ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 34, Number 2, Summer 2004

A LIOUVILLE-GELFAND EQUATION FOR k-HESSIAN OPERATORS

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ABSTRACT. In this paper we establish existence and multiplicity results for a class of fully nonlinear elliptic equations of k-Hessian type with exponential nonlinearity. In particular, we characterize the precise dependence of the multiplicity of solutions with respect to both the space dimension and the value of k. The choice of exponential nonlinearity is motivated by the classical Liouville-Gelfand problem from combustible gas dynamics and prescribed curvature problems.

1. Introduction. Let Ω denote a bounded domain in $\mathbf{R}^N$. If $k \in \{1, \dots, N\}$ and $u \in C^2(\Omega)$, then the k-Hessian operator is defined by

$$S_k(D^2u) = S_k(\lambda[D^2u]) = \sum_{1 \le i_1 < \cdots < i_k \le N} \lambda_{i_1} \cdots \lambda_{i_k}, \tag{1}$$

where $\lambda[r] = (\lambda_1, \cdots, \lambda_N)$ denotes the eigenvalues of the symmetric matrix r and S_k is the kth elementary symmetric polynomial in N letters. Alternatively, $S_k(D^2u)$ is the sum of all principal $k \times k$ minors of the Hessian matrix D^2u. For instance, $S_1(D^2u) = \Delta u$ and $S_N(D^2u) = \det D^2u$. The k-Hessian operators form a discrete collection of partial differential operators which includes both the Laplace and the Monge-Ampère operator.

In this note we are concerned with solutions of the following k-Hessian equation

$$\begin{cases} S_k(D^2u) = \lambda e^{|u|} & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{2}$$

for $\lambda > 0$. The choice of $|u|$ is motivated by the fact that, for $k > 1$, solutions of elliptic k-Hessian equations, with Dirichlet boundary conditions, are negative (in fact, subharmonic) in Ω.

2000 AMS *Mathematics Subject Classification.* Primary 35J65, 35B20, Secondary 80A25.
Key words and phrases. k-Hessian, Liouville, Gelfand, multiplicity.
Received by the editors on July 9, 2001.

Equations of Monge-Ampère type have enjoyed considerable attention recently for numerous reasons, including their many applications to both pure and applied geometry, e.g., wear due to contact with an abrasive plane [**17, 2**], Gauss curvature flow [**2, 44, 45, 32**], meteorology and oceanography [**13, 6**], geometric optics [**23**], and reflector antenna design [**9**]. The recent monograph [**10**] is devoted to applications of Monge-Ampère equations to geometry and optimization.

Beginning with Ivočkina [**24, 25**], Caffarelli, Nirenberg and Spruck [**8**] and Krylov [**28**], the Monge-Ampère equation was also studied as a special case of a more general class of nonlinear elliptic equations defined by functions of the eigenvalues of the Hessian matrix, including (1). This connection between the Laplacian and the Monge-Ampère operator raises many interesting questions. For instance, it is known that there is no critical exponent for the Monge-Ampère operator, thus one may wonder for which k does a critical exponent exist? This question was answered by Tso [**43**], who employed a variational identity due to Pucci and Serrin [**33**] to establish a critical exponent of $(N+2)k/(N-2k)$, when $1 \leq k < N/2$. He also includes a complementary existence result to show there is no critical exponent when $N/2 \leq k \leq N$. It is in this same spirit that we investigate the Liouville-Gelfand problem (2).

The classical Liouville-Gelfand problem is concerned with positive solutions of the equation

$$\begin{cases} \Delta u + \lambda e^u = 0 & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{3}$$

for $\lambda > 0$, which arises in the study of combustible gas dynamics [**5, 20, 18**] and prescribed curvature problems [**4, 3**]. If $\Omega = B_1(0)$ is the unit ball in $\mathbf{R}^N$, then by the classical result of Gidas, Ni and Nirenberg [**21**], all positive solutions of (3) are radially symmetric, reducing (3) to the boundary value problem

$$\begin{cases} u'' + (N-1)/ru' + \lambda e^u = 0 & r \in (0,1), \\ u'(0) = u(1) = 0, \end{cases} \tag{4}$$

for the profile $u(r) = u(|x|)$. As one might expect, for $\lambda > 0$ sufficiently small, equation (4) has at least one solution. The multiplicity of such solutions is however surprising varied according to the value of the (now continuous) parameter N.

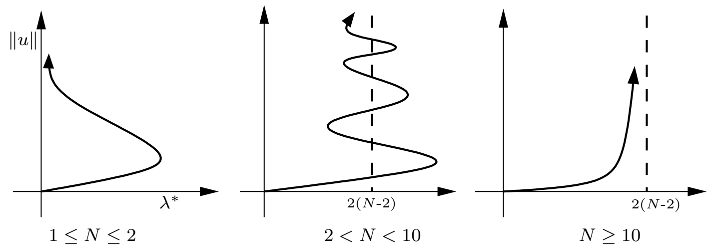


FIGURE 1. Global continua for (4) depend on N.

