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LINEAR MAPS BETWEEN BANACH ALGEBRAS COMPRESSING CERTAIN SPECTRAL FUNCTIONS

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ABSTRACT. In this paper, we discuss the linear maps between semi-simple Banach algebras which compress any one of the spectrum, the left spectrum, the right spectrum, the intersection of left spectrum and right spectrum, the boundary of spectrum and the full spectrum. We prove that such linear maps are idempotent preserving. As applications, we characterize such maps in terms of Jordan homomorphisms on C*-algebras of real rank zero. In particular, we give several characterizations of isomorphisms between standard operator algebras by using such spectral function compressing linear maps and surjectivity spectrum compressing or approximate point spectrum compressing linear maps.

1. Introduction. Over the past decade, there has been a considerable interest in the study of linear maps on operator algebras that preserve certain properties of operators. In particular, a problem of how to characterize linear maps that preserve the spectrum of each operator has attracted the attention of many mathematicians. In [**16**], Jafarian and Sourour proved that a surjective linear map preserving spectrum from $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$ is either an isomorphism or an anti-isomorphism, where X and Y are complex Banach spaces, and $\mathcal{B}(X)$ is the Banach algebra of all bounded linear operators acting on X. Aupetit and Mouton [**3**] extended the result of Jafarian and Sourour to primitive Banach algebras with minimal ideals. In [**21**], Sourour characterized the linear bijective maps preserving invertibility from $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$ and obtained a similar result when linear map is unital. Bresar and Semrl [**6**] proved that a linear surjective map preserving spectral radius on $\mathcal{B}(X)$ is either an automorphism or an anti-automorphism multiplied by a scalar with modulus 1. It is shown in [**20**] that every point spectrum

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preserving and surjective linear map on $\mathcal{B}(X)$ is an automorphism and when X is a Hilbert space, every surjective linear map preserving surjectivity spectrum is an automorphism. Recently, Aupetit [**1**] showed that a spectrum preserving linear surjection from a von Neumann algebra onto another is a Jordan isomorphism. For some other papers concerning this type of linear preserver, see [**13, 14, 23, 24**]. We also mention [**4, 9, 17, 24**] about invertibility preservers and spectrum compressers between semi-simple Banach algebras.

Let $\mathcal{A}$ and $\mathcal{B}$ be two unital semi-simple complex Banach algebras. We always denote by I the unit in both $\mathcal{A}$ and $\mathcal{B}$. For any $T \in \mathcal{A}$, the set $\sigma(T)$, $\sigma_l(T)$, $\sigma_r(T)$, $\partial\sigma(T)$ and $r(T)$ will denote the spectrum, the left spectrum, the right spectrum, the boundary of spectrum and the spectral radius of T, respectively. The polynomial convex hull $\eta\sigma(T) = \eta(\sigma(T))$ of $\sigma(T)$ is called the full spectrum of T. If T is an operator on a Banach space, $\sigma_{ap}(T)$ and $\sigma_s(T)$ will denote the approximate point spectrum and the surjectivity spectrum of T, respectively. Recall that the surjectivity spectrum $\sigma_s(T)$ of T is the set $\{\lambda \in \mathbf{C} \mid T - \lambda \text{ is not surjective}\}$. Let $\Delta(\cdot)$ denote any one of nine symbols $\sigma(\cdot)$, $\sigma_l(\cdot)$, $\sigma_r(\cdot)$, $\sigma_l(\cdot) \cap \sigma_r(\cdot)$, $\partial\sigma(\cdot)$, $\eta\sigma(\cdot)$, $\sigma_{ap}(\cdot)$, $\sigma_s(\cdot)$ and $\sigma_{ap}(\cdot) \cap \sigma_s(\cdot)$, then $\Delta(\cdot)$ is a map from $\mathcal{A}$ (or $\mathcal{B}(X)$) into $2^{\mathbf{C}}$, which is called a spectral function on $\mathcal{A}$ (or $\mathcal{B}(X)$). If Δ and Λ are two spectral functions and if $\Lambda(T) \subseteq \Delta(T)$ for all T, we say that Λ is a subspectral function of Δ. A linear map $\Phi : \mathcal{A} \to B$ is said to be $\Delta(\cdot)$ preserving (or, compressing) if $\Delta(\Phi(T)) = \Delta(T)$ (or, $\Delta(\Phi(T)) \subseteq \Delta(T)$) for all $T \in \mathcal{A}$.

A more general, natural and interesting question is to ask what we can say for a surjective linear map $\Phi : \mathcal{A} \to \mathcal{B}$ which preserves or compresses certain spectral function $\Delta(\cdot)$ mentioned above. Another question is how to characterize the linear maps which preserve the left (or, right) invertibility or the semi-invertibility, here an element $T \in \mathcal{A}$ is called semi-invertible if it is either left invertible or right invertible. The purpose of the present paper is to solve these questions, by developing a method which works for all spectral functions, and generalize most of the results mentioned above as special cases of our results. Note that the results in [**1, 3, 4, 9, 20, 21, 24**] concern only the spectral functions $\sigma(\cdot)$ and $\sigma_p(\cdot)$, the point spectrum.

The paper is arranged as follows. In Section 2 we discuss the linear maps compressing the spectral function $\Delta(\cdot)$ between semi-simple complex Banach algebras, where $\Delta(\cdot)$ is any one of six spectral functions $\sigma(\cdot)$, $\sigma_l(\cdot)$, $\sigma_r(\cdot)$, $\sigma_l(\cdot)\cap\sigma_r(\cdot)$, $\partial\sigma(\cdot)$ and $\eta\sigma(\cdot)$. We show that such surjections are idempotent preserving (Theorem 2.2) which generalizes a corresponding result in [**1**] greatly, where the similar assertion was proved for linear surjection *preserving* the spectral function $\Delta(\cdot)=\sigma(\cdot)$. In Section 3, we characterize the spectral function compressing or semi-invertibility preserving linear maps on C*-algebras of real rank zero in terms of Jordan homomorphisms. The main result is Theorem 3.1 which states that every $\Delta(\cdot)$ compressing surjective linear map on a C*-algebras of real rank zero is a Jordan homomorphism. We also give some characterizations of isomorphisms from a C*-algebra of real rank zero into a prime and semi-simple Banach algebra (Theorem 3.3, Corollary 3.4, Corollary 3.5). Section 4 is devoted to the characterizations of $\Delta(\cdot)$ compressing surjective linear maps on standard operator algebras on Banach spaces (Theorem 4.1, Theorem 4.4), where $\Delta(\cdot)$ is any one of nine spectral functions $\sigma(\cdot)$, $\sigma_l(\cdot)$, $\sigma_r(\cdot)$, $\sigma_l(\cdot)\cap\sigma_r(\cdot)$, $\partial\sigma(\cdot)$, $\eta\sigma(\cdot)$, $\sigma_{ap}(\cdot)$, $\sigma_s(\cdot)$ and $\sigma_{ap}(\cdot)\cap\sigma_s(\cdot)$. These results also allow us to get some characterizations of linear bijective maps which preserve left (or, right) invertibility or, semi-invertibility (Corollaries 3.2, 4.6 and 4.7), and of linear surjective maps which preserve the surjectivity of operators or the lower-boundedness of operators (Corollary 4.8).

