ and $N(n\tau)$, for certain positive integers $n$, as sums of quotients of Dedekind eta-functions. These particular quotients, called Hauptmoduls, frequently arise in the theory and applications of modular forms and elliptic functions. In particular, on pages 50–51 of his lost notebook [**14**], Ramanujan claimed

**Theorem 3.** *Let $\eta(\tau)$, $L(\tau)$, $M(\tau)$ and $N(\tau)$ be defined by* (1.1), (1.6), (1.7) *and* (1.8), *respectively. Then we have*

$$M(\tau) = \frac{\eta^{10}(\tau)}{\eta^2(5\tau)} + 250\eta^4(\tau)\eta^4(5\tau) + 3125\frac{\eta^{10}(5\tau)}{\eta^2(\tau)}, \tag{1.12}$$

$$M(5\tau) = \frac{\eta^{10}(\tau)}{\eta^2(5\tau)} + 10\eta^4(\tau)\eta^4(5\tau) + 5\frac{\eta^{10}(5\tau)}{\eta^2(\tau)}, \tag{1.13}$$

$$\begin{aligned} N(\tau) &= \left\{\frac{\eta^{10}(\tau)}{\eta^2(5\tau)} - 500\eta^4(\tau)\eta^4(5\tau) - 15625\frac{\eta^{10}(5\tau)}{\eta^2(\tau)}\right\} \\ &\quad \times \left\{\frac{\eta^{10}(\tau)}{\eta^2(5\tau)} + 22\eta^4(\tau)\eta^4(5\tau) + 125\frac{\eta^{10}(5\tau)}{\eta^2(\tau)}\right\}^{1/2}, \end{aligned} \tag{1.14}$$

$$\begin{aligned} N(5\tau) &= \left\{\frac{\eta^{10}(\tau)}{\eta^2(5\tau)} + 4\eta^4(\tau)\eta^4(5\tau) - \frac{\eta^{10}(5\tau)}{\eta^2(\tau)}\right\} \\ &\quad \times \left\{\frac{\eta^{10}(\tau)}{\eta^2(5\tau)} + 22\eta^4(\tau)\eta^4(5\tau) + 125\frac{\eta^{10}(5\tau)}{\eta^2(\tau)}\right\}^{1/2}. \end{aligned} \tag{1.15}$$

As usual, he gave no proofs. The first published proofs of these four identities are due to Raghavan and Rangachari [**13**], who used the theory of modular forms with which Ramanujan was likely unfamiliar. These proofs are very short but do not provide any insight into how Ramanujan discovered the identities. Berndt, Chan, Sohn and Son [**3**] recently found proofs of (1.12)–(1.15) based entirely on results found in Ramanujan's notebooks [**15**] and lost notebook [**14**]. In the present paper, we present a third, quite different approach that uses the identity (1.5) of Ramanujan and the complex theory of elliptic functions. The Jacobi theta functions play a pivotal role in our investigation. And they are defined by [**17**, p. 464]

(1.16)
$$\theta_1(z|\tau) = 2q^{1/8}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}\sin(2n+1)z,$$

(1.17)
$$\theta_2(z|\tau) = 2q^{1/8}\sum_{n=0}^{\infty} q^{n(n+1)/2}\cos(2n+1)z,$$

(1.18)
$$\theta_3(z|\tau) = 1 + 2\sum_{n=0}^{\infty} q^{n^2/2}\cos 2nz,$$

(1.19)
$$\theta_4(z|\tau) = 1 + 2\sum_{n=0}^{\infty}(-1)^n q^{n^2/2}\cos 2nz.$$

The remainder of this paper is organized as follows. In Section 2 we briefly recall some basic facts of Jacobi's theta functions. In Section 3 we prove two general theta function identities (3.1) and (3.2) below. In Section 4 we prove (1.12)–(1.15) using (3.1), (3.2) and (1.5).

In Section 5 of the paper we prove the following two theta function identities related to modular equations of degree 5.

**Theorem 4.** *Let* $\theta_2(z|\tau), \theta_3(z|\tau)$ *and* $\theta_4(z|\tau)$ *be defined by* (1.17), (1.18) *and* (1.19), *respectively. Then we have*

(1.20)
$$\frac{\theta_2^9(0|5\tau)}{\theta_2(0|\tau)} - \frac{\theta_3^9(0|5\tau)}{\theta_3(0|\tau)} + \frac{\theta_4^9(0|5\tau)}{\theta_4(0|\tau)} = 44\eta(\tau)\eta^7(5\tau) + 4\eta^7(\tau)\eta(5\tau),$$

*and*

(1.21)
$$\frac{\theta_2^9(0|\tau)}{\theta_2(0|5\tau)}-\frac{\theta_3^9(0|\tau)}{\theta_3(0|5\tau)}+\frac{\theta_4^9(0|\tau)}{\theta_4(0|5\tau)}=2500\eta(\tau)\eta^7(5\tau)+220\eta^7(\tau)\eta(5\tau).$$

Our proofs are constructive and may shed some light on the origin of identities (1.12)–(1.15).