For $N = 1$ this equation was first solved by Liouville in 1853 [**31**], using reduction of order methods. Indeed, multiplying (4) by u' one may show that, for the free parameter $u(0) = \alpha > 0$, the boundary conditions $u'(0) = u(1) = 0$ are satisfied, provided

$$\lambda = \lambda(\alpha) = \frac{1}{2e^{\alpha}}[\log(2e^{\alpha} + 2\sqrt{e^{\alpha}(e^{\alpha} - 1)} - 1)]^2. \tag{5}$$

It follows that there exists $\lambda^* \approx 0.88$, for which (4) has a unique solution and for each $\lambda \in (0, \lambda^*)$, (4) has precisely two solutions. In 1914, Bratu [**7**] found an explicit solution of (4) when $N = 2$, which obeys the same multiplicity as Liouville's result (now with $\lambda^* = 2$). Elusive to analytical solutions when $N = 3$, numerical progress was made in 1934 by both Frank-Kamenetskii [**18**] in his study of combustion theory and Chandrasekhar [**11**] in his study of isothermal gas stars. In 1963, Gelfand [**20**] published a comprehensive paper that included a review of (4) for $N = 1, 2$ and 3. Curiously, unlike the one- and two-dimensional cases, Gelfand showed that, when $N = 3$, there exists a value of λ for which (4) has infinitely many nontrivial solutions. Approximately ten years later Joseph and Lundgren [**27**] determined the multiplicity of solutions for all N. Let us briefly recall their main result concerning (4), see Figure 1.

Case 1. $1 \le N \le 2$. There exists $\lambda^* > 0$ such that (4) has exactly one solution for $\lambda = \lambda^*$ and exactly two solutions for each $\lambda \in (0, \lambda^*)$.

Case 2. $2 < N < 10$. Equation (4) has an unbounded continuum of solutions which oscillates around the line $\lambda = 2(N-2)$ with the amplitude of oscillations tending to zero, as $u(0) = \|u\| \to \infty$.

Case 3. $N \geq 10$. Equation (4) has a unique solution for each $\lambda \in (0, 2(N-2))$ and no solutions for $\lambda \geq 2(N-2)$. Moreover, $\|u\| \to \infty$ as $\lambda \to 2(N-2)$.

For more on the Liouville-Gelfand problem, we refer the reader to [**5, 16, 35, 18, 36**].

If k is odd, then the mapping $u \mapsto -u$ defines a correspondence between negative solutions of (2) and positive solutions of

$$\begin{cases} S_k(D^2u) + \lambda e^u = 0 & x \in \Omega, \\ u = 0 & x \in \partial\Omega. \end{cases} \tag{6}$$

In particular, all positive solutions of (3) are captured by (2), and our setup includes the original Liouville-Gelfand problem.

From the viewpoint of the k-Hessian operators, the results of Joseph and Lundgren provide a complete description for (2) when $k = 1$ and Ω is a ball in $\mathbf{R}^N$. For $k > 1$, only fragmentary information is known. In [**12**], the authors consider, among other topics, a variant of (2) with Ω a ball in $\mathbf{R}^{2n}$ and $k = n$. Surprisingly, they find the two-fold multiplicity (e.g., $k = n = 1$) holds for all n. In [**26**], the author considers (2) in the Monge-Ampère case, $k = N$, and finds qualitative results similar to the case $N = 1$. However, unlike the Joseph and Lundgren result, it is shown that, regardless of the space dimension, all continua eventually tend to infinity at the origin, even for nonradial domains (some geometric restrictions on Ω apply, see Section 2). This result depends on the a priori convexity of solutions to Monge-Ampère equations.

The work to date concerning (2) paints an interesting picture. The papers [**26**] and [**27**] essentially cover the extreme cases $k = N$ and $k = 1$. For the Monge-Ampére case the multiplicity of solutions is independent of N, in contrast to the Joseph-Lundgren result for $k = 1$. The result of [**12**] fills in the middle where, like the Monge-Ampère case, the multiplicity is independent of N. Intuitively, one would expect the cases $k \geq N/2$ to all share the two-fold multiplicity. On the other hand, it is not clear how the lower Hessian operators, i.e., $1 < k < N/2$, interpolate to the case $k = 1$.

Our purpose here is to extend and unify these results to the general k-Hessian equation (2). For $k > N/2$ we obtain qualitative results assuming only the natural restriction on the domain Ω required by elliptic k-Hessian equations (see Section 2). For general k we obtain precise existence and multiplicity results when Ω is the unit ball. In particular, we demonstrate that, in some sense, the situation described by Figure 1 still captures all possible behavior for (2) (see Figure 3).