2. General results. Suppose that $\mathcal{A}$ is a complex Banach algebra. In this and the next section, $\Delta(\cdot)$ will denote any one of the following six spectral functions $\sigma(\cdot)$, $\sigma_l(\cdot)$, $\sigma_r(\cdot)$, $\sigma_l(\cdot)\cap\sigma_r(\cdot)$, $\partial\sigma(\cdot)$ and $\eta\sigma(\cdot)$. The main result is to induce an important property of the linear maps compressing any one of these six spectral functions between semi-simple complex Banach algebras. The proof of this result relies on a characterization of idempotents in semi-simple complex Banach algebras by Aupetit [**1**]. We state the result in the following lemma and refer to [**1**] for its proof. Recall that an idempotent is an element P so that $P^2=P$.

Lemma 2.1. *Let $\mathcal{A}$ be a unital complex semi-simple Banach algebra. Then an element $P\in\mathcal{A}$ is an idempotent if and only if $\sigma(P)\subseteq\{0,1\}$ and there are positive numbers r and c such that $\sigma(T)\subseteq\sigma(P)+c\|P-T\|$ whenever $\|P-T\|<r$, where the set $\sigma(P)+c\|P-T\|$ denotes the union of the circular disks centered at points of $\sigma(P)$ with radius $c\|P-T\|$.*

A linear map is said to be idempotent preserving if it maps every idempotent to an idempotent. It is easily seen that an idempotent preserving linear map must send a set of mutually orthogonal idempotents to a set of mutually orthogonal idempotents (two idempotents P_1 and P_2 are orthogonal if $P_1P_2 = P_2P_1 = 0$).

Now we give the main result in this section.

Theorem 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital semi-simple complex Banach algebras and let $\Delta(\cdot)$ denote any one of the spectral functions $\sigma(\cdot)$, $\sigma_l(\cdot)$, $\sigma_r(\cdot)$, $\sigma_l(\cdot)\cap\sigma_r(\cdot)$, $\partial\sigma(\cdot)$ and $\eta\sigma(\cdot)$. Suppose that $\Phi : \mathcal{A} \to \mathcal{B}$ is a surjective linear map. If $\Delta(\Phi(T)) \subseteq \Delta(T)$ for every $T \in \mathcal{A}$ then Φ is idempotent preserving and $\Phi(I) = I$.*

Proof. Assume that $\Delta(\cdot) = \partial\sigma(\cdot)$, that is, assume that Φ is $\partial\sigma(\cdot)$-compressing. Firstly we show that Φ is continuous. Since $\partial\sigma(\Phi(T)) \subseteq \partial\sigma(T)$ for every $T \in \mathcal{A}$, we have $r(\Phi(T)) \leq r(T)$ for every $T \in \mathcal{A}$. From our assumptions that Φ is onto and $\mathcal{B}$ is semi-simple, applying [**2**; Theorem 5.5.2], we obtain that Φ is continuous. Therefore $\ker\Phi$ is a closed linear subspace of $\mathcal{A}$. Let $\pi : \mathcal{A} \to \mathcal{A}/\ker\Phi$ be the quotient map. Then Φ induces a continuous linear bijective map $\widehat{\Phi} : \mathcal{A}/\ker\Phi \to \mathcal{B}$ determined by $\widehat{\Phi} \circ \pi = \Phi$. Consequently, there are two positive constants α and β such that $\alpha\|\pi(T)\| \leq \|\widehat{\Phi}(\pi(T))\| = \|\Phi(T)\| \leq \beta\|\pi(T)\|$ for all T, where $\|\pi(T)\| = \inf\{\|T - A\| \mid A \in \ker\Phi\}$. Let P in $\mathcal{A}$ be any idempotent. Then $\sigma(P) \subseteq \{0, 1\}$, so $\partial\sigma(P) \subseteq \{0, 1\}$. If $\Phi(P) = 0$, then $\Phi(P)$ is already an idempotent. Now we assume $\Phi(P) \neq 0$. We shall prove $\Phi(P)^2 = \Phi(P)$. Since $\partial\sigma(\Phi(P)) \subseteq \partial\sigma(P)$, $\partial\sigma(\Phi(P)) \subseteq \{0, 1\}$ and consequently, $\eta\sigma(\Phi(P)) \subseteq \{0, 1\}$, hence $\sigma(\Phi(P)) \subseteq \{0, 1\}$. By Lemma 2.1, there are r, $c > 0$ such that $\sigma(T) \subseteq \{0, 1\} + c\|P - T\|$ whenever $\|P - T\| < r$. In particular, $\sigma(T + A) \subseteq \{0, 1\} + c\|P - T - A\|$ whenever $\|P - T - A\| < r$. For any sufficient small $\varepsilon > 0$, there is an element A in $\ker\Phi$ such that $\|P-T-A\| < \|\pi(P-T)\|+\varepsilon < r$. Thus we have, when $\|\pi(P-T)\| < r$,

$$\partial\sigma(\Phi(T)) \subseteq \partial\sigma(T + A) \subseteq \{0, 1\} + c\|\pi(P - T)\| + c\varepsilon.$$

Letting $\varepsilon \to 0$, we get

$$\partial\sigma(\Phi(T)) \subseteq \{0, 1\} + c\|\pi(P - T)\|,$$

so

$$\partial\sigma(\Phi(T)) \subseteq \{0,1\} + \frac{c}{\alpha}\|\Phi(P) - \Phi(T)\|,$$

and therefore,

$$\sigma(\Phi(T)) \subseteq \eta\sigma(\Phi(T)) \subseteq \{0,1\} + \frac{c}{\alpha}\|\Phi(P) - \Phi(T)\|$$

for $\|\pi(P - T)\| < r$. Again, since $\widehat{\Phi}$ is a surjective open map, there is a $r_1 > 0$ such that $\sigma(B) \subseteq \{0,1\} + (c/\alpha)\|\Phi(P) - B\|$, whenever $\|\Phi(P) - B\| < r_1$. Therefore, applying Lemma 2.1 again, we get $\Phi(P)^2 = \Phi(P)$. So Φ preserves idempotents.