**2. Some basic facts about the Jacobi theta functions.** In this section we will discuss some basic properties about the Jacobi theta functions. Using the Jacobi triple product formula one can obtain the infinite product representations for the Jacobi theta functions [**17**, p. 470],

(2.1)
$$\theta_1(z|\tau)=2q^{1/8}(\sin z)(q;q)_\infty(qe^{-2zi};q)_\infty(qe^{2zi};q)_\infty,$$
(2.2)
$$\theta_2(z|\tau)=2q^{1/8}(\cos z)(q;q)_\infty(-qe^{-2zi};q)_\infty(-qe^{2zi};q)_\infty,$$
(2.3)
$$\theta_3(z|\tau)=(q;q)_\infty(-q^{1/2}e^{-2zi};q)_\infty(-q^{1/2}e^{2zi};q)_\infty,$$
(2.4)
$$\theta_4(z|\tau)=(q;q)_\infty(q^{1/2}e^{-2zi};q)_\infty(q^{1/2}e^{2zi};q)_\infty.$$

By simple computations, we find that

(2.5) $$\theta_1\Big(z+\frac{1}{2}\pi|\tau\Big)=\theta_2(z|\tau),$$

(2.6) $$\theta_1\Big(z+\frac{1}{2}(\pi+\pi\tau)|\tau\Big)=q^{-1/8}e^{-iz}\theta_3(z|\tau),$$

(2.7) $$\theta_1\Big(z+\frac{1}{2}\pi\tau|\tau\Big)=iq^{-1/8}e^{-iz}\theta_4(z|\tau).$$

From the definition of $\theta_1(z|\tau)$, we can simply derive that, for any integer $n$,

(2.8) $$\theta_1(z+n\pi|\tau)=(-1)^n\theta_1(z|\tau),$$

(2.9) $$\theta_1(z+n\pi\tau|\tau)=(-1)^nq^{-n^2/2}e^{-2niz}\theta_1(z|\tau).$$

Taking $n = 1$ in (2.9) and then replacing $\tau$ by $5\tau$ and finally letting $z = -\pi\tau$ and $z = -2\pi\tau$ respectively, we derive that

(2.10)
$$\theta_1(4\pi\tau|5\tau) = q^{-3/2}\theta_1(\pi\tau|5\tau) \quad \text{and} \quad \theta_1(3\pi\tau|5\tau) = q^{-1/2}\theta_1(2\pi\tau|5\tau).$$

We use $\theta_1'(z|\tau)$ to denote the partial derivative with respect to the variable $z$. Differentiating (2.1) with respect to $z$ and then, putting $z = 0$, we obtain

$$\theta_1'(0|\tau) = 2q^{1/8}(q;q)_\infty^3 = 2\eta^3(\tau). \tag{2.11}$$

Differentiating (2.8) and (2.9) respectively with respect to $z$ and then setting $z = 0$, we find that

$$\theta_1'(n\pi|\tau) = (-1)^n\theta_1'(0|\tau), \tag{2.12}$$

$$\theta_1'(n\pi\tau|\tau) = (-1)^n q^{-n^2/2}\theta_1'(0|\tau). \tag{2.13}$$

Replacing $\tau$ by $5\tau$ in (2.1) and then, taking $z = \pi\tau$ and $2\pi\tau$ in the resulting equation, respectively we find that

$$\theta_1(\pi\tau|5\tau) = iq^{1/8}(q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty, \tag{2.14}$$

(2.15)
$$\theta_1(2\pi\tau|5\tau) = iq^{-3/8}(q^2;q^5)_\infty(q^3;q^5)_\infty(q^5;q^5)_\infty.$$

It follows that

(2.16)
$$\theta_1(\pi\tau|5\tau)\theta_1(2\pi\tau|5\tau) = -q^{-1/4}(q;q)_\infty(q^5;q^5)_\infty = -q^{-1/2}\eta(\tau)\eta(5\tau)$$

and

$$\frac{\theta_1(\pi\tau|5\tau)}{\theta_1(2\pi\tau|5\tau)} = q^{3/10}R(q). \tag{2.17}$$

Similarly, from (2.2)–(2.4), we can derive that, see also [**9**, p. 133]

$$\theta_j(\pi\tau|5\tau)\theta_j(2\pi\tau|5\tau) = \sqrt{\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty}}\sqrt{\frac{\theta_j(0|\tau)}{\theta_j(0|5\tau)}}, \quad j = 2, 3, 4. \tag{2.18}$$