This paper is organized as follows. Section 2 collects some preliminary results concerning k-Hessian equations. In Section 3 we prove that (2) has a global continuum of nontrivial solutions that is unbounded in the Banach space $E = C(\overline{\Omega})$. For general domains, with $k > N/2$, we show that all continua satisfy $\lambda \to 0^+$ as $\|u\| \to \infty$. Finally, in Section 4, we determine the precise structure of the continuum for all k, when Ω is the unit ball.

2. Preliminaries. In this section we recall some fundamental results concerning the equation

$$\begin{cases} S_k(D^2u) = f(x) & x \in \Omega, \\ u = 0 & x \in \partial\Omega. \end{cases} \tag{7}$$

A function $u \in C^2(\Omega)$ is called *k-convex*, or *k-subharmonic*, in Ω if $\lambda[D^2u] \in \overline{\Gamma}_k$ for each $x \in \Omega$, where

$$\Gamma_k = \{\lambda \in \mathbf{R}^N : S_j(\lambda) > 0 \text{ for each } j = 1, \dots, k\}. \tag{8}$$

The set $\overline{\Gamma}_k$ is a closed convex cone which corresponds to the connected component of the set $\{\lambda \in \mathbf{R}^N : S_k(\lambda) \geq 0\}$ that contains $(1, 1, \dots, 1)$. For example, 1-convex functions satisfy $\Delta u \geq 0$, hence subharmonic. Similarly, N-convex functions satisfy $\det D^2u \geq 0, \dots, \Delta u \geq 0$, and are convex.

The k-convex functions play a distinguished role in the theory of k-Hessian equations in that, for $k > 1$, the linearized operators $\partial S_k/\partial r_{ij}$ are elliptic precisely on this class of functions, i.e., if $\lambda[D^2u] \in \Gamma_k$ for all $x \in \Omega$, then

$$\frac{\partial S_k}{\partial r_{ij}}(D^2u)\xi_i\xi_j > 0, \quad \text{for all } \xi \in \mathbf{R}^N \setminus \{0\} \text{ and } x \in \Omega. \tag{9}$$

By using the notion of viscosity solutions one may extend these definitions to include upper semi-continuous functions [**40, 41, 42**]. More precisely, an upper semi-continuous function $u : \Omega \to [-\infty, \infty)$ is called k-convex on Ω if, whenever ϕ is a quadratic polynomial that touches u from above (i.e., there exists $x_0 \in \Omega$ such that $\phi(x_0) = u(x_0)$ and $u - \phi$ has a local maximum at x_0), then $S_k(D^2\phi) \geq 0$. A k-convex function is *proper* if it is not identically equal to ∞ on any component of Ω. We shall denote the class of proper k-convex functions on Ω by $\Phi^k(\Omega)$. As before, $u \in \Phi^1(\Omega)$ if and only if it is subharmonic and $u \in \Phi^N(\Omega)$ if and only if it is convex. Notice $\Phi^k(\Omega) \subseteq \Phi^l(\Omega)$ for $l \leq k$ and, in particular, elements of $\Phi^k(\Omega)$ are subharmonic for each k.

The value $k = N/2$ defines a notable border in the theory of k-convex functions. One way to discover this frontier is by looking for solutions of the form $\phi = r^\alpha$ to the equation

$$S_k(D^2 u) = 0. \tag{10}$$

In an appropriate coordinate system (e.g., see (22)) the Hessian matrix $A = D^2\phi$ is diagonal with nonzero entries $a_{11} = \alpha(\alpha - 1)r^{\alpha-2}$ and $a_{ii} = \alpha r^{\alpha-2}$ for $i > 1$. Applying (1) and cancelling the common factor $\alpha r^{\alpha-2}$, one finds $S_k(A) = 0$ provided

$$\binom{N-1}{k-1}(\alpha - 1) + \binom{N-1}{k} = 0. \tag{11}$$

In particular, if $k > N/2$, then (11) has the positive solution $\alpha = (2k - N)/k$. In other words, when $k > N/2$, equation (10) has Hölder continuous fundamental solutions $\phi = r^{2-N/k}$. For $1 \leq k \leq N/2$ the fundamental solutions have a singularity at the origin ($\phi = \log r$ if $k = N/2$ and $r^{2-N/k}$ otherwise [**41**]).

Heuristically, one can expect that when $k < N/2$, k-convex functions behave more like subharmonic functions, whereas for $k > N/2$, k-convex functions behave more like convex functions. This point is further emphasized by the following regularity result of Trudinger and Wang [**41**].