Note that, for an arbitrary element $T \in \mathcal{A}$, $\partial\sigma(T)$ is a subset of any one of the sets $\sigma_l(T) \cap \sigma_r(T)$, $\sigma_l(T)$, $\sigma_r(T)$, $\sigma(T)$ or $\eta\sigma(T)$. If $\Delta(\cdot)$ takes any one of $\sigma_l(\cdot) \cap \sigma_r(\cdot)$, $\sigma_l(\cdot)$, $\sigma_r(\cdot)$, $\sigma(\cdot)$ and $\eta\sigma(\cdot)$, and if Φ compresses the spectral function $\Delta(\cdot)$, then, from the above argument, one easily sees that Φ preserves idempotents. Because $\Delta(\Phi(I)) \subseteq \{1\}$, hence $\sigma(\Phi(I)) \subseteq \{1\}$. Now it follows from $\Phi(I)^2 = \Phi(I)$ that $\Phi(I) = I$. The proof is completed. □

Corollary 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital semi-simple complex Banach algebras and let $\Delta(\cdot)$ denote any one of the spectral functions $\sigma(\cdot)$, $\sigma_l(\cdot)$, $\sigma_r(\cdot)$, $\sigma_l(\cdot)\cap\sigma_r(\cdot)$, $\partial\sigma(\cdot)$ and $\eta\sigma(\cdot)$. Suppose that $\Phi : \mathcal{A} \to \mathcal{B}$ is a surjective linear map. If $\Delta(\Phi(T)) = \Delta(T)$ for every $T \in \mathcal{A}$, then Φ is a bijective linear map preserving idempotents.*

Proof. By Theorem 2.2 we only need to show that Φ is injective. Let us consider the case that $\Delta(\cdot) = \partial\sigma(\cdot)$. For $T \in \mathcal{A}$, if $\Phi(T) = 0$, then, for every quasi-nilpotent element $A \in \mathcal{A}$, we have $\partial\sigma(T + A) = \partial\sigma(\Phi(T+A)) = \partial\sigma(\Phi(A)) = \partial\sigma(A) = \{0\}$. It follows that $r(T+A) = 0$ holds for all quasi-nilpotent element $A \in \mathcal{A}$ and hence $T \in \mathrm{rad}\,(\mathcal{A})$, where $\mathrm{rad}\,(\mathcal{A})$ denotes the Jacobson radical of $\mathcal{A}$, [**2**]. This implies that $T = 0$ as $\mathcal{A}$ is semi-simple, that is, Φ is injective. The other five cases may be dealt with similarly. □

Remark 2.4. The assumptions that $\mathcal{A}$ and $\mathcal{B}$ are semi-simple and that Φ is surjective can not be omitted simply. For the first one, it is easily seen if we take $\mathcal{A} = \mathcal{B} = \tau_n$, the $n \times n$ upper triangular matrix algebra. For the second one, let $\mathcal{A} = \mathcal{B}(H)$, $\mathcal{B} = B(H \oplus H)$ and let $\Phi(T) = \left(\begin{smallmatrix} T & \varphi(T)I \\ 0 & T \end{smallmatrix}\right)$, where H is a Hilbert space and φ is a linear functional on $\mathcal{B}(H)$.

3. The case for C*-algebras of real rank zero. In this section, we apply the results in Section 2 to characterize the spectral function compressing linear maps on C*-algebras of real rank zero in terms of Jordan homomorphisms. Firstly we recall that a C*-algebra $\mathcal{A}$ is of real rank zero if the set of all real linear combinations of orthogonal Hermitian idempotents is dense in the set of all Hermitian elements of $\mathcal{A}$, [**7**]. It is clear that every von Neumann algebra is a C*-algebra of real rank zero. In particular, $\mathcal{B}(H)$, the algebra of all bounded linear operators on a complex Hilbert space, has real rank zero.

Theorem 3.1. *Let $\mathcal{A}$ be a unital C^*-algebra of real rank zero and $\mathcal{B}$ a unital semi-simple complex Banach algebra. Let $\Delta(\cdot)$ denote any one of the spectral functions $\sigma(\cdot)$, $\sigma_l(\cdot)$, $\sigma_r(\cdot)$, $\sigma_l(\cdot)\cap\sigma_r(\cdot)$, $\partial\sigma(\cdot)$ and $\eta\sigma(\cdot)$. Suppose $\Phi : \mathcal{A} \to \mathcal{B}$ is a surjective linear map. If $\Delta(\Phi(T)) \subseteq \Delta(T)$ for every $T \in \mathcal{A}$, then Φ is a Jordan homomorphism. Furthermore, if $\mathcal{B}$ is prime, then Φ is either a homomorphism or an anti-homomorphism.*

Proof. Pick a Hermitian element A which is a real linear combination of orthogonal Hermitian idempotents, i.e., $A = \sum_{i=1}^n t_i P_i$ with $t_i \in \mathbf{R}$, $P_i^2 = P_i = P_i^*$ and $P_i P_j = 0$ if $i \neq j$. By Theorem 2.2, Φ is continuous and maps mutually orthogonal Hermitian idempotents to mutually orthogonal idempotents, so $\Phi(A^2) = \Phi(A)^2$. Now, since $\mathcal{A}$ is a C*-algebra of real rank aero, the set of Hermitian elements, which are finite real linear combinations of orthogonal Hermitian idempotents, is dense in the set of all Hermitian elements in $\mathcal{A}$, we see that $\Phi(A^2) = \Phi(A)^2$ hold for all Hermitian elements A. Replacing A by $A + B$, where both A and B are Hermitian, we get $\Phi(AB+BA) = \Phi(A)\Phi(B)+\Phi(B)\Phi(A)$. Since every $T \in \mathcal{A}$ can be written in the form $T = A + iB$ with A and B Hermitian, the last relation implies that $\Phi(T^2) = \Phi(T)^2$; that is, Φ is Jordan.

The last assertion of the theorem is obvious because it is well known (for example, see [**12**, pp. 47–51]) that if $\mathcal{B}$ is a prime ring, then every Jordan homomorphism Φ from a ring $\mathcal{A}$ onto $\mathcal{B}$ is either a homomorphism or an anti-homomorphism. This completes the proof. $\square$

Recall that the left invertible or right invertible elements are called semi-invertible. We say that a linear map Φ is semi-invertibility preserving if $\Phi(T)$ is semi-invertible whenever T is. Using Theorem 3.1, we can easily obtain the following corollary.

Corollary 3.2. *Let $\mathcal{A}$ be a unital C^*-algebra of real rank zero and $\mathcal{B}$ a unital semi-simple complex Banach algebra. Suppose that $\Phi : \mathcal{A} \to \mathcal{B}$ is a surjective linear map and $\Phi(I)$ is invertible. If Φ preserves any one of the invertibility, left invertibility, right invertibility and semi-invertibility, then Φ is a Jordan homomorphism multiplied by an invertible element.*

Proof. For $B \in \mathcal{B}$, denote L_B the linear map from $\mathcal{B}$ into itself defined by multiplying by B from the left hand, that is, $L_B S = BS$ for every $S \in \mathcal{B}$. Let $\Psi = L_{\Phi(I)^{-1}} \circ \Phi$, then $\Psi(I) = I$. As a preserver, Ψ has the same property as Φ has. Let $\Delta(\cdot)$ be one of the spectral functions $\sigma(\cdot)$, $\sigma_l(\cdot)$, $\sigma_r(\cdot)$ and $\sigma_l(\cdot) \cap \sigma_r(\cdot)$ accordingly as Φ preserves one of the following invertibility; left invertibility, right invertibility and semi-invertibility (for instance, let $\Delta(\cdot) = \sigma_l(\cdot)$ if Φ preserves the left invertibility). Then it is easy to check $\Delta(\Psi(T)) \subseteq \Delta(T)$ for every $T \in \mathcal{A}$. Now by Theorem 3.1, Ψ is a Jordan homomorphism and $\Phi = L_{\Phi(I)} \circ \Psi$. $\square$

The next result gives some characterizations of isomorphisms from a C*-algebra of real rank zero onto a prime Banach algebra containing a nontrivial left invertible element.