The Jacobi imaginary transformation formulas [**4**, p. 76] are

$$\theta_2(0|-1/\tau)=\sqrt{-\tau i}\,\theta_4(0|\tau), \tag{2.19}$$

$$\theta_3(0|-1/\tau)=\sqrt{-\tau i}\,\theta_3(0|\tau), \tag{2.20}$$

$$\theta_4(0|-1/\tau)=\sqrt{-\tau i}\,\theta_2(0|\tau). \tag{2.21}$$

From [**1**, pp. 24, 48, 69], we know that

$$\eta(-1/\tau)=\sqrt{-i\tau}\,\eta(\tau), \tag{2.22}$$

$$L(-1/\tau)=-\frac{6i\tau}{\pi}+\tau^2L(\tau), \tag{2.23}$$

$$M(-1/\tau)=\tau^4M(\tau), \tag{2.24}$$

$$N(-1/\tau)=\tau^6N(\tau). \tag{2.25}$$

It follows that

$$\eta(-1/5\tau)=\sqrt{-5i\tau}\,\eta(\tau), \tag{2.26}$$

$$B(-1/5\tau)=-5\tau^2B(\tau), \tag{2.27}$$

$$M(-1/5\tau)=5^4\tau^4M(5\tau), \tag{2.28}$$

$$N(-1/5\tau)=5^6\tau^6N(5\tau). \tag{2.29}$$

The trigonometric series expansion for the logarithmic derivative of $\theta_1(z|\tau)$ [**17**, p. 489] is

$$\frac{\theta_1'}{\theta_1}(z|\tau)=\cot z+4\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sin 2nz. \tag{2.30}$$

The Bernoulli numbers $B_k$ are defined as the coefficients in the power series

$$\frac{x}{e^x-1}=\sum_{k=0}^{\infty}B_k\frac{x^k}{k!},\quad |x|<2\pi. \tag{2.31}$$

It is easy to show that $B_{2k+1}=0$ for $k\geq 1$, and the first few values of $B_k$ are

$$\begin{gathered}B_0=1,\quad B_1=-\frac{1}{2},\quad B_2=\frac{1}{6},\quad B_4=-\frac{1}{30},\\ B_6=\frac{1}{42},\quad B_8=-\frac{1}{30},\quad B_{10}=\frac{5}{66},\quad B_{12}=-\frac{691}{2730}.\end{gathered} \tag{2.32}$$

The normalized Eisenstein series $E_{2k}(\tau)$ are defined by

$$E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \frac{n^{2k-1}q^n}{1-q^n} = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n, \tag{2.33}$$

where $\sigma_k(n)$ stands for the standard function and is defined as

$$\sigma_k(n) = \sum_{d|n} d^k, \tag{2.34}$$

and it is also understood that $\sigma_k(x) = 0$ if $x$ is not an integer. The first few $E_{2k}(\tau)$ are

$$E_2(\tau) = L(\tau) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n, \tag{2.35}$$

$$E_4(\tau) = M(\tau) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3q^n}{1-q^n} = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \tag{2.36}$$

$$E_6(\tau) = N(\tau) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5q^n}{1-q^n} = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n. \tag{2.37}$$

The Laurent series expansion of $\cot z$ about $z = 0$ is

$$\begin{aligned} \cot z &= \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \frac{2z^5}{945} - \cdots \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k} B_{2k}}{(2k)!} z^{2k-1}, \end{aligned} \tag{2.38}$$

Substituting this and

$$\sin z = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{z^{2k-1}}{(2k-1)!}, \tag{2.39}$$

into (2.30) and inverting the order of summation, we find that

$$\begin{aligned} \frac{\theta_1'}{\theta_1}(z|\tau) &= \frac{1}{z} - \frac{1}{3} L(\tau) z - \frac{1}{45} M(\tau) z^3 - \frac{2}{945} N(\tau) z^5 + \cdots \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k} B_{2k}}{(2k)!} E_{2k}(\tau) z^{2k-1}. \end{aligned} \tag{2.40}$$

From (2.1) by some elementary calculation, we obtain

$$\theta_1(z+\pi\tau|5\tau)\theta_1(z-\pi\tau|5\tau)\theta_1(z+2\pi\tau|5\tau)\theta_1(z-2\pi\tau|5\tau) = \frac{(q^5;q^5)_\infty^5}{(q;q)_\infty}\frac{\theta_1(z|\tau)}{\theta_1(z|5\tau)}. \tag{2.41}$$

For brevity, we define

$$H(z|\tau) = \frac{\theta_1'}{\theta_1}(z|\tau). \tag{2.42}$$

Taking the logarithmic derivative of both sides of (2.41) with respect to $z$, we find that

(2.43)

$$H(z+\pi\tau|5\tau)+H(z-\pi\tau|5\tau)+H(z+2\pi\tau|5\tau)+H(z-2\pi\tau|5\tau) = H(z|\tau)-H(z|5\tau).$$

Using (2.40) in the right side of this equation, we obtain

(2.44)

$$\begin{aligned}H(z+\pi\tau|5\tau)+H(z-\pi\tau|5\tau)+H(z+2\pi\tau|5\tau)+H(z-2\pi\tau|5\tau)\\ = -\frac{1}{3}(L(\tau)-L(5\tau))z-\frac{1}{45}(M(\tau)-M(5\tau))z^3\\ -\frac{2}{945}(N(\tau)-N(5\tau))z^5+\cdots\end{aligned}$$

Differentiating this equation with respect to $z$ and then setting $z=0$ gives

$$H'(\pi\tau|5\tau)+H'(2\pi\tau|5\tau) = -\frac{1}{6}(L(\tau)-L(5\tau)). \tag{2.45}$$

Similarly we have

(2.46)

$$H'''(\pi\tau|5\tau)+H'''(2\pi\tau|5\tau) = -\frac{1}{15}(M(\tau)-M(5\tau)),$$

(2.47)

$$H^{(5)}(\pi\tau|5\tau)+H^{(5)}(2\pi\tau|5\tau) = -\frac{8}{63}(N(\tau)-N(5\tau)).$$

**3. Two theta function identities.** The starting point of this section is the following fundamental theorem of elliptic function theory [**4**, p. 22].