Lemma 2.1. *For $k > N/2$, $\Phi^k(\Omega) \subset C^\alpha(\Omega)$ for $\alpha = 2 - N/k$.*

Certain restrictions on the geometry of the domain are necessary when working with k-convex functions. For instance, it is shown in

[**8**] that if $u \in \Phi^k(\Omega)$ solves (7), with $f > 0$ and C^2 on $\overline{\Omega}$, then necessarily $\partial\Omega$ is $(k-1)$-convex, namely, if $\kappa = \kappa(x) = (\kappa_1, \dots, \kappa_{N-1})$ denotes the vector of principal curvatures of $\partial\Omega$ at x, then $S_{k-1}(\kappa) \geq 0$. Equivalently, a domain is $(k-1)$-convex provided the jth mean curvatures of $\partial\Omega$ are nonnegative for each $j = 1, \dots, k-1$. If $S_{k-1}(\kappa) \geq \varepsilon > 0$ for all $x \in \partial\Omega$, then we say Ω is uniformly $(k-1)$-convex. For instance, a uniformly $(N-1)$-convex domain is strictly convex.

The following fundamental existence theorem is due to Trudinger.

Theorem 2.2 (Trudinger [**39**]). *Let Ω be a uniformly $(k-1)$-convex domain in $\mathbf{R}^N$ where $k \in \{2, \dots, N\}$. For any nonnegative $f \in L^p(\Omega)$ with $p > N/2k$, there exists a unique $u \in \Phi^k(\Omega) \cap C(\overline{\Omega})$ solving*

$$\begin{cases} S_k(D^2u) = f(x) & x \in \Omega, \\ u = 0 & x \in \partial\Omega. \end{cases} \tag{12}$$

Furthermore, $u \in C^\alpha(\Omega)$ for any exponent $\alpha < 1$ such that $\alpha \leq 2 - N/kp$.

Let $F_k = (S_k)^{1/k}$ and $F_{ij} = (\partial F_k / \partial r_{ij})$. In [**8**], Caffarelli, Nirenberg and Spruck show how an inequality due to Gårding [**19**] implies that F_k is concave on the set of $N \times N$ symmetric matrices whose eigenvalues lie in Γ_k, implying the following lemma.

Lemma 2.3. *If $k \in \{1, \dots, N\}$ and $w \in \Phi^k(\Omega)$, then the linear operator*

$$L_k = \sum_{i,j=1}^{N} F_{ij}(D^2w) D_{ij},$$

is elliptic. Moreover, if $u, v \in \Phi^k(\Omega)$, then $L_k(u-v) \geq F_k(D^2u) - F_k(D^2v)$.

Proof. The inequality follows from the concavity of F_k on $\Phi^k(\Omega)$:

$$L_k(u-v) = \sum_{i,j=1}^{N} F_{ij}(D^2w)(D_{ij}u - D_{ij}v) \geq F_k(D^2u) - F_k(D^2v). \qquad \square$$

3. Global solution continuum. In this section we prove the equation

$$\begin{cases} S_k(D^2u) = \lambda e^{|u|} & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{13}$$

has an unbounded global continuum of solution pairs $(\lambda, u) \in [0, \infty) \times C(\overline{\Omega})$. We demonstrate that the allowable values of λ remain bounded, implying that the continuum is unbounded in $C(\overline{\Omega})$. It follows that there exists $\lambda^* > 0$ such that (13) has at least one nontrivial solution $u \in \Phi^k(\Omega)$ for each $\lambda \in (0, \lambda^*]$ and no solution for $\lambda > \lambda^*$. Finally, if $k > N/2$ we prove that $\lambda \to 0$ as $\|u\| \to \infty$.

We begin by recalling an abstract result concerning global solution continua:

Theorem 3.1 (Global solution theorem [**37, 34**]). *Let $\mathcal{O}$ be an open bounded subset of the real Banach space $\mathcal{E}$ and assume that $F : \mathbf{R} \times \mathcal{E} \to \mathcal{E}$ is completely continuous. If for $\lambda = \lambda_0$, the equation*

$$u - F(\lambda, u) = 0 \tag{14}$$

has a unique solution u_0, and

$$d(I - F(\lambda_0, \cdot), \mathcal{O}, 0) \neq 0, \tag{15}$$

then for

$$\mathcal{S}^+ = \{(\lambda, u) \in [\lambda_0, \infty) \times \mathcal{E} : u = F(\lambda, u)\}$$

there exist a continuum $\mathcal{C}^+ \subseteq \mathcal{S}^+$, such that

1. $\mathcal{C}^+ \cap \{\lambda_0\} \times \mathcal{O} = \{u_0\}$;
2. *Either $\mathcal{C}^+$ is unbounded or else $\mathcal{C}^+ \cap \{\lambda_0\} \times \mathcal{E} \setminus \overline{\mathcal{O}} \neq \varnothing$.*

Here $d(I - F, \mathcal{O}, 0)$ is the Leray-Schauder degree defined for completely continuous perturbations of the identity (see, e.g., [**14**]). The reader may view Theorem 3.1 as a degree theoretic globalization of the implicit function theorem, where the nonzero degree condition (15) effectively replaces the requirement that a Fréchet derivative be a homeomorphism. In return, we are no longer guaranteed that locally solutions are described by a function.

