ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 34, Number 2, Summer 2004

ON ASSOCIATIVE SUPERALGEBRAS OF MATRICES

S. DĂSCĂLESCU, P.D. JARVIS, A.V. KELAREV AND C. NĂSTĂSESCU

1. Introduction. This work is a contribution to ongoing investigations of algebraic structures in relation to the theoretical description of physical systems. Matrix methods have been used by various mathematical physicists in the late nineteenth century and have been applied, for example, in the exploration of quaternions and other division algebras as a generalization of the complex number system for quantum physics, see, for example, [**25**]. One of the deepest results permeating physics is the spin-statistics theorem, see [**27**], according to which the space-time properties (spin) of elementary particles are correlated with their quantum statistical description. The two classes of particle statistics, Bose-Einstein and Fermi-Dirac, respectively, can be accommodated naturally in a larger algebraic scheme incorporating the notion of *grading* to accommodate various sign factors in defining relations (in this regard see, for example, [**3, 8, 15, 23, 24, 29, 30**]).

At the level of nonassociative algebras, the structure and representation theory of $\mathbf{Z}_2$-graded *Lie superalgebras* have been extensively studied as symmetry algebras of physical systems (for examples of applications we refer to [**2, 7, 11, 13, 14**]). In recent years the study of two-dimensional systems has led to the realization that richer algebraic schemes such as the so-called *quantum algebras* may be relevant (the spin-statistics theorem is also weaker in the two-dimensional case).

In the present paper we relax the notion of a superalgebra and investigate associative rings graded by semigroups. Retaining, in this paper, a bipartite decomposition of the underlying space into a 'Bose-like' and a 'Fermi-like' piece, we therefore study the five classes of two-element semigroups. Since matrix rings play important roles in this research direction (see [**1, 5, 10, 12, 26**]), the first natural step is to investigate the matrix algebras graded by the two-element semigroups.

Let S be a semigroup. An associative ring R is said to be S-*graded*,

Received by the editors on November 14, 2000, and in revised form on February 26, 2002.

if $R = \oplus_{s \in sS} R_s$ is a direct sum and $R_s R_t \subseteq R_{st}$ for all $s, t \in S$. If R is an F-algebra, we say that R is an S-graded F-algebra if R is an S-graded ring such that all the homogeneous components R_s are F-vector subspaces of R. For earlier results on semigroup-graded rings and groupoid-graded rings, we refer to the surveys [**19, 20, 21**] and to the monograph [**22**].

Let F be a field. An interesting example of a semigroup grading of a matrix algebra was given in [**28**] by Wedderburn, who showed that the full matrix algebra $M_n(F)$ over a field F of characteristic zero can be graded by a rectangular band, i.e., by a semigroup satisfying the identities $xyx = x$ and $x^2 = x$, so that all the components are isomorphic to the field F.

The general problem of describing all semigroup gradings of a full matrix algebra was posed by Zel'manov, see [**19**]. An obvious type of grading of $M_n(F)$ to look at is one for which all the matrix units e_{ij} are homogeneous elements. Such a grading is called a good grading. Good gradings were studied in the group-grading case in [**9**] and in a different setting in [**16**] and [**17**], where they were constructed from weight functions on the complete graph Γ on n points, using the fact that $M_n(F)$ is a quotient of the path algebra of the quiver Γ.

The problem of finding all, not necessarily good, gradings of the F-algebra $M_n(F)$ by a semigroup S has already been considered in the literature. The special case of $\mathbf{Z}_2$-gradings was solved in [**9**], in particular providing examples of gradings which are not good gradings. If F is algebraically closed, it was shown that any $\mathbf{Z}_2$-grading of $M_2(F)$ is isomorphic to a good grading. However, if F is not algebraically closed, it may be possible to find $\mathbf{Z}_2$-gradings of $M_2(F)$ which are not isomorphic to a good grading. In Section 2 we describe all gradings of $M_2(F)$ by semigroups with two elements which are not groups and determine the isomorphism types of such gradings. It is interesting that all these gradings are isomorphic to good gradings, independently on the structure of the field F. In Section 3 we look at gradings of matrix algebras which are not full; more precisely, we describe gradings of an upper triangular 2×2 matrix algebra T by all semigroups with two elements. In this case the structure of the field F does not have any influence on the number of isomorphism types of such gradings. The situation is different from the full matrix algebra case, any $\mathbf{Z}_2$-grading of T being isomorphic to a good grading. The same fact is true for gradings by a left zero semigroup. However, we find gradings of T by the semi-lattice with two elements which are not isomorphic to a good grading.