**Theorem 5.** *The sum of all the residues of an elliptic function at the poles inside a period-parallelogram is zero.*

It should be pointed out that in [**6–12**] this fact has been used to derive many wonderful theta function identities. In this section we will use Theorem 5 to prove the following two theorems.

**Theorem 6.** *Let $H(z|\tau)$ be defined by* (2.42) *and $x, y$ be two different complex numbers that are not integral multiples of $\pi$. Then we have*

(3.1)

$$8M(\tau)+3H'''(x|\tau)+3H'''(y|\tau)-2\left\{2L(\tau)+3H'(x|\tau)+3H'(y|\tau)\right\}^2 = \frac{144\eta^{12}(\tau)\theta_1^2(x|\tau)\theta_1^2(y|\tau)}{\theta_1(x-y|\tau)\theta_1(x+y|\tau)}\left\{\frac{\theta_1^2(2x|\tau)}{\theta_1^8(x|\tau)}-\frac{\theta_1^2(2y|\tau)}{\theta_1^8(y|\tau)}\right\}.$$

**Theorem 7.** *Let $H(z|\tau)$ be defined by* (2.42) *and $x, y$ be two complex numbers that are not integral multiples of $\pi$. Then we have*

(3.2)

$$\begin{aligned}&64N(\tau)-9H^{(5)}(x|\tau)-9H^{(5)}(y|\tau)\\ &= 20\left\{2L(\tau)+3H'(x|\tau)+3H'(y|\tau)\right\}^3\\ &\quad -6\left\{2L(\tau)+3H'(x|\tau)+3H'(y|\tau)\right\}\left\{8M(\tau)-15H'''(x|\tau)-15H'''(y|\tau)\right\}.\end{aligned}$$

*Proof.* To prove Theorem 6, we introduce the function

$$f(z)=\frac{\theta_1^3(2z|\tau)}{\theta_1^8(z|\tau)\theta_1(z+x|\tau)\theta_1(z-x|\tau)\theta_1(z+y|\tau)\theta_1(z-y|\tau)}, \tag{3.3}$$

where $0 < x,\, y < \pi$ and $x \neq y$.

By using (2.8) and (2.9), we can directly verify that $f(z)$ is an elliptic function with periods $\pi$ and $\pi\tau$. It has five poles in the fundamental period parallelogram and these poles are $x$, $y$, $\pi - x$, $\pi - y$ and 0. Moreover, $x$, $y$, $\pi - x$, $\pi - y$ are its simple poles and 0 is a pole of order 5. Let $\operatorname{res}(\alpha)$ denote the residue of $f(z)$ at $\alpha$. Now we begin to compute the residue of $f(z)$. It is evident that

$$\begin{aligned}\operatorname{res}(x) &= \lim_{z\to x}(z-x)f(z)\\ &= \lim_{z\to x}\frac{(z-x)}{\theta_1(z-x|\tau)}\\ &\quad\times\lim_{z\to x}\frac{\theta_1^3(2z|\tau)}{\theta_1^8(z|\tau)\theta_1(z+x|\tau)\theta_1(z+y|\tau)\theta_1(z-y|\tau)}.\end{aligned}\tag{3.4}$$

It is plain that

$$\lim_{z\to x}\frac{\theta_1^3(2z|\tau)}{\theta_1^8(z|\tau)\theta_1(z+x|\tau)\theta_1(z+y|\tau)\theta_1(z-y|\tau)} = \frac{\theta_1^2(2x|\tau)}{\theta_1^8(x|\tau)\theta_1(x+y|\tau)\theta_1(x-y|\tau)}.\tag{3.5}$$

By first using L' Hôpital's rule and then using (2.11), we have

$$\lim_{z\to x}\frac{(z-x)}{\theta_1(z-x|\tau)} = \frac{1}{\theta_1'(0|\tau)} = \frac{1}{2\eta^3(\tau)}.\tag{3.6}$$

Combining these two equations, we deduce that

$$\operatorname{res}(x) = \frac{\theta_1^2(2x|\tau)}{2\eta^3(\tau)\theta_1^8(x|\tau)\theta_1(x+y|\tau)\theta_1(x-y|\tau)}.\tag{3.7}$$