Let us now apply Theorem 3.1 to equation (13). For a fixed $k \in \{1, \ldots, N\}$, let $\Omega \subset \mathbf{R}^N$ denote a uniformly $(k-1)$-convex domain and $E = C(\overline{\Omega})$. For each $f \in E$, it follows from Theorem 2.2 that there exists a unique $u \in \Phi^k(\Omega) \cap E$ such that

$$\begin{cases} S_k(D^2u) = |f| & x \in \Omega, \\ u = 0 & x \in \partial\Omega. \end{cases} \tag{16}$$

In this way we may define the solution operator $T_k : E \to E$; i.e., $T_k(f) = u$, where u solves (16). It is established in [**26**, Proposition 3.2] that T_k is a completely continuous operator. Define $G_k : \mathbf{R} \times E \to E$ by

$$G_k(\lambda, f) = |\lambda|^{1/k}(T_k \circ N)(f),$$

where $N : E \to E$ is a Nemystkiĭ operator defined by $N(f) = e^{|f|}$. The operator G_k, being the composition of a bounded continuous map with a completely continuous map, is completely continuous on $\mathbf{R} \times E$.

If $(\lambda_0, u) \in \mathbf{R} \times E$ such that $u - G_k(\lambda_0, u) = 0$, then u solves (13) with $\lambda = |\lambda_0|$. Since $G_k(0, u) = 0$ for all $u \in E$, it follows that

$$d(I - G_k(0, \cdot), B_r(0), 0) = d(I, B_r(0), 0) = 1,$$

for any $r > 0$. Therefore, by Theorem 3,1 there exists a global continuum $\mathcal{C} \subset [0, \infty) \times E$ whose components satisfy (13). Since $u = 0$ is the unique solution corresponding to $\lambda = 0$, the continuum must be unbounded in $(0, \infty) \times E$. Moreover, for $\lambda > 0$, the function $u = 0$ is not a solution of (13), thus the solutions obtained are nontrivial. The behavior of the continuum is further refined by the following lemma.

Lemma 3.2. *If (λ, u) is a solution of (13) with $\lambda > 0$, then $\lambda < k!\lambda_1^k$ where λ_1 is the principal eigenvalue associated with the k-Hessian operator for the domain Ω.*[1]

Proof. Assume that there exists a solution (λ, u) of (13) with $\lambda \geq k!\lambda_1^k$. Let ϕ be an eigenfunction for λ_1. By scaling, if necessary, we may assume $u(x) < \phi(x)$ for all $x \in \Omega$. Let $\delta^* > 0$ be maximal such that $(u - \delta^*\phi)(x) \leq 0$ for all $x \in \Omega$ and consider the linear second order elliptic operator

$$L = F_{ij}(D^2w)D_{ij},$$

where $w = \delta^* \phi$. As $w \in \Phi^k(\Omega)$, we may apply Lemma 2.3 to conclude

$$\begin{aligned} L_k(u-w) &\geq F_k(D^2u) - F_k(D^2w) \\ &= [\lambda e^{-u}]^{1/k} - [|\lambda_1 w|^k]^{1/k}. \end{aligned} \tag{17}$$

Since $-u(x) \geq -w(x) \geq 0$ on Ω, it follows that

$$e^{-u} \geq e^{-w} \geq \frac{|w|^k}{k!}.$$

Therefore,

$$|\lambda_1 w|^k \leq k! \lambda_1^k e^{-u} \leq \lambda e^{-u},$$

which, combined with (17), implies $L_k(u-w) \geq 0$. By the maximum principle we conclude $u = w$ for all $x \in \Omega$. It follows that

$$\lambda e^{-u} = |\lambda_1 w|^k = |\lambda_1 u|^k,$$

which is impossible unless $u = w$ is a constant, however $w < 0$ in Ω and $w = 0$ on $\partial\Omega$. This completes the proof. □

Thus we have established the following theorem.

Theorem 3.3. *Equation* (13) *has an unbounded continuum of solutions* $(\lambda, u) \in \mathcal{C} \subset [0, k!\lambda_1^k) \times E$.

In particular, it follows that there exists a constant $\lambda^* > 0$ such that (13) has at least one nontrivial solution $u \in \Phi^k(\Omega)$ for each $\lambda \in (0, \lambda^*]$ and no solution when $\lambda > \lambda^*$. A possible continuum is drawn in Figure 2.

If $k > N/2$, we can deduce one further property of the solution continuum.