Corollary 3.3. *Let $\mathcal{A}$ be a unital C^*-algebra of real rank zero and $\mathcal{B}$ a unital prime semi-simple complex Banach algebra. Suppose that $\Phi : \mathcal{A} \to \mathcal{B}$ is a surjective linear map. If $\mathcal{B}$ contains a left invertible element which is not invertible, then the following are equivalent:*

(1) Φ *is injective and left spectrum* (*or, right spectrum*) *compressing.*

(2) Φ *is injective, unital and left* (*or, right*) *invertibility preserving.*

(3) Φ *is left spectrum* (*or, right spectrum*) *preserving.*

(4) Φ *is an isomorphism.*

Proof. It is clear that we only need to prove (1)$\Rightarrow$(4), (2)$\Rightarrow$(4) and (3)$\Rightarrow$(4). For (2)$\Rightarrow$(4), assume that Φ is left invertibility preserving. Since $\Phi(I) = I$, Φ is left spectrum compressing. By Theorem 3.1, Φ is either an homomorphism or an anti-homomorphism since $\mathcal{B}$ is prime. Let $A \in \mathcal{A}$ be an element that is left invertible but not invertible and $B \in \mathcal{A}$ be a left inverse of A. If Φ is an anti-homomorphism, then $I = \Phi(BA) = \Phi(A)\Phi(B)$, which implies that $\Phi(A)$ is right invertible. Hence $\Phi(A)$ is invertible and $I = \Phi(B)\Phi(A) = \Phi(AB)$. Since Φ is injective and $\Phi(I) = I$, we get $AB = I$. Thus A is invertible, a contradiction. If Φ preserves right invertibility, similarly, one can show that Φ is an isomorphism. The arguments of (1)$\Rightarrow$(4) and (3)$\Rightarrow$(4) are similar. The proof is completed. $\square$

For linear maps compressing $\Delta(\cdot)$ from $\mathcal{B}(H)$ onto $\mathcal{B}(K)$, where H and K are Hilbert spaces, we have more concrete characterizations.

Corollary 3.4. *Let H and K be complex Hilbert spaces, and let $\Delta(\cdot)$ denote any one of the spectral functions $\sigma(\cdot)$, $\sigma_l(\cdot) \cap \sigma_r(\cdot)$, $\partial\sigma(\cdot)$ and $\eta\sigma(\cdot)$. Suppose that $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is a surjective linear map, then the following are equivalent*:

(1) Φ *is injective and $\Delta(\Phi(T)) \subseteq \Delta(T)$ for every $T \in \mathcal{B}(H)$.*

(2) $\Delta(\Phi(T)) = \Delta(T)$ *for every $T \in \mathcal{B}(H)$.*

(3) Φ *is injective, unital and invertibility preserving.*

(4) Φ *is injective, unital and semi-invertibility preserving.*

(5) Φ *is a Jordan isomorphism.*

(6) Φ *is either an isomorphism or an anti-isomorphism.*

(7) *There exists an invertible operator $A \in \mathcal{B}(H, K)$ such that either $\Phi(T) = ATA^{-1}$ for every $T \in \mathcal{B}(H)$ or $\Phi(T) = AT^{tr}A^{-1}$ for every $T \in \mathcal{B}(H)$, where T^{tr} denotes the transpose of T relative to an arbitrary but fixed orthonormal base of H.*

Proof. By Theorem 3.1 and Corollary 3.2, (1)–(6) are equivalent. Now (7) follows a classical result [**8**] that every isomorphism or anti-isomorphism Φ between $\mathcal{B}(H)$ and $\mathcal{B}(K)$ is spatial, i.e., there is an invertible operator $A \in \mathcal{B}(H,K)$ such that $\Phi(T) = ATA^{-1}$ for all $T \in \mathcal{B}(H)$, or $\Phi(T) = AT^{tr}A^{-1}$ for all $T \in \mathcal{B}(H)$. The proof is completed. $\square$

If Φ is a left (or right) spectrum compressing bijection, then it is easily seen from Corollary 3.2 and Corollary 3.3 that the following corollary holds true.

Corollary 3.5. *Let H and K be complex Hilbert spaces and let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a surjective linear map. Then the following are equivalent*:

(1) *$\sigma_l(\Phi(T)) = \sigma_l(T)$ for every $T \in \mathcal{A}$.*

(1′) *$\sigma_r(\Phi(T)) = \sigma_r(T)$ for every $T \in \mathcal{A}$.*

(2) *Φ is injective and $\sigma_l(\Phi(T)) \subseteq \sigma_l(T)$ for every $T \in \mathcal{A}$.*

(2′) *Φ is injective and $\sigma_r(\Phi(T)) \subseteq \sigma_r(T)$ for every $T \in \mathcal{A}$.*

(3) *Φ is injective, unital and left invertibility preserving.*

(3′) *Φ is injective, unital and right invertibility preserving.*

(4) *Φ is an isomorphism.*

(5) *There exists an invertible operator $A \in \mathcal{B}(H,K)$ such that $\Phi(T) = ATA^{-1}$ for every $T \in \mathcal{B}(H)$.*

4. The case for standard operator algebras acting on Banach spaces. Let $\mathcal{B}(X)$ denote the Banach algebra of all bounded linear operators on a complex Banach space X. Recall that a standard operator algebra acting on Banach space X is a closed subalgebra of $\mathcal{B}(X)$ containing the identity and all finite rank operators. In this section, we characterize the linear maps compressing the spectral function $\Delta(\cdot)$ on standard operator algebras acting on Banach spaces, where $\Delta(\cdot)$ stands for any one of nine spectral functions $\sigma(\cdot)$, $\sigma_l(\cdot)$, $\sigma_r(\cdot)$, $\sigma_{ap}(\cdot)$, $\sigma_s(\cdot)$, $\sigma_{ap}(\cdot) \cap \sigma_s(\cdot)$, $\sigma_l(\cdot) \cap \sigma_r(\cdot)$, $\partial\sigma(\cdot)$ and $\eta\sigma(\cdot)$. For every $T \in \mathcal{B}(X)$, it is clear that $\sigma_{ap}(T) \subset \sigma_l(T)$, $\sigma_s(T) \subset \sigma_r(T)$ and $\partial\sigma(T) \subset \sigma_{ap}(T) \cap \sigma_s(T)$. Indeed, let $\lambda \in \partial\sigma(T)$ but $\lambda \notin \sigma_s(T)$, then $\lambda - T$ is surjective but not injective, by [**22**, p. 285], λ belongs to the interior of $\sigma(T)$, this is a contradiction. So $\partial\sigma(T) \subset \sigma_s(T)$. $\partial\sigma(T) \subset \sigma_{ap}(T)$ follows [**10**, p. 215]. The main result in this section is the following theorem which generalizes Theorem 4 in [**20**] when $\Delta(\cdot) = \sigma_s(\cdot)$.