Direct computation shows that

$$\operatorname{res}(\pi-x) = \operatorname{res}(x) = \frac{\theta_1^2(2x|\tau)}{2\eta^3(\tau)\theta_1^8(x|\tau)\theta_1(x+y|\tau)\theta_1(x-y|\tau)}.\tag{3.8}$$

In the same, way we have

$$\operatorname{res}(y) = \operatorname{res}(\pi-y) = -\frac{\theta_1^2(2y|\tau)}{2\eta^3(\tau)\theta_1^8(y|\tau)\theta_1(x+y|\tau)\theta_1(x-y|\tau)}.\tag{3.9}$$

It is plain that

$$\operatorname{res}(0) = \frac{1}{24}\left[\frac{d^4(z^5 f(z))}{d^4 z}\right]_{z=0}. \tag{3.10}$$

To compute $\operatorname{res}(0)$, we set

$$F(z) = z^5 f(z) \quad \text{and} \quad \phi(z) = \frac{F'(z)}{F(z)}. \tag{3.11}$$

By an elementary evaluation, we find that

$$\begin{aligned}\frac{d^4 F(z)}{d^4 z} = F(z)\Big\{&\phi^4(z) + 6\phi^2(z)\phi'(z)\\ &+ 4\phi(z)\phi''(z) + 3\phi'(z)^2 + \phi'''(z)\Big\}.\end{aligned} \tag{3.12}$$

It is obvious that

$$F(0) = \lim_{z\to 0} z^5 f(z) = \frac{1}{4\eta^{15}(\tau)\theta_1^2(x|\tau)\theta_1^2(y|\tau)}. \tag{3.13}$$

Using (2.40) and (2.42), we find that

$$\begin{aligned}\phi(z) &= \frac{5}{z} + 6H(2z|\tau) - 8H(z|\tau) - H(z+x|\tau)\\ &\quad - H(z-x|\tau) - H(z+y|\tau) - H(z-y|\tau)\\ &= -\frac{4}{3}L(\tau)z - \frac{8}{9}M(\tau)z^3 - H(z+x|\tau)\\ &\quad - H(z-x|\tau) - H(z+y|\tau) - H(z-y|\tau) + O(z^5).\end{aligned} \tag{3.14}$$

Therefore we have

$$\phi(0) = 0, \quad \phi'(0) = -\frac{4}{3}\left\{L(\tau) + \frac{3}{2}H'(x|\tau) + \frac{3}{2}H'(y|\tau)\right\}, \tag{3.15}$$

and

(3.16)

$$\phi''(0) = 0, \quad \phi'''(0) = -\frac{16}{3}\left\{M(\tau) + \frac{3}{8}H'''(x|\tau) + \frac{3}{8}H'''(y|\tau)\right\}.$$

Combining (3.10), (3.12), (3.13), (3.14), and (3.15), we obtain

(3.17)

$$\begin{aligned}\operatorname{res}(0) &= \frac{1}{24}F(0)\left(3\phi'(0)^2+\phi'''(0)\right)\\ &= \frac{1}{18\eta^{15}(\tau)\theta_1^2(x|\tau)\theta_1^2(y|\tau)}\Bigg\{\left(L(\tau)+\frac{3}{2}H'(x|\tau)+\frac{3}{2}H'(y|\tau)\right)^2\\ &\qquad - M(\tau)-\frac{3}{8}H'''(x|\tau)-\frac{3}{8}H'''(y|\tau)\Bigg\}.\end{aligned}$$

From Theorem 5 we have

$$\operatorname{res}(x)+\operatorname{res}(\pi-x)+\operatorname{res}(y)+\operatorname{res}(\pi-y)+\operatorname{res}(0)=0. \tag{3.18}$$

Substituting (3.8), (3.9) and (3.17) into this identity we know that (3.1) holds for $0 < x,\, y < \pi$ and $x \neq y$. By analytic continuation, we complete the proof of Theorem 6.

To prove Theorem 7, we consider the function

$$f(z)=\frac{\theta_1(2z|\tau)\theta_1(z+x|\tau)\theta_1(z-x|\tau)\theta_1(z+y|\tau)\theta_1(z-y|\tau)}{\theta_1^8(z|\tau)}, \tag{3.19}$$

where $0 < x,\, y < \pi$.

By using (2.8) and (2.9), we can verify that $f(z)$ is an elliptic function with periods $\pi$ and $\pi\tau$. It has only one pole in the fundamental period parallelogram and this pole is 0. Furthermore 0 is a pole of order 7. Let $F(z)=z^7 f(z)$ and $\phi(z)=F'(z)/F(z)$. By an elementary calculation, we find that

(3.20)

$$\begin{aligned}\operatorname{res}(0) &= \frac{1}{720}\left[\frac{d^6F(z)}{d^6z}\right]_{z=0}\\ &= \frac{1}{720}F(0)\Big\{\phi^6(0)+15\phi^4(0)\phi'(0)+20\phi^3(0)\phi''(0)\\ &\quad +15\phi^2(0)\phi'''(0)+45\phi^2(0)\phi'(0)^2+60\phi(0)\phi'(0)\phi''(0)\\ &\quad +6\phi(0)\phi^{(4)}(0)+15\phi'(0)^3+15\phi'(0)\phi'''(0)\\ &\qquad +10\phi''(0)^2+\phi^{(5)}(0)\Big\}.\end{aligned}$$