Theorem 3.4. *If* $N/2 < k \leq N$, *then* $\lambda \to 0^+$ *as* $\|u\| \to \infty$.

Proof. Suppose $\{(\lambda_n, u_n)\}$ is a sequence of solutions to (13) such that $\|u\| \to \infty$ and $\lambda \to \mu \neq 0$. From this we may produce a sequence of unit vectors $v_n = u_n/\|u_n\|$ satisfying the equation

$$S_k(D^2 v_n) = \lambda_n \frac{e^{|u_n|}}{\|u_n\|^k}, \quad x \in \Omega, \tag{18}$$

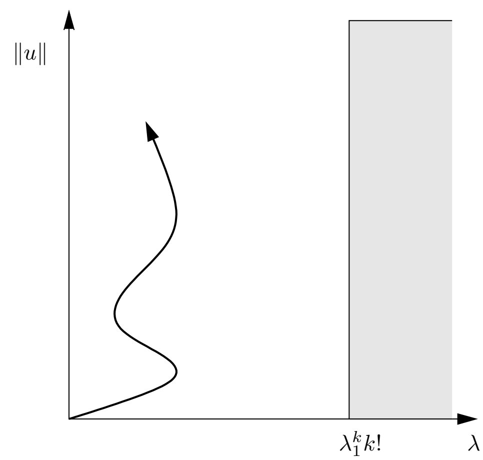


FIGURE 2. Possible continuum of solutions to (13).

which we may rewrite as

$$S_k(D^2 v_n) = \lambda_n \frac{e^{|u_n|}}{|u_n|^k} |v_n|^k, \quad x \in \Omega. \tag{19}$$

Let $\Omega' \subset\subset \Omega$. From Lemma 2.1 it follows that $|u_n| \to \infty$ uniformly on Ω'.[2] In particular, the functions $v_n \to 1$ uniformly on Ω', implying $S_k(D^2 v_n) \to 0$ for each $x \in \Omega'$. On the other hand, from (19) and the fact that the values of λ_n and $|v_n|$ are bounded away from zero on Ω', we conclude that $S_k(D^2 v_n) \to \infty$ for each $x \in \Omega'$, a contradiction. Therefore, $\lambda \to 0$ as $\|u\| \to \infty$, and the proof is complete. □

A value $\mu \in \mathbf{R}$ is said to be an *asymptotic bifurcation point* of the equation $F(\lambda, u) = 0$ if there exists a sequence of solutions $\{(\lambda_n, u_n)\}$ such that $(\lambda_n, \|u_n\|) \to (\mu, \infty)$. In this terminology Theorem 3.4 may be restated as "if $k > N/2$, then 0 is the only asymptotic bifurcation for (13)." For instance, 0 is the only asymptotic bifurcation point for the Monge-Ampère equation

$$\begin{cases} \det D^2 u = \lambda e^{|u|} & x \in \Omega, \\ u = 0 & x \in \partial\Omega. \end{cases} \tag{20}$$

Theorem 3.4 remains true when $k = N/2$ and Ω is a ball (see Theorem 4.1).

4. The radial case. In this section we assume $\Omega = B_1(0)$ is the unit ball in $\mathbf{R}^N$. In this case, the method of moving planes due to Alexandrov [**1**] and Serrin [**38**] extends to (2) (see [**15**] for the Monge-Ampère case) reducing (2) to the boundary value problem

$$\begin{cases} r^{1-N}(r^{N-k}(u')^k)' - \lambda e^{-u} = 0 & 0 < r < 1, \\ u'(0) = u(1) = 0, \end{cases} \tag{21}$$

for the profile $u(r) = u(|x|)$. Indeed, if $u : \Omega \to \mathbf{R}$ is radially symmetric, then a calculation shows

$$\frac{\partial u}{\partial x_i} = u'(r)\frac{x_i}{r} \tag{22}$$

and

$$\frac{\partial^2 u}{\partial x^2} = u''(r)\frac{x_i x_j}{r^2} + u'(r)\left[\frac{r^2\delta_{ij} - x_i x_j}{r^3}\right],$$

for $i, j = 1, \dots, N$. At the point $x = (r, 0, \dots, 0)$ the Hessian matrix D^2u is diagonal with $u_{11} = u''(r)$ and $u_{ii} = u'(r)/r$ for $i > 1$. Since the operator S_k is invariant with respect to rotations, it follows that

$$\begin{aligned} S_k(D^2u) &= u'' \binom{N-1}{k-1} \frac{u'}{r} + \binom{N-1}{k} \frac{u'}{r} \\ &= \binom{N-1}{k-1} \frac{r^{1-N}}{k} (r^{N-k}(u')^k)', \end{aligned}$$

where $\binom{n}{k}$ is the binomial coefficient. It follows from regularity results of [**46, 22**] that any solution to (13) is at least $C^1(\Omega)$, implying the symmetric boundary condition $u'(0) = 0$ is satisfied.