2. Gradings of full matrix algebras. It is a folklore, and easily follows, for example, from [**6**, Lemma 2.26 and Theorem 3.5] that there exist five isomorphism types of semigroups with two elements: the group $\mathbf{Z}_2$ with two elements, the semi-lattice, i.e., a monoid which is not a group, the left zero semigroup, i.e., the semigroup satisfying the identities $xy = x$ and $x^2 = x$, the right zero semigroup, i.e., the semigroup satisfying the identities $xy = y$ and $x^2 = x$, and the null semigroup, i.e., the semigroup with zero satisfying the identity $xy = 0$. The left zero semigroup and the right zero semigroup cases are similar and so in the first theorem we consider only one of these cases.

Theorem 1. *Let $S = \{s, r\}$ be a left zero semigroup with two elements. Then any S-grading of the algebra $A = M_2(F)$ is of one of the following three types*:

(i) $A_s = A$, $A_r = 0$;

(ii) $A_s = 0$, $A_r = A$;

(iii)

$$A_s = \left\{ \begin{pmatrix} u & v \\ \lambda u & \lambda v \end{pmatrix} \middle| u, v \in F \right\}, \quad A_r = \left\{ \begin{pmatrix} \mu u & \mu v \\ u & v \end{pmatrix} \middle| u, v \in F \right\},$$

for some $\lambda, \mu \in F$ such that $\lambda\mu \neq 1$;

(iv)

$$A_s = \left\{ \begin{pmatrix} 0 & 0 \\ u & v \end{pmatrix} \middle| u, v \in F \right\}, \quad A_r = \left\{ \begin{pmatrix} u & v \\ \mu u & \mu v \end{pmatrix} \middle| u, v \in F \right\},$$

for some $\mu \in F$;

(v)

$$A_s = \left\{ \begin{pmatrix} \lambda u & \lambda v \\ u & v \end{pmatrix} \middle| u, v \in F \right\}, \quad A_r = \left\{ \begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix} \middle| u, v \in F \right\},$$

for some $\lambda \in F$.

Apart from the gradings obtained from $\mu = 0$ in type (iv) *and $\lambda = 0$ in type* (v)*, there are no other identical gradings in the list. Any grading of type* (iii) *is isomorphic to the grading*

$$A_s = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}, \quad A_r = \begin{pmatrix} 0 & 0 \\ F & F \end{pmatrix},$$

and any grading of type (iv) *or* (v) *is isomorphic to the grading*

$$A_s = \begin{pmatrix} 0 & 0 \\ F & F \end{pmatrix}, \quad A_r = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}.$$

In particular, there exist four isomorphism types of S-algebra gradings on $M_2(F)$.

Proof. For any $a, b \in A$ we have that

$$\begin{aligned}(ab)_s &= a_s b_s + a_s b_r \\ &= a_s b_s + a_s(b - b_s) \\ &= a_s b.\end{aligned}$$

Thus the map $\varphi : A \to A$, $\varphi(a) = a_s$, is a morphism of right A-modules. Therefore it is of the form $\varphi(a) = ha$ for some $h \in A$. Moreover, since $\varphi^2 = \varphi$, we must have $h^2 = h$. Let

$$h = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Then $h^2 = h$ is equivalent to

$$\alpha^2 + \beta\gamma = \alpha,\ \beta(\alpha + \delta) = \beta,\ \gamma(\alpha + \delta) = \gamma,\ \delta^2 + \beta\gamma = \delta.$$

If $\alpha + \delta \neq 1$, then $\beta = \gamma = 0$ and $\alpha, \delta \in \{0, 1\}$; thus, either $\alpha = \delta = 0$ or $\alpha = \delta = 1$. In this case we obtain two solutions, $h = 0$ and $h = I_2$. If $h = 0$, then we obtain the trivial grading $A_s = 0$, $A_r = A$. If $h = I_2$, then we have the other trivial grading $A_s = A$, $A_r = 0$.

If $\alpha + \gamma = 1$, the solutions are of the form

$$h = \begin{pmatrix} \alpha & \beta \\ \gamma & 1 - \alpha \end{pmatrix},$$

where $\alpha \in F$ and $\beta, \gamma \in F$ such that $\beta\gamma = \alpha - \alpha^2$. In this case, if

$$a = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in A,$$

then the homogeneous component of degree s of a is

$$a_s = \begin{pmatrix} \alpha x + \beta z & \alpha y + \beta t \\ \gamma x + (1 - \alpha z) & \gamma y + (1 - \alpha)t \end{pmatrix}.$$

Note that

$$\gamma(\alpha x + \beta z) = \alpha(\gamma x + (1 - \alpha)z), \gamma(\alpha y + \beta t) = \alpha(\gamma y + (1 - \alpha)t).$$

Let us consider first the case when $\alpha \neq 0$. For any $u, v \in F$ choose some $z, t \in F$ and $x = u - \beta z/\alpha$, $y = v - \beta t/\alpha$. Then $\alpha x + \beta z = u$ and $\alpha y + \beta t = v$. It follows that

$$A_s = \left\{ \begin{pmatrix} u & v \\ (\gamma/\alpha)u & (\gamma/\alpha)v \end{pmatrix} \middle| u, v \in F \right\}.$$

By a similar computation we find that

$$A_r = \left\{ \begin{pmatrix} -(\beta/\alpha)u & -(\beta/\alpha)v \\ u & v \end{pmatrix} \middle| u, v \in F \right\}.$$

We claim that the pair $[(\gamma/\alpha), -(\beta/\alpha)]$ can take any value $(\lambda, \mu) \in F^2$ with $\lambda\mu \neq 1$. Indeed, if $\alpha, \beta, \gamma \in F$ such that $\alpha \neq 0$ and $\beta\gamma = \alpha(1-\alpha)$, then $(\gamma/\alpha)(-\beta/\alpha) = \alpha - 1/\alpha \neq 1$. Conversely, if $\lambda, \mu \in F$ satisfy $\lambda\mu \neq 1$, then take

$$\alpha = \frac{1}{1 - \lambda\mu}, \quad \beta = \frac{-\mu}{1 - \lambda\mu}, \quad \alpha = \frac{\lambda}{1 - \lambda\mu}.$$

Then clearly $\alpha \neq 0$, $\beta\gamma = \alpha(1 - \alpha)$ and $\lambda = \gamma/\alpha$, $\mu = -\beta/\alpha$. Thus we obtain gradings of type (iii).

If $\alpha = 0$, then either $\beta = 0$ or $\gamma = 0$. In the first case we obtain gradings of type (iv), while in the second case we find gradings of type (v).

Let us show that a grading of type (iii), given by the parameters λ, μ with $\lambda\mu \neq 1$, is isomorphic to the grading

$$A_s = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}, \quad A_r = \begin{pmatrix} 0 & 0 \\ F & F \end{pmatrix}.$$

Take

$$X = \begin{pmatrix} 1 & -\mu \\ -\lambda & 1 \end{pmatrix},$$

which is clearly an invertible matrix. Then the map $f : A \to A$ defined by $f(a) = XaX^{-1}$ for any $a \in A$ is an algebra isomorphism, and it is straightforward to check that

$$f\left(\begin{pmatrix} u & v \\ \lambda u & \lambda v \end{pmatrix}\right) \in \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix} \text{ and } f\left(\begin{pmatrix} \mu u & \mu v \\ u & v \end{pmatrix}\right) \in \begin{pmatrix} 0 & 0 \\ F & F \end{pmatrix}.$$

Thus f is an isomorphism of S-graded algebras. Similarly, the gradings of type (iv) and (v) are isomorphic to the grading

$$A_s = \begin{pmatrix} 0 & 0 \\ F & F \end{pmatrix}, \quad A_r = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}. \qquad \square$$

Proposition 2. *Let $S = \{1, s\}$ be a semi-lattice, and let $A = M_2(F)$. Then there exist exactly two S-algebra gradings of A, namely,*

(i) $A_1 = A$, $A_s = 0$, *and*

(ii) $A_1 = 0$, $A_s = A$.

Proof. For any structure of an S-graded algebra of A, we clearly have that A_s is a two-sided ideal of A. Then either $A_s = 0$ and the grading is of type (i), or $A_s = A$ and the grading is of type (ii). $\square$

Finally, if S is the zero group, it is obvious what an S-grading must be.

Proposition 3. *Let S be a null semigroup, not necessarily with two elements, and A an F-algebra. Then there exists only one S-grading of A; this is $A_0 = A$ and $A_s = 0$ for any $s \neq 0$.*

Proof. We have

$$A = AA = \left(\sum_{s\in S} A_s\right)\left(\sum_{t\in S} A_t\right) = \sum_{s,t\in S} A_s A_t \subseteq \sum_{s,t\in S} A_{st} = A_0,$$

and so $A = A_0$. Then, obviously A_s must be 0 for all $s \neq 0$. □

3. Gradings of upper triangular matrix algebras. Let us consider the F-algebra of 2×2 upper triangular matrices

$$T = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}.$$

Theorem 4. *Assume that the field F has a characteristic different from 2. Then any grading of the F-algebra T by the group $\mathbf{Z}_2 = \{e, g\}$ is of one of the following two types.*

(i) *The trivial grading, i.e., $T_e = T$, $T_g = 0$.*

(ii)

$$T_e = \left\{ \begin{pmatrix} x & a(x-z) \\ 0 & z \end{pmatrix} \middle| x, z \in F \right\},\ T_g = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix},$$

for some $a \in F$.

Moreover, any grading of type (ii) *is isomorphic to the grading*

$$T_e = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}, \quad T_g = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}.$$

In particular, there exist two isomorphism types of $\mathbf{Z}_s$-graded algebra structure on T.

The following lemma can be proved by a straightforward but tedious computation.

Lemma 5. *Let $\phi : T \to T$ be an F-linear map. Then ϕ is an algebra automorphism of T if and only if there exist $a \in F$ and $b \in F^*$ such that*

$$\phi(e_{11}) = e_{11} + ae_{12}, \phi(e_{12}) = be_{12}, \phi(e_{22}) = -ae_{12} + e_{22}.$$

In this case $\phi^2 = \mathrm{Id}$ *if and only if either* $b = 1$ *and* $a = 0$, *when* $\phi = \mathrm{Id}$ *or* $b = -1$ *and* $a \in F$.

Proof of Theorem 4. If $T = T_e \oplus T_g$ is a $\mathbf{Z}_2$-grading of T, let us define the map $\phi : T \to T$ by $\phi(A) = A_e - A_g$ for any $A \in T$. Then

$$\begin{aligned}\phi(AB) &= (AB)_e - (AB)_g \\ &= A_eB_e + A_gB_g - A_eB_g - A_gB_e \\ &= \phi(A)\phi(B)\end{aligned}$$

and since ϕ is clearly bijective we obtain that ϕ is an algebra automorphism of T. Moreover, $\phi^2(A) = \phi(A_e - A_g) = A_e + A_g = A$, thus $\phi^2 = \mathrm{Id}$. In terms of the automorphism ϕ, the grading is

$$\begin{aligned}T_e &= \{A_e \mid A \in T\} = \left\{\frac{1}{2}(A + \phi(A)) \mid A \in T\right\}, \\ T_g &= \{A_g \mid A \in T\} = \left\{\frac{1}{2}(A - \phi(A)) \mid A \in T\right\}.\end{aligned}$$

Lemma 5 shows that either $\phi = \mathrm{Id}$ or $\phi(e_{11}) = e_{11} + ae_{12}$, $\phi(e_{12}) = -e_{12}$, $\phi(e_{22}) = -ae_{12} + e_{22}$ for some $a \in F$. In the first case we obtain the trivial grading. In the second case, for

$$A = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$$

we have

$$\phi(A) = \begin{pmatrix} x & a(x-z) - y \\ 0 & z \end{pmatrix},$$

and then

$$T_e = \left\{ \begin{pmatrix} x & (a/2)(x-z) \\ 0 & z \end{pmatrix} \middle| x, z \in F \right\}, \quad T_g = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix},$$

which proves the first part of the statement.

If for any $a \in F$ we denote by $T(a)$ the algebra T with the grading

$$T(A)_e = \left\{ \begin{pmatrix} x & a(x-z) \\ 0 & z \end{pmatrix} \middle| x, z \in F \right\}, \quad T(a)_g = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix},$$

then the map $\psi : T(a) \to T(0)$ defined by

$$\phi\left(\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\right) = \begin{pmatrix} x & a(x-z)-y \\ 0 & z \end{pmatrix}$$

is an isomorphism of $\mathbf{Z}_2$-graded algebras. ◻

Theorem 6. *Let $S = \{s, r\}$ be the left zero semigroup. Then an S-grading of the F-algebra T is of one of the following types.*

(i) $T_s = 0$, $T_r = T$.

(ii) $T_s = T$, $T_r = 0$.

(iii)

$$T_s = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}, \quad T_r = \left\{ \begin{pmatrix} 0 & cz \\ 0 & z \end{pmatrix} \middle| z \in F \right\}$$

for some $c \in F$.

(iv)

$$T_s = \left\{ \begin{pmatrix} 0 & cz \\ 0 & z \end{pmatrix} \middle| z \in F \right\}, \quad T_r = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$$

for some $c \in F$.

Moreover, any grading of type (iii) *is isomorphic to the grading*

$$T_s = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}, \quad T_r = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix},$$

and any grading of type (iv) *is isomorphic to the grading*

$$T_s = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}, \quad T_r = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}.$$

In particular, there exist four isomorphism types of S-gradings of the F-algebra T.

Proof. If $T = T_s \oplus T_r$ is an S-grading of T, define $\phi : T \to T$ by $\phi(A) = A_s$ for any $A \in T$. Then, as in the proof of Theorem 1, we see that there exists $h \in T$ with $h^2 = h$ such that $\phi(A) = hA$ for any $A \in T$, and

$$T_s = \{\phi(A) \mid A \in T\}, \quad T_r = \{A - \phi(A) \mid A \in T\}.$$

Straightforward computations show that h must be one of the following matrices

$$0, I_2, \begin{pmatrix} 0 & c \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & c \\ 0 & 0 \end{pmatrix},$$

for some $c \in F$. If $h = 0$, we obtain $T_s = 0$, $T_r = T$. If $h = I_2$, we have $T_s = T$, $T_r = 0$. If $h = \left(\begin{smallmatrix} 1 & c \\ 0 & 0 \end{smallmatrix}\right)$, the homogeneous components of the matrix $A = \left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right) \in T$ are

$$A_s = \phi(A) = hA = \begin{pmatrix} x & y - cz \\ 0 & \end{pmatrix} \quad \text{and} \quad A_r = A - \phi(A) = \begin{pmatrix} x & cz \\ 0 & c \end{pmatrix},$$

producing a grading of type (iii). Similarly, $h = \left(\begin{smallmatrix} 0 & c \\ 0 & 1 \end{smallmatrix}\right)$ produces a grading of type (iv).

Finally, if we denote by $T(c)$ the algebra T with the grading

$$T_s = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}, \quad T_r = \left\{ \begin{pmatrix} 0 & cz \\ 0 & z \end{pmatrix} \middle| z \in F \right\},$$

we have the map $f : T(0) \to T(c)$ defined by

$$f\left(\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \right) = \begin{pmatrix} x & -c(x - z) + y \\ 0 & z \end{pmatrix}$$

is an isomorphism of S-graded algebras. Similarly for gradings of type (iv). $\square$

Theorem 7. *Let $S = \{1, s\}$ be a semi-lattice. Then an S-grading of the F-algebra T is of one of the following types.*

(i) $T_1 = T$, $T_s = 0$.

(ii) $T_1 = 0$, $T_s = T$.

(iii)

$$T_1 = \left\{ \begin{pmatrix} 0 & cx \\ 0 & x \end{pmatrix} \middle| x \in F \right\}, \quad T_s = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix},$$

for some $c \in F$.

(iv)

$$T_1 = FI_2, T_s = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$$

(v)

$$T_1 = \left\{ \begin{pmatrix} x & cx \\ 0 & 0 \end{pmatrix} \middle| x \in F \right\}, \quad T_s = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix},$$

for some $c \in F$.

(vi)

$$T_1 = FI_2, \quad T_s = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}.$$

(vii)

$$T_1 = \left\{ \begin{pmatrix} x & \alpha(x-y) \\ 0 & y \end{pmatrix} \middle| x, y \in F \right\}, \quad T_s = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix},$$

for some $c \in F$.

Moreover, a grading of type (iii) *is isomorphic to the grading*

$$T_1 = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}, \quad T_s = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix},$$

a grading of type (v) *is isomorphic to the grading*

$$T_1 = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}, \quad T_s = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix},$$

a grading of type (vii) *is isomorphic to the grading*

$$T_1 = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}, \quad T_s = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix},$$

and the isomorphism types (i)–(vi) *and* (vii) *of* S*-gradings are different. In particular, there exist seven isomorphism types of* S*-gradings of the* F*-algebra* T*, five of them being good gradings, and the other two not isomorphic to good gradings.*

Proof. As in the proof of Proposition 2, T_s is a two-sided ideal of T. Thus, T_s is one of

$$0, T, \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}, \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}.$$

If $T_s = 0$, we obtain $T_1 = T$. If $T_s = T$, we get $T_1 = 0$. If

$$T_s = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix},$$

then T_1 has dimension one over F, more precisely,

$$T_1 = F \begin{pmatrix} a & c \\ 0 & b \end{pmatrix},$$

for some $a, b, c \in F$ with $c \neq 0$. Since $T_1 T_1 \subseteq T_1$, we see that either $a = b$ and $c = 0$ or $a = 0$. In the first situation we obtain a grading of type (iv), in the second one a grading of type (iii).

Similarly, if $T_s = \left(\begin{smallmatrix} 0 & F \\ 0 & F \end{smallmatrix}\right)$, then we obtain gradings of types (v) and (vi).

Assume now that $T_s = \left(\begin{smallmatrix} 0 & F \\ 0 & 0 \end{smallmatrix}\right)$. Then it is easy to see that T_1 has a basis consisting of the matrices

$$\begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \beta \\ 0 & 1 \end{pmatrix}$$

for some $\alpha, \beta \in F$. If we write that the product of these two matrices is in T_1, thus spanned by the two matrices, we obtain that $\beta = -\alpha$. Then

$$\begin{aligned} T_1 &= \left\{ x \begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix} + y \begin{pmatrix} 1 & -\alpha \\ 0 & 0 \end{pmatrix} \middle| x, y \in F \right\} \\ &= \left\{ \begin{pmatrix} x & \alpha(x-y) \\ 0 & y \end{pmatrix} \middle| x, y \in F \right\}, \end{aligned}$$

i.e., we have a grading of type (vii). The rest of the claim follows now as in the proof of Theorem 4. $\square$

In conclusion, we note that gradings of T by the null semigroup have been already described in Proposition 3.

Acknowledgments. The authors are grateful to the referee for several corrections.

REFERENCES

1. J. Avan, *Current algebra realization of R-matrices associated to* Z_2*-graded Lie algebras*, Phys. Lett. B **252** (1990), 230–236.

2. J. Avan and M. Talon, *Graded R-matrices for integrable systems*, Nuclear Phys. B **352** (1991), 215–249.

3. Yu.A. Bahturin, A.A. Mikhalev, V.M. Petrogradsky and M.V. Zaicev, *Infinite-dimensional Lie superalgebras*, Berlin, 1992.

4. C. Boboc and S. Dăscălescu, *Gradings of matrix algebras by cyclic groups*, preprint.

5. E. Celeghini, R. Giachetti, P.P. Kulish, E. Sorace and M. Tarlini, *Hopf superalgebra contractions and R-matrix for fermions*, J. Phys. A. **24** (1991), 5675–5682.

6. A.H. Clifford and G.B. Preston, *The algebraic theory of semigroups* II, Amer. Math. Soc., Providence, 1967.

7. R. Coquereaux, G. Esposito-Farese and F. Scheck, *Noncommutative geometry and graded algebras in electroweak interactions*, Internat. J. Modern Phys. **7** (1992), 6555–6593.

8. L. Corwin, Y. Ne'erman and S. Sternberg, *Graded Lie algebras in mathematics and physics* (*Bose-Fermi symmetry*), Rev. Modern Phys. **47** (1975), 573–603.

9. S. Dăscălescu, B. Ion, C. Năstăsescu and J. Rios Montes, *Group gradings on full matrix rings*, J. Algebra **220** (1999), 709–728.

10. M. Dubois-Violette, J. Madore and R. Kerner, *Super matrix geometry*, Classical Quantum Gravity **8** (1991), 1077–1089.

11. T.M.W. Eyre, *Chaotic decompositions in* Z_2*-graded quantum stochastic calculus*, in *Quantum probability*, Gdansk, 1997.

12. R.M. Gade, *Universal R-matrix and graded Hopf algebra structure of* $U_q(\widehat{gl}(2|2))$, J. Phys. A **31** (1998), 4909–4925.

13. M.D. Gould, R.B. Zhang and A.J. Bracken, *Quantum double construction for graded Hopf algebras*, Bull. Austral. Math. Soc. **47** (1993), 353–375.

14. ———, *Lie bi-superalgebras and the graded classical Yang-Baxter equation*, Rev. Math. Phys. **3** (1991), 223–240.

15. H.S. Green and P.D. Jarvis, *Casimir invariants, characteristic identities and Young diagrams for colour algebras and superalgebras*, J. Math. Phys. **24** (1983), 1681–1687.

16. E.L. Green, *Graphs with relations, coverings and group-graded algebras*, Trans. Amer. Math. Soc. **279** (1983), 297–310.

17. E.L. Green and E.N. Marcos, *Graded quotients of path algebras*: *A local theory*, J. Pure Appl. Algebra **93** (1994), 195–226.

18. J.M. Howie, *Fundamentals of semigroup theory*, Clarendon Press, Oxford, 1995.

19. A.V. Kelarev, *Applications of epigroups to graded ring theory*, Semigroup Forum **50** (1995), 327–350.

20. ———, *Recent results and open questions on radicals of semigroup-graded rings*, Fund. Appl. Math. **4** (1998), 1115–1139.

21. ———, *Band-graded rings*, Math. Japonica **49** (1999), 467–479.

22. ———, *Ring constructions and applications*, World Scientific, Singapore, 2002.

23. R. Kerner, *Ternary structures and the Z_3-grading*, in *Quantum groups*, Karpacz, 1994.

24. B. Le Roy, *A Z_3-graded generalization of supermatrices*, J. Math. Phys. **37** (1996), 474–483.

25. J. Von Neumann, *Mathematische Grundlagen der Quantenmechanik*, Springer, Berlin, 1968.

26. E.H. Saidi, *Matrix representation of higher integer conformal spin symmetries*, J. Math. Phys. **36** (1995), 4461–4475.

27. R.F. Streater and A.S. Wightman, *PCT, spin and statistics and all that*, W.A. Benjamin, New York, 1964.

28. J.H.M. Wedderburn, *Problem 3700 in "The Otto Dunkel Memorial Problem Book,"* Amer. Math. Monthly **64** (1957), 22.

29. Y.Z. Zhang, *On the graded quantum Yang-Baxter and reflection equations*, Comm. Theor. Phys. **29** (1998), 377–380.

30. R.B. Zhang, A.J. Bracken and M.D. Gould, *Solution of the graded Yang-Baxter equation associated with the vector representation of $U_q(osp(M/2n))$*, Phys. Lett. B **257** (1991), 133–139.

University of Bucharest, Faculty of Mathematics, Str. Academiei 14, RO-70109 Bucharest 1, Romania

Discipline of Physics, University of Tasmania, G.P.O. Box 252-37, Hobart, Tasmania 7001, Australia

Computing, University of Tasmania, Private Bag 100, Hobart, Tasmania 7001, Australia
E-mail address: `Andrei.Kelarev@utas.edu.au`

University of Bucharest, Faculty of Mathematics, Str. Academiei 14, RO-70109 Bucharest 1, Romania