From the definition of $F(z)$, we immediately have

$$F(0) = \lim_{z\to 0} z^7 f(z) = \frac{2\theta_1^2(x|\tau)\theta_1^2(y|\tau)}{128\eta^{21}(\tau)} \neq 0. \tag{3.21}$$

By (2.40) and (2.42),

$$\begin{aligned} \phi(z) &= \frac{7}{z} - 8H(z|\tau) + 2H(2z|\tau) + H(z+x|\tau) \\ &\quad + H(z-x|\tau) + H(z+y|\tau) + H(z-y|\tau) \\ &= \frac{4}{3}L(\tau)z - \frac{8}{45}M(\tau)z^3 - \frac{16}{135}N(\tau)z^5 + H(z+x|\tau) \\ &\quad + H(z-x|\tau) + H(z+y|\tau) + H(z-y|\tau) + O(z^7). \end{aligned} \tag{3.22}$$

Hence we have

$$\phi(0) = 0, \quad \phi'(0) = \frac{4}{3}\left\{L(\tau) + \frac{3}{2}H'(x|\tau) + \frac{3}{2}H'(y|\tau)\right\}, \tag{3.23}$$

$$\phi''(0) = 0, \quad \phi'''(0) = -\frac{16}{15}\left\{M(\tau) - \frac{15}{8}H'''(x|\tau) - \frac{15}{8}H'''(y|\tau)\right\}, \tag{3.24}$$

and

$$\phi^{(4)}(0) = 0, \quad \phi^{(5)}(0) = -\frac{128}{9}N(\tau) + 2H^{(5)}(x|\tau) + 2H^{(5)}(y|\tau). \tag{3.25}$$

From the fundamental theorem of elliptic functions, Theorem 5, we have

$$\operatorname{res}(0) = 0. \tag{3.26}$$

Substituting (3.21), (3.23), (3.24) and (3.25) into (3.20) and combining (3.26) we readily obtain (3.2) for $0 < x, y < \pi$. By analytic continuation we know that Theorem 7 holds.

**4. The proofs of Ramanujan's identities.** Replacing $\tau$ by $5\tau$ and then choosing $x = \pi\tau$ and $y = 2\pi\tau$ in (3.1), we obtain

$$\begin{aligned} &8M(5\tau) + 3H'''(\pi\tau|5\tau) + 3H'''(2\pi\tau|5\tau) \\ &\qquad - 2\left\{2L(5\tau) + 3H'(\pi\tau|5\tau) + 3H'(2\pi\tau|5\tau)\right\}^2 \\ &\quad = -144\eta^{12}(5\tau)\frac{\theta_1^2(\pi\tau|5\tau)\theta_1^2(2\pi\tau|5\tau)}{\theta_1(\pi\tau|5\tau)\theta_1(3\pi\tau|5\tau)}\left\{\frac{\theta_1^2(2\pi\tau|5\tau)}{\theta_1^8(\pi\tau|5\tau)} - \frac{\theta_1^2(4\pi\tau|5\tau)}{\theta_1^8(2\pi\tau|5\tau)}\right\}. \end{aligned} \tag{4.1}$$

By (2.45), we have

(4.2)

$$2L(5\tau)+3H'(\pi\tau|5\tau)+3H'(2\pi\tau|5\tau)=\frac{1}{2}(5L(5\tau)-L(\tau))=\frac{1}{2}B(\tau).$$

Using (2.46), we obtain

$$8M(5\tau)+3H'''(\pi\tau|5\tau)+3H'''(2\pi\tau|5\tau)=\frac{1}{5}(41M(5\tau)-M(\tau)). \tag{4.3}$$

From (2.10), (2.16), (2.17) and (1.5), we find that the left-hand side of (4.1) equals

$$\begin{aligned}\frac{-144q^{1/2}\eta^{12}(5\tau)}{\theta_1^2(\pi\tau|5\tau)\theta_1^2(2\pi\tau|5\tau)}&\left\{\frac{\theta_1^5(2\pi\tau|5\tau)}{\theta_1^5(\pi\tau|5\tau)}-q^{-3}\frac{\theta_1^5(\pi\tau|5\tau)}{\theta_1^5(2\pi\tau|5\tau)}\right\}\\ &=-\frac{144\eta^{15}(5\tau)}{\eta^2(\tau)\eta^2(5\tau)}\left\{R^{-5}(q)-R^5(q)\right\}\\ &=-144\eta^4(\tau)\eta^4(5\tau)-1584\frac{\eta^{10}(5\tau)}{\eta^2(\tau)}.\end{aligned} \tag{4.4}$$

Substituting (4.2), (4.3) and (4.4) into (4.1) gives

$$2M(\tau)-82M(5\tau)=1440\eta^4(\tau)\eta^4(5\tau)+15840\frac{\eta^{10}(5\tau)}{\eta^2(\tau)}-5B^2(\tau). \tag{4.5}$$