It follows from Theorem 3.3 that (21) has an unbounded solution continuum $\mathcal{C} \subset [0, k!\lambda_1^k) \times E$. To determine the precise structure of this continuum we employ a change of variables that essentially dates back to Emden and Fowler, e.g., see [**11**]. Consider the transformation

$$\begin{cases} s = k \ln r & r \in (0,1), \\ v = (du/ds), \\ w = \lambda e^{2s} e^{-u}, \end{cases} \tag{23}$$

for $\lambda > 0$. A calculation shows

$$\begin{aligned}\frac{dv}{ds} &= \frac{d}{ds}\left[u'(r)\frac{e^{s/k}}{k}\right] = \frac{1}{k^2}[r^2u''(r) + ru'(r)] \\ &= \frac{1}{k^2}[r^2u''(r) + kv]\end{aligned} \tag{24}$$

and

$$\frac{dw}{ds} = \frac{d}{ds}[\lambda e^{2s}e^{-u}] = 2w + w(-v) = w(2-v). \tag{25}$$

In developed form, (21) is

$$\begin{cases} kr^{1-k}(u')^{k-1}u'' + (N-k)r^{-k}(u')^k - \lambda e^{-u} = 0, \quad 0 < r < 1, \\ u'(0) = u(1) = 0. \end{cases} \tag{26}$$

Multiplying (26) by r^{k+1} and simplifying one finds

$$r^2u'' = \frac{k-N}{k}ru' + \frac{\lambda}{k(u')^{k-1}}r^{k+1}e^{-u},$$

which coupled with (24) yields

$$\frac{dv}{ds} = \left[\frac{2k-N}{k^2}\right]v + \frac{\lambda}{k^{k+2}v^{k-1}}r^{2k}e^{-u}. \tag{27}$$

From (25) and (27) we obtain the main system

$$\begin{cases} (dw/ds) = w(2-v), \\ (dv/ds) = [(2k-N)/k^2]v + (1/k^{k+2})wv^{1-k}, \end{cases} \tag{28}$$

with the additional conditions

$$w(0) = \lambda \tag{29}$$

$$v(0) = u'(1)/k, \tag{30}$$

$$w(-\infty) = 0, \tag{31}$$

$$v(-\infty) = 0. \tag{32}$$

Notice that when $k = 1$ we obtain a system equivalent to the Joseph and Lundgren system (see [**5**]):

$$\begin{cases} dw/ds = w(2-v) \\ dv/ds = (2-N)v + w. \end{cases} \tag{33}$$

An element $(\lambda, u) \in \mathcal{C}$, therefore, corresponds to an integral curve of (28) which emanates from $(0,0) = (w(-\infty), v(-\infty))$ and passes through $(\lambda, \beta) = (w(0), v(0))$. In particular, each crossing of this integral curve with the line $w = \lambda_0$ corresponds to a solution of (21) with $\lambda = \lambda_0$, $u'(1) = k\beta$ and $u(0) = \alpha = \int_{-\infty}^{0} v(s)\, ds$. We may therefore determine the multiplicity of solutions to (21) through an analysis of the first order system (28) in the w-v plane.

The system (28) has a critical point at

$$(w, v) = ((2k)^k (N-2k), 2). \tag{34}$$

For $1 \leq k < N/2$, this critical point lies in the first quadrant while, for $k > N/2$, it lies in the second quadrant. If $k = 1$, then the origin is also a critical point, otherwise it is a singular point. When $k = N/2 > 1$, the v axis is comprised of singular points.

We may now state the main theorem characterizing the multiplicity of solutions to (2) in the radial case:

Theorem 4.1. *Let $k \in \{1, \dots, N\}$. Equation* (21) *has the following properties*:

Case I. $N \leq 2k$. *There exists a $\lambda^* > 0$ such that* (21) *has one solution for $\lambda = \lambda^*$ and two solutions for $0 < \lambda < \lambda^*$.*

Case II. $2k < N < 2k+8$. *Equation* (21) *has a continuum of solutions which oscillates around the line $\lambda = (2k)^k(N-2k)$, with the amplitude of oscillations tending to zero, as $\|u\| \to \infty$.*

Case III. $N \geq 2k+8$. *Equation* (21) *has a unique solution for each $\lambda \in (0, (2k)^k(N-2k))$ and no solutions for $\lambda \geq (2k)^k(N-2k)$.*

Proof. It follows from Theorem 3.3 that (28) has an integral curve which emanates from the origin and is confined in the strip $[0, \lambda_1^k k!] \times \mathbf{R}$ in the w-v plane. It is evident from (28) that this curve is confined to the first quadrant. The multiplicity of solutions depends on the behavior of the curve as $s \to \infty$, which falls into two cases.