Theorem 4.1. *Let X and Y be Banach spaces over the complex field, and let $\mathcal{A}$ and $\mathcal{B}$ be standard operator algebras on X and Y, respectively. Suppose that $\Phi : \mathcal{A} \to \mathcal{B}$ is a bijective linear map and $\Delta(\cdot)$ stands for any one of the spectral functions $\sigma(\cdot)$, $\sigma_l(\cdot)$, $\sigma_r(\cdot)$, $\sigma_{ap}(\cdot)$, $\sigma_s(\cdot)$, $\sigma_{ap}(\cdot) \cap \sigma_s(\cdot)$, $\sigma_l(\cdot) \cap \sigma_r(\cdot)$, $\partial\sigma(\cdot)$ and $\eta\sigma(\cdot)$. Then the following are equivalent*:

(1) *Φ is $\Delta(\cdot)$ compressing.*

(2) *Either there exists an invertible operator $A \in \mathcal{B}(X, Y)$ such that $\Phi(T) = ATA^{-1}$ for every $T \in \mathcal{A}$ or there exists an invertible operator $A \in \mathcal{B}(X^*, Y)$ such that $\Phi(T) = AT^*A^{-1}$ for every $T \in \mathcal{A}$. The last case cannot occur if X or Y is not reflexive, or if there exists a semi-invertible but not invertible element in $\mathcal{A}$.*

To prove Theorem 4.1, we need the following lemmas.

Lemma 4.2. *Let X be a Banach space over the complex field and $\Delta(\cdot)$ denote any one of the spectral functions $\sigma(\cdot)$, $\sigma_l(\cdot)$, $\sigma_r(\cdot)$, $\sigma_{ap}(\cdot)$, $\sigma_s(\cdot)$, $\sigma_l(\cdot) \cap \sigma_r(\cdot)$, $\sigma_{ap}(\cdot) \cap \sigma_s(\cdot)$, $\partial\sigma(\cdot)$ and $\eta\sigma(\cdot)$. Let $T \in \mathcal{B}(X)$, $x \in X$, $f \in X^*$ and $\lambda \in \mathbf{C}$. If $\lambda \notin \eta\sigma(T)$, then $\lambda \in \Delta(T + x \otimes f)$ if and only if $\langle(\lambda - T)^{-1}x, f\rangle = 1$.*

Proof. Assume that $\lambda \notin \eta\sigma(T)$ but $\lambda \in \Delta(T + x \otimes f)$. We claim that $\lambda \in \sigma(T + x \otimes f)$. This is clear if $\Delta(\cdot)$ is any one of $\sigma(\cdot)$, $\sigma_l(\cdot)$, $\sigma_r(\cdot)$, $\sigma_l(\cdot) \cap \sigma_r(\cdot)$, $\sigma_{ap}(\cdot)$, $\sigma_s(\cdot)$, $\sigma_{ap}(\cdot) \cap \sigma_s(\cdot)$ and $\partial\sigma(\cdot)$. Now we consider the case $\Delta(\cdot) = \eta\sigma(\cdot)$. Since $\partial\sigma(T)\backslash \operatorname{iso}\sigma(T) \subseteq \sigma(T + C)$ holds for every compact operator C (refer to [**10**], for example), we see that $\eta\sigma(T)\backslash \operatorname{iso}\eta\sigma(T) \subseteq \eta(\partial\sigma(T)\backslash \operatorname{iso}\sigma(T)) \subseteq \eta\sigma(T + C)$ for every compact operator C, where $\operatorname{iso}\sigma(T)$ denotes the isolated points of $\sigma(T)$. Therefore $\eta\sigma(T + C)\backslash\eta\sigma(T)$ is a set consisting of some isolated points in $\eta\sigma(T + C)$. Hence $\lambda \in \eta\sigma(T + x \otimes f)$ and $\lambda \notin \eta\sigma(T)$ imply that $\lambda \in \sigma(T + x \otimes f)$, as desired. It is clear now that $1 \in \sigma((\lambda - T)^{-1}x \otimes f)$, and consequently, $\langle (\lambda - T)^{-1}x, f\rangle = 1$.

Conversely, assume that $\langle (\lambda - T)^{-1}x, f\rangle = 1$. Then $1 \in \sigma((\lambda - T)^{-1}x \otimes f)$ and therefore $\lambda \in \sigma(T + x\otimes f) \subseteq \eta\sigma(T + x \otimes f)$. Since $\lambda \notin \eta\sigma(T)$, by virtue of the argument above, we have

$$\begin{aligned}\lambda &\in \operatorname{iso}\sigma(T + x\otimes f) \subseteq \partial\sigma(T + x\otimes f)\\ &\subseteq \sigma_l(T + x\otimes f) \cap \sigma_r(T + x\otimes f)\\ &\subseteq \sigma_l(T + x\otimes f) \text{ (or } \sigma_r(T + x\otimes f)) \subseteq \sigma(T + x\otimes f).\end{aligned}$$

So, in any case, we always have $\lambda \in \Delta(T + x\otimes f)$. $\square$

Lemma 4.3. *Let X be a Banach space over the complex field, and let $\Delta(\cdot)$ denote any one of the spectral functions $\sigma(\cdot)$, $\sigma_l(\cdot)$, $\sigma_r(\cdot)$, $\sigma_{ap}(\cdot)$, $\sigma_s(\cdot)$, $\sigma_l(\cdot)\cap\sigma_r(\cdot)$, $\sigma_{ap}(\cdot)\cap\sigma_s(\cdot)$, $\partial\sigma(\cdot)$ and $\eta\sigma(\cdot)$. Then, for an operator $R \in \mathcal{B}(X)$, the following conditions are equivalent*:

(1) $\operatorname{rank} R \leq 1$.

(2) *For every $T \in \mathcal{B}(X)$ and all distinct scalars α and β,*

$$\Delta(T + \alpha R) \cap \Delta(T + \beta R) \subseteq \eta\sigma(T).$$

(3) *Condition* (2) *is satisfied for every $T \in \mathcal{B}(X)$ of rank at most* 2.

(4) *For every $T \in \mathcal{B}(X)$, there exists a compact subset K_T of the complex plane containing $\eta\sigma(T)$, such that*

$$\Delta(T + \alpha R) \cap \Delta(T + \beta R) \subseteq K_T$$

for all distinct scalars α and β.

(5) Condition (4) is satisfied for every $T \in \mathcal{B}(X)$ of rank at most 2.

Proof. (1)$\Rightarrow$ (2). Assume that there exists a rank-1 operator R, an operator $T \in \mathcal{B}(X)$ and α_1, $\alpha_2 \in \mathbf{C}$ with $\alpha_1 \neq \alpha_2$ such that $\lambda \in \Delta(T + \alpha_1 R) \cap \Delta(T + \alpha_2 R)$ but $\lambda \notin \eta\sigma(T)$. Then by the proof of Lemma 4.2, we always have $\lambda \in \sigma(T + \alpha_1 R) \cap \sigma(T + \alpha_2 R)$. It follows that $\alpha_i \neq 0$ and $\lambda - T - \alpha_i R = (I - \alpha_i R(\lambda - T)^{-1})(\lambda - T)$ is not invertible, $i = 1, 2$. Therefore $I - \alpha_i R(\lambda - T)^{-1}$ is not invertible and α_i^{-1}, $i = 1, 2$, belong to the spectrum of the rank-1 operator $R(\lambda - T)^{-1}$, which is impossible since the spectrum of a rank one operator cannot contain two distinct nonzero points. So (1) implies (2).