Writing $\tau$ as $-1/5\tau$ and then using (2.26), (2.27) and (2.28), we obtain

(4.6)

$$82M(\tau)-1250M(5\tau)=-36000\eta^4(\tau)\eta^4(5\tau)-3168\frac{\eta^{10}(\tau)}{\eta^2(5\tau)}+125B^2(\tau).$$

From (4.5) and (4.6), we find that

(4.7)

$$32M(\tau)=125B^2(\tau)-1968\frac{\eta^{10}(\tau)}{\eta^2(5\tau)}-36000\eta^4(\tau)\eta^4(5\tau)-15000\frac{\eta^{10}(5\tau)}{\eta^2(\tau)},$$

(4.8)

$$32M(5\tau)=5B^2(\tau)-48\frac{\eta^{10}(\tau)}{\eta^2(5\tau)}-1440\eta^4(\tau)\eta^4(5\tau)-9840\frac{\eta^{10}(5\tau)}{\eta^2(\tau)}.$$

Substituting (1.11) into (4.7) and (4.8), we obtain (1.12) and (1.13) respectively.

Now we begin to derive (1.14) and (1.15). By (2.46) and (2.47), we have

$$8M(5\tau) - 15H'''(\pi\tau|5\tau) - 15H'''(2\pi\tau|5\tau) = M(\tau) + 7M(5\tau) \tag{4.9}$$

and

$$64N(5\tau) - 9H^{(5)}(\pi\tau|5\tau) - 9H^{(5)}(2\pi\tau|5\tau) = \frac{8}{7}\left\{N(\tau) + 55N(5\tau)\right\}. \tag{4.10}$$

Replacing $\tau$ by $5\tau$ in (3.2) and then substituting (4.2), (4.9) and (4.10) in the resulting equation, we obtain

$$16N(\tau) + 880N(5\tau) = 35B^3(\tau) - 42B(\tau)(M(\tau) + 7M(5\tau)). \tag{4.11}$$

Replacing $\tau$ by $-1/5\tau$ and then using (2.27), (2.28) and (2.29) in the resulting equation, we get

$$88N(\tau) + 25000N(5\tau) = -1750B^3(\tau) + 21B(\tau)(7M(\tau) + 625M(5\tau)). \tag{4.12}$$

From (4.11) and (4.12), we infer that

$$32N(\tau) = 125B^3(\tau) - 3B(\tau)(39M(\tau) + 625M(5\tau)), \tag{4.13}$$

$$160N(\tau) = -5B^3(\tau) + 3B(\tau)(M(\tau) + 39M(5\tau)). \tag{4.14}$$

Substituting (1.11), (1.12) and (1.13) into (4.13) and (4.14), we obtain (1.14) and (1.15) respectively. Therefore we complete the proofs of Ramanujan's identities (1.12)–(1.15).

**5. Two new theta function identities.** We start this section by proving the following theta function identity.

**Theorem 8.** *Let $x \pm y$ not be integral multiples of $\pi$. Then we have*

$$\begin{aligned} &\frac{\theta_2^8(0|\tau)}{\theta_2^2(x|\tau)\theta_2^2(y|\tau)} - \frac{\theta_3^8(0|\tau)}{\theta_3^2(x|\tau)\theta_3^2(y|\tau)} + \frac{\theta_4^8(0|\tau)}{\theta_4^2(x|\tau)\theta_4^2(y|\tau)} \\ &\qquad = \frac{4}{\theta_1(x-y|\tau)\theta_1(x+y|\tau)} \left\{ \frac{\theta_1^8(x|\tau)}{\theta_1^2(2x|\tau)} - \frac{\theta_1^8(y|\tau)}{\theta_1^2(2y|\tau)} \right\}. \end{aligned} \tag{5.1}$$

*Proof.* We consider the function

$$f(z) = \frac{\theta_1^8(z|\tau)}{\theta_1(2z|\tau)\theta_1(z-x|\tau)\theta_1(z+x|\tau)\theta_1(z-y|\tau)\theta_1(z+y|\tau)}, \tag{5.2}$$

where $0 < x$, $y < \pi$ and $x \neq y$.

It is easy to check that $f(z)$ is an elliptic function with periods $\pi$ and $\pi\tau$. It has seven poles in the fundamental period parallelogram and these poles are $x$, $\pi - x$, $y$, $\pi - y$, $\pi/2$, $(\pi + \pi\tau)/2$, $\pi\tau/2$, and all these poles are simple poles. From Theorem 5, we have

$$\begin{aligned} &\operatorname{res}(x) + \operatorname{res}(\pi - x) + \operatorname{res}(y) + \operatorname{res}(\pi - y) \\ &\qquad + \operatorname{res}(\pi/2) + \operatorname{res}\left(\frac{\pi + \pi\tau}{2}\right) + \operatorname{res}\left(\frac{\pi\tau}{2}\right) = 0. \end{aligned} \tag{5.3}$$