Case I. $N/2 \le k \le N$. In this case one can see from (28) that $dv/ds > 0$ for all s, whereas $dw/ds > 0$ for $v < 2$ and negative for $v > 2$. These observations determine the behavior of the curve: it leaves the origin with both w and v increasing, when v reaches 2 the dynamics of w shift from increasing to decreasing. Since the curve has already passed the level $v = 2$, it cannot reach the critical point and therefore proceeds asymptotically to the v axis. We conclude the multiplicity in this case behaves as described. Note that this case is in harmony with the more general result of Theorem 3.4.

Case II. $1 \le k < N/2$. In this case, more complicated dynamics are allowed since the sign of dv/ds can change. In particular, for v sufficiently large (28) implies $dv/ds < 0$. It follows that the curve remains bounded for all $s > 0$ and must either converge to the critical point or a periodic limit cycle as $s \to \infty$. However, the nature of the vector field allows us to rule out a periodic orbit. Indeed, if a periodic orbit γ exists, then on this orbit $w'(s)\,dv - v'(s)\,dw = 0$, or

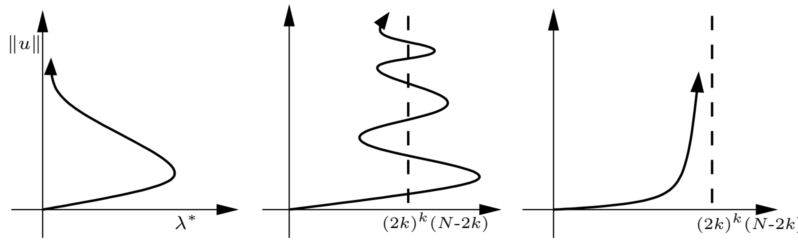
$$\frac{w'(s)}{w(s)}\,dv - \frac{v'(s)}{w(s)}\,dw = 0.$$

By Green's theorem,

$$\begin{aligned} 0 &= \oint_\gamma \frac{w'}{w}\,dv - \frac{v'}{w}\,dw \\ &= \oint_\Gamma \left(\frac{w'}{w}\right)_w + \left(\frac{v'}{w}\right)_v dw\,dv \\ &= \int_\Gamma \frac{N-2k}{k^2 w} + \frac{1-k}{k^{k+2} v^k}\,dw\,dv \\ &< 0, \end{aligned}$$

a contradiction. Therefore, there are no periodic limit cycles to (28) and the solution curve must converge to the critical point (34) as $s \to \infty$.

To determine the multiplicity of solutions we consider the linearization at the critical point $((2k)^k(N-2k), 2)$. The trace and determinant



(a) $N \leq 2k$ (b) $2k < N < 2k+8$ (c) $2k+8 \leq N$

FIGURE 3. Global continua for (21) depend on k and N.

of the Jacobian matrix are given by the equations

$$\operatorname{tr} J = \frac{(2k-N)}{k}, \qquad \det J = -\frac{2(2k-N)}{k^2} = -\frac{2}{k}\operatorname{tr} J.$$

The eigenvalues of the linearized system at the critical point (34) are given by the equation

$$\lambda = \frac{\operatorname{tr} J}{2} \pm \frac{\sqrt{(\operatorname{tr} J)^2 - 4\det J}}{2}.$$

In particular, the eigenvalues will be complex with negative real part when

$$2k < N < 2k+8. \tag{35}$$

For $N = 2k+8$, the eigenvalue is a unique negative number, and for $N > 2k+8$ the eigenvalues are distinct negative numbers, thus the critical point is a stable attracting node. This completes the proof. □

Theorem 4.1 completes the picture for the generalized Liouville-Gelfand problem defined by (2). It extends the previously known cases of $k = 1$ [**27**], $k = N/2$ [**12**] and $k = N$ [**26**], and affirmatively answers the conjecture from [**26**] for $k > N/2$.

It is tempting to conclude from Figure 3 that each k-Hessian operator has exactly seven oscillatory dimensions; however, this is false due to the inability to identify the k-Hessian operator without also knowing N. For instance, for the Monge-Ampère operator $k = N$, thus the multiplicity is always as in Figure 3(a). In fact, all k-Hessian operators close to the Monge-Ampère operator (in the sense $k \geq N/2$), obey the multiplicity of Figure 3(a). As another example illustrating Theorem 4.1, if $N > 2$ even, say $N = 2n$, then the $(n-1)$-Hessian always has infinitely many nontrivial solutions for $\lambda = 2^n(n-1)^{n-1}$.

ENDNOTES

1. The existence of a principal eigenvalue for S_k is due to P.L. Lions [**30**] for $k = N$ and X.-J. Wang [**46**] for $2 \leq k < N$. See also [**26**].

2. See [**29, 42**] for an interesting development of a capacity theory when $k \leq N/2$.

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