(2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (4) $\Rightarrow$ (5) are trivial.

(5) $\Rightarrow$ (1). Assume that R satisfies condition (5). Firstly we show that $\Delta(R)$ contains at most one nonzero complex number by taking $T = 0$. Indeed, if $\Delta(R)$ contains two distinct nonzero complex numbers λ and μ, then $w \in \Delta(\lambda^{-1}wR) \cap \Delta(\mu^{-1}wR)$ holds for all complex numbers w, contradicting the assumption that $\Delta(\lambda^{-1}wR)\cap\Delta(\mu^{-1}wR)$ is contained in a compact subset of $\mathbf{C}$. So there is a nonzero complex number c such that $\Delta(R) \subseteq \{0, c\}$. Thus we always have $\partial\sigma(R) \subseteq \{0, c\}$ as $\partial\sigma(R) \subseteq \Delta(R)$, and therefore $\sigma(R) \subseteq \{0, c\}$.

Let $x \in X$ and $f \in X^*$ be nonzero and let $G(z) = \langle (I-zR)^{-1}x, f\rangle$ for $z \in \mathbf{C} \setminus \{c^{-1}\}$. Applying condition (5) with $T = x\otimes f$, we will prove that the equation $G(z) = w$ has at most one solution for every w with $|w|$ large enough. Assume that $G(z) = w$, then $\langle (w - zwR)^{-1}x, f\rangle = 1$. It follows from Lemma 4.2 that $w \in \Delta(T+zwR)$. Take w so that $w \notin K_T$, thus we have $w \notin \eta\sigma(T)$. Now if $z_1 \neq z_2$ and if $G(z_1) = G(z_2) = w$. Then

$$w \in \Delta(T + z_1wR) \cap \Delta(T + z_2wR) \subseteq K_T,$$

which is a contradiction. Thus, by Picard's Big theorem [**18**], the function G has poles, or removable singularity, at each of $1/c$ and ∞. So G must be a rational function. Let $G(z) = P(z)/Q(z)$ where P and Q are polynomials, $Q(z) = (cz - 1)^n$ for some nonnegative integer n and $P(1/c) \neq 0$. We will show $\deg P \leq 1$ and $\deg Q \leq 1$. For all w large enough, say $|w| > r$, we know that the equation

$$P(z) - wQ(z) = 0 \tag{4.1}$$

has at most one solution. If $\deg (P - wQ)$ is larger than one, then any solution of (4.1) must also satisfy the equation

$$P^{'}(z) - wQ^{'}(z) = 0. \tag{4.2}$$

It follows that, for any such w, the corresponding solution z satisfies the polynomial equation

$$Q(z)P^{'}(z) - Q^{'}(z)P(z) = 0. \tag{4.3}$$

If the polynomials $QP^{'} - PQ^{'}$ are identically zero, then G is constant and the proof is completed. If not, then equation (4.3) has only finitely many solutions and so the polynomial $P - wQ$ has degree larger than one for only finitely many w among $\{w \mid |w| > r\}$. Hence $P - wQ$ has degree at most one for all w larger enough. This implies that $\deg P \leq 1$ and $\deg Q \leq 1$. Hence

$$G(z) = az + b \quad \text{or} \quad \frac{az + b}{cz - 1}$$

for some $a, b \in \mathbf{C}$. The form of the function G implies that G satisfies one of the following differential equations:

$$G^{''}(z) = 0 \quad \text{or} \quad (cz - 1)G^{''}(z) + 2cG^{'}(z) = 0.$$

In particular $G^{''}(0) = 0$ or $G^{''}(0) = 2cG^{'}(0)$. Direct computation yields that $G^{'}(0) = \langle Rx, f\rangle$ and $G^{''}(0) = 2\langle R^2x, f\rangle$. So for every $x \in X$ and $f \in X^*$, we have $\langle R^2x, f\rangle \equiv 0$ or $\langle (R^2 - cR)x, f\rangle \equiv 0$. Now it is easy to check that either $\langle R^2x, f\rangle = 0$ for all $x \in X$ and $f \in X^*$, or $\langle (R^2 - cR)x, f\rangle = 0$ for all $x \in X$ and $f \in X^*$. Hence $R^2x = 0$ for all $x \in X$ or $(R^2 - cR)x = 0$ for all $x \in X$, thus we have $R^2 = 0$ or $R^2 = cR$.

We are now in a position to prove that R has rank one. If $R^2 = cR$ and $\text{rank}R > 1$, let u and v be two linearly independent vectors in the range of R. Thus $Ru = cu$ and $Rv = cv$. Let $f \in X^*$ be such that $\langle u, f\rangle = 0$ and $\langle v, f\rangle = 1$ and let $T = v \otimes f$. Since $\sigma(R) = \{0, c\}$, we have $\Delta(T + \alpha R) = \eta\sigma(T + \alpha R)$ for all scalars α. Let λ be any complex number outside K_T, since $(T + (\lambda/c)R)u = \lambda u$ and $(T + (\lambda - 1/c)R)v = \lambda v$, we get

$$\lambda \in \Delta\Big(T + \frac{\lambda}{c}R\Big) \cap \Delta\Big(T + \frac{\lambda - 1}{c}R\Big) \subseteq K_T;$$

this is a contradiction. Therefore $\text{rank}\, R \leq 1$.

If $R^2 = 0$ and $\text{rank}R > 1$, let u_2 and u_4 be two linearly independent vectors in the range of R and u_1 and $u_3 \in X$ be such that $Ru_1 = u_2$ and $Ru_3 = u_4$. Thus u_1, u_2, u_3, u_4 are linearly independent and $W = \text{span}\,\{u_1, u_2, u_3, u_4\}$ is a subspace invariant under R with $R|_W$ having matrix representation

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

relative to the basis $\{u_1, u_2, u_3, u_4\}$. Let f_2 and $f_4 \in X^*$ be such that $\langle u_j, f_i \rangle = 1$ if $i = j$ and 0 if $i \neq j$ and define a rank two operator T by $T = u_1 \otimes f_2 + 4u_3 \otimes f_4$. Thus T leaves W invariant and $T|_W$ having matrix representation

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}.$$

Similarly, we can prove that every complex number λ belongs to $\Delta(T+\lambda^2 R) \cap \Delta(T+(\lambda^2/4)R)$. This contradiction establishes the desired conclusion, i.e., $\operatorname{rank} R \leq 1$.

Now we characterize the general surjective linear maps compressing the spectral function $\Delta(\cdot)$ on standard operator algebras.

Theorem 4.4. *Let X and Y be Banach spaces over the complex field, and let $\mathcal{A}$ and $\mathcal{B}$ be standard operator algebras on X and Y, respectively. Let $\Delta(\cdot)$ stand for any one of the spectral functions $\sigma(\cdot)$, $\sigma_l(\cdot)$, $\sigma_r(\cdot)$, $\sigma_{ap}(\cdot)$, $\sigma_s(\cdot)$, $\sigma_{ap}(\cdot) \cap \sigma_s(\cdot)$, $\sigma_l(\cdot) \cap \sigma_r(\cdot)$, $\partial\sigma(\cdot)$ and $\eta\sigma(\cdot)$. Suppose that $\Phi : \mathcal{A} \to \mathcal{B}$ is a surjective linear map. If Φ is $\Delta(\cdot)$ compressing, then either $\Phi(F) = 0$ for every finite rank operator $F \in \mathcal{A}$, or Φ is injective. In the latter case, either*

(1) *there exists an invertible operator $A \in \mathcal{B}(X, Y)$ such that $\Phi(T) = ATA^{-1}$ for every $T \in \mathcal{A}$*; *or*

(2) *there exists an invertible operator $A \in \mathcal{B}(X^*, Y)$ such that $\Phi(T) = AT^*A^{-1}$ for every $T \in \mathcal{A}$. This case cannot occur if X or Y is not reflexive, or if there exists a semi-invertible but not invertible element in $\mathcal{A}$ and if $\Delta(\cdot)$ is any one of $\sigma_l(\cdot)$, $\sigma_r(\cdot)$, $\sigma_{ap}(\cdot)$ and $\sigma_s(\cdot)$.*

Proof. Assume that Φ is $\Delta(\cdot)$ compressing. Since $\mathcal{A}$ and $\mathcal{B}$ are semi-simple, for the cases $\Delta(\cdot) = \sigma_{ap}(\cdot)$, $\sigma_s(\cdot)$ or $\sigma_{ap} \cap \sigma_s(\cdot)$, it can be also checked that Φ preserves idempotents and $\Phi(I) = I$ as in Theorem 2.2.

We claim that Φ is rank-1 nonincreasing. Let $R \in \mathcal{A}$ and $\operatorname{rank} R = 1$, then condition (2) of Lemma 4.3 is satisfied; this, together with the fact that Φ is spectral function $\Delta(\cdot)$ compressing, implies that

$$\Delta(\Phi(T) + \alpha\Phi(R)) \cap \Delta(\Phi(T) + \beta\Phi(R)) \subseteq \eta\sigma(T),$$

for every operator $T \in \mathcal{A}$ and $\alpha \neq \beta$. Since $\eta(\Delta(S)) = \eta\sigma(S)$ for every operator S and $\Delta(\Phi(T)) \subseteq \Delta(T)$ for each $T \in \mathcal{A}$, we have

$\eta\sigma(\Phi(T)) \subseteq \eta\sigma(T)$. Thus by Lemma 4.3 (4), rank $(\Phi(R)) \leq 1$ since Φ is surjective and $\eta\sigma(T)$ is a compact set containing $\eta\sigma(\Phi(T))$. That is, Φ is rank-1 nonincreasing. By [**14**], we know that either (i) there exist linear transformations $A : X \to Y$and $C : X^* \to Y^*$ such that $\Phi(x \otimes f) = Ax \otimes Cf$ for every $x \in X$ and $f \in X^*$ or (ii) there exist linear transformations $A : X^* \to Y$ and $C : X \to Y^*$ such that $\Phi(x \otimes f) = Af \otimes Cx$ for every $x \in X$ and $f \in X^*$.

Here we take an argument following the line of that in [**21**]. Assume that Φ has the form (i). Let T be an arbitrary operator in $\mathcal{A}$, then

$$\Phi(T + x \otimes f) = \Phi(T) + Ax \otimes Cf.$$

Pick a complex number λ with $\lambda \notin \eta\sigma(T)$, then $\lambda \notin \eta\sigma(\Phi(T))$. If

$$\langle(\lambda - \Phi(T))^{-1}Ax, Cf\rangle = 1,$$

by Lemma 4.2 and the spectral function $\Delta(\cdot)$ compressing property of Φ, it is obvious that $\langle(\lambda - T)^{-1}x, f\rangle = 1$. So, by linearity, we have, as functions in $\lambda \in \mathbf{C}\backslash\eta\sigma(T)$, either

$$\langle(\lambda - \Phi(T))^{-1}Ax, Cf\rangle \equiv \langle(\lambda - T)^{-1}x, f\rangle$$

or

$$\langle(\lambda - \Phi(T))^{-1}Ax, Cf\rangle \equiv 0$$

for every $x \in X$ and $f \in X^*$. Take $r = \min\{\|T\|^{-1}, \|\Phi(T)\|^{-1}\}$. If $|z| < r$, then $I - zT$ is invertible in $\mathcal{A}$ and $I - z\Phi(T)$ is invertible in $\mathcal{B}$. Replacing λ with $1/z$ in the above equality, we have, in the disk $\{z \in \mathbf{C} \mid |z| < r\}$, either

$$\langle(I - z\Phi(T))^{-1}Ax, Cf\rangle \equiv \langle(I - zT)^{-1}x, f\rangle \tag{4.4}$$

or

$$\langle(I - z\Phi(T))^{-1}Ax, Cf\rangle \equiv 0 \tag{4.5}$$

for every $x \in X$, $f \in X^*$. It is easily checked that either the equation (4.4) holds for all $x \in X$ and $f \in X^*$, or the equation (4.5) holds for all $x \in X$ and $f \in X^*$.

Taking derivatives at 0 in (4.4) and (4.5) respectively, we obtain that, for every $T \in \mathcal{A}$, either $\langle Tx, f\rangle \equiv \langle \Phi(T)Ax, Cf\rangle$ for all $x \in X$ and $f \in X^*$ or $\langle \Phi(T)Ax, Cf\rangle \equiv 0$ for all $x \in X$ and $f \in X^*$. It follows from the linearity of Φ that either

$$\langle Tx, f\rangle \equiv \langle \Phi(T)Ax, Cf\rangle$$

holds for all $T \in \mathcal{A}$, $x \in X$ and $f \in X^*$, or

$$\langle \Phi(T)Ax, Cf\rangle \equiv 0$$

holds for all $T \in \mathcal{A}$, $x \in X$ and $f \in X^*$. In the latter case we must have $A = 0$ or $C = 0$ since the range of Φ contains every finite rank operator. Hence $\Phi(x \otimes f) = Ax \otimes Cf = 0$ and consequently, $\Phi(F) = 0$ for every finite rank operator $F \in \mathcal{A}$.

If the first case occurs, that is, for all $T \in \mathcal{A}$, $x \in X$ and $f \in X^*$, we have $\langle Tx, f\rangle = \langle \Phi(T)Ax, Cf\rangle$, then for every $y \in Y$, there exist $T_y \in \mathcal{B}(X)$ and $x_y \in X$ such that $\Phi(T_y)Ax_y = y$ and hence $\langle y, Cf\rangle = \langle T_y x_y, f\rangle$. This shows that there exists a linear transformation $B : Y \to X$ such that

$$\langle By, f\rangle = \langle y, Cf\rangle. \tag{4.6}$$

So $C = B^*$ and $\langle Tx, f\rangle = \langle B\Phi(T)Ax, f\rangle$ for every $x \in X$ and $f \in X^*$. Thus $T = B\Phi(T)A$ for all $T \in \mathcal{A}$. It follows immediately that Φ is injective. The surjectivity of B can be easily checked. Now we prove that B is injective. Assume that $By = 0$ for some $y \in Y$. For any $g \in Y^*$ with $g \neq 0$, there exists $T_1 \in \mathcal{A}$ such that $\Phi(T_1) = y \otimes g$. Therefore $B\Phi(T_1) = 0$ and consequently, $T_1 = 0$. Hence $y \otimes g = \Phi(T_1) = 0$ and $y = 0$. This proves that B is injective. Since $\Phi(I) = I$, $BA = I$. This implies that B is bijective and $B = A^{-1}$. It is easily seen from equation (4.6) that B is closed, and therefore is bounded by closed graph theorem. Hence $\Phi(T) = ATA^{-1}$.

If Φ has the form (ii), then by a similar argument, we have that either $\Phi(F) = 0$ for every finite rank operator $F \in \mathcal{A}$ or $\langle Tx, f\rangle = \langle \Phi(T)Af, Cx\rangle$ for every $x \in X$, $f \in X^*$ and $T \in \mathcal{A}$. Let $y \in Y$. Since the range of Φ contains every finite rank operator, there exist $f_y \in X^*$ and $T_y \in \mathcal{B}(X)$ such that $\Phi(T_y)Af_y = y$, and so

$$\langle y, Cx\rangle = \langle T_y x, f_y\rangle = \langle x, T_y^* f_y\rangle$$

Therefore there exists a linear transformation $B : Y \to X^*$ such that

$$\langle y, Cx\rangle = \langle x, By\rangle. \tag{4.7}$$

Equation (4.7) implies that B is bounded, so $\langle x, T^*f\rangle = \langle x, B\Phi(T)Af\rangle$ and $B\Phi(T)A = T^*$. Now the same argument as that of case (i) above shows that A and B are bijective and $B = A^{-1}$. Hence $\Phi(T) = AT^*A^{-1}$. It is clear that in this case both X and Y are reflexive. If $\mathcal{A}$ contains a semi-invertible element T_0 which is not invertible, this case can not occur whenever $\Delta(\cdot)$ takes any one of $\sigma_l(\cdot)$, $\sigma_r(\cdot)$, $\sigma_{ap}(\cdot)$ and $\sigma_s(\cdot)$. For instance, if T_0 is left invertible and $\Delta(\cdot) = \sigma_r(\cdot)$, then there is a $S_0 \in \mathcal{A}$ such that $S_0T_0 = I$. The $\sigma_r(\cdot)$-compressibility of Φ implies that $\Phi(S_0) = AS_0^*A^{-1}$ is right invertible. So there is an element W in $\mathcal{A}$ such that $\Phi(S_0)\Phi(W) = I$, which leads to $S_0W = I$. Hence S_0 is invertible, a contradiction. □

Proof of Theorem 4.1. It is an immediate consequence of Theorem 4.4. □

If Φ is $\Delta(\cdot)$ preserving, then, by Corollary 2.3, Φ is injective, so we have the following corollaries.

Corollary 4.5. *Let X and Y be Banach spaces over the complex field, and let $\mathcal{A}$ and $\mathcal{B}$ be standard operator algebras on X and Y, respectively. Let $\Delta(\cdot)$ stand for any one of the spectral functions $\sigma(\cdot)$, $\sigma_l(\cdot)$, $\sigma_r(\cdot)$, $\sigma_{ap}(\cdot)$, $\sigma_s(\cdot)$, $\sigma_{ap}(\cdot) \cap \sigma_s(\cdot)$, $\sigma_l(\cdot) \cap \sigma_r(\cdot)$, $\partial\sigma(\cdot)$ and $\eta\sigma(\cdot)$. Suppose that $\Phi : \mathcal{A} \to \mathcal{B}$ is a surjective linear map, then the following are equivalent*:

(1) *Φ is $\Delta(\cdot)$ preserving.*

(2) *Either there exists an invertible operator $A \in \mathcal{B}(X, Y)$ such that $\Phi(T) = ATA^{-1}$ for every $T \in \mathcal{A}$ or there exists an invertible operator $A \in \mathcal{B}(X^*, Y)$ such that $\Phi(T) = AT^*A^{-1}$ for every $T \in \mathcal{A}$. The last case cannot occur if X or Y is not reflexive, or if there exists a semi-invertible but not invertible element in $\mathcal{A}$ and $\Delta(\cdot)$ takes one of $\sigma_l(\cdot)$, $\sigma_r(\cdot)$, $\sigma_{ap}(\cdot)$ and $\sigma_s(\cdot)$.*

The following corollary characterizes the linear maps preserving the left invertibility, the right invertibility, the lower-boundedness of operators or the surjectivity of operators between standard operator algebras, where the equivalence of (3) and (5) generalizes one of the main results in [20] replacing Hilbert space by Banach space and omitting the assumption "in both directions." Recall that an operator T is said to be bounded below if there is a positive number α such that $\|Tx\| \geq \alpha\|x\|$ for all vectors x. It is clear that T is bounded below if and only if T is injective and has closed range.

Corollary 4.6. *Let X and Y be complex Banach spaces, and let $\mathcal{A}$ and $\mathcal{B}$ be standard operator algebras on X and Y, respectively. Suppose that $\Phi : \mathcal{A} \to \mathcal{B}$ is a bijective linear map and $\Phi(I)$ is invertible. If there exists a semi-invertible but not invertible element in $\mathcal{A}$, then the following are equivalent*:

(1) Φ *preserves left invertibility.*

(2) Φ *preserves right invertibility.*

(3) Φ *preserves the surjectivity of operators.*

(4) Φ *maps operators which are bounded below to operators which are bounded below.*

(5) *There exist invertible operators $A \in \mathcal{B}(X, Y)$ and $B \in \mathcal{B}(Y, X)$ such that $\Phi(T) = ATB$ for every $T \in \mathcal{A}$.*

Proof. Obviously, we need prove that each one of the conditions (1), (2), (3) and (4) implies the condition (5). Assume that (1) holds. Let $\Psi = \Phi(I)^{-1}\Phi$; then $\Psi(I) = I$. It is easy to check that Ψ preserves the spectral function $\sigma_l(\cdot)$. Now (1) $\Rightarrow$ (5) follows from Corollary 3.3 and Theorem 4.4. The remain parts can be dealt with similarly. $\square$

In the following corollary, the equivalence of (1) and (3) was also obtained by Sourour in [**21**].

Corollary 4.7. *Let X and Y be complex Banach spaces, and let $\mathcal{A}$ and $\mathcal{B}$ be standard operator algebras on X and Y, respectively. Suppose that $\Phi : \mathcal{A} \to \mathcal{B}$ is a bijective linear map. Then the following are equivalent*:

(1) Φ *preserves invertibility.*

(2) Φ *preserves semi-invertibility and