By using L' Hôpital's rule and (2.11), we have

$$\operatorname{res}(x) = \operatorname{res}(\pi - x) = \frac{\theta_1^8(x|\tau)}{2\eta^3(\tau)\theta_1^2(2x|\tau)\theta_1(x-y|\tau)\theta_1(x+y|\tau)}, \tag{5.4}$$

$$\operatorname{res}(y) = \operatorname{res}(\pi - y) = -\frac{\theta_1^8(y|\tau)}{2\eta^3(\tau)\theta_1^2(2y|\tau)\theta_1(x-y|\tau)\theta_1(x+y|\tau)}. \tag{5.5}$$

By using L' Hôpital's rule, (2.5) and (2.12), we have

$$\operatorname{res}\left(\frac{\pi}{2}\right) = -\frac{\theta_2^8(0|\tau)}{4\eta^3(\tau)\theta_2^2(x|\tau)\theta_2^2(y|\tau)}. \tag{5.6}$$

In the same way, we find that

$$\operatorname{res}\left(\frac{\pi+\pi\tau}{2}\right)=\frac{\theta_3^8(0|\tau)}{4\eta^3(\tau)\theta_3^2(x|\tau)\theta_3^2(y|\tau)}, \tag{5.7}$$

$$\operatorname{res}\left(\frac{\pi\tau}{2}\right)=-\frac{\theta_4^8(0|\tau)}{4\eta^3(\tau)\theta_4^2(x|\tau)\theta_4^2(y|\tau)}. \tag{5.8}$$

Substituting (5.4)–(5.8) into (5.3), we find that (5.1) holds for $0<x$, $y<\pi$ and $x\neq y$. Using analytic continuation we complete the proof of the Theorem.

Replacing $\tau$ by $5\tau$ in (5.1) and then taking $x=\pi\tau$ and $x=2\pi\tau$ in the resulting equation, we obtain

(5.9)

$$\begin{aligned}
&\frac{\theta_2^8(0|5\tau)}{\theta_2^2(\pi\tau|5\tau)\theta_2^2(2\pi\tau|5\tau)}-\frac{\theta_3^8(0|5\tau)}{\theta_3^2(\pi\tau|5\tau)\theta_3^2(2\pi\tau|5\tau)}+\frac{\theta_4^8(0|5\tau)}{\theta_4^2(\pi\tau|5\tau)\theta_4^2(2\pi\tau|5\tau)}\\
&\quad=-\frac{4}{\theta_1(\pi\tau|5\tau)\theta_1(3\pi\tau|5\tau)}\left\{\frac{\theta_1^8(\pi\tau|\tau)}{\theta_1^2(2\pi\tau|5\tau)}-\frac{\theta_1^8(2\pi\tau|\tau)}{\theta_1^2(4\pi\tau|5\tau)}\right\}\\
&\quad=-\frac{4q^{1/2}}{\theta_1(\pi\tau|5\tau)\theta_1(2\pi\tau|5\tau)}\left\{\frac{\theta_1^8(\pi\tau|\tau)}{\theta_1^2(2\pi\tau|5\tau)}-q^3\frac{\theta_1^8(2\pi\tau|\tau)}{\theta_1^2(\pi\tau|5\tau)}\right\}\\
&\quad=-4q^{1/2}\theta_1^2(\pi\tau|5\tau)\theta_1^2(2\pi\tau|5\tau)\left\{\frac{\theta_1^5(\pi\tau|\tau)}{\theta_1^5(2\pi\tau|5\tau)}-q^3\frac{\theta_1^5(2\pi\tau|\tau)}{\theta_1^5(\pi\tau|5\tau)}\right\}\\
&\quad=4q^2\theta_1^2(\pi\tau|5\tau)\theta_1^2(2\pi\tau|5\tau)(R^{-5}(q)-R^5(q)).
\end{aligned}$$

Using (1.5), (2.16) and (2.18) in the equation, we obtain (2.20). Replacing $\tau$ by $-1/5\tau$ in (1.20) and then using (2.19)–(2.22), we obtain (1.21). We complete the proof of Theorem 4.

It is worthwhile to point out that if we let $y\to x$ in (5.1), we obtain

$$\begin{aligned}
&\frac{\theta_2^8(0|\tau)}{\theta_2^4(x|\tau)}-\frac{\theta_3^8(0|\tau)}{\theta_3^4(x|\tau)}+\frac{\theta_4^8(0|\tau)}{\theta_4^4(x|\tau)}\\
&\qquad=\frac{8\theta_1^8(x|\tau)}{\eta^3(\tau)\theta_1^3(2x|\tau)}\left(2H(x|\tau)-H(2x|\tau)\right),
\end{aligned} \tag{5.10}$$

where $H(x|\tau)$ is defined by (2.42). Letting $x\to 0$ in this equation and using (2.40), we derive the Jacobi quartic identity, see for example [**9**, p. 136],

$$\theta_2^4(0|\tau)+\theta_4^4(0|\tau)=\theta_3^4(0|\tau). \tag{5.11}$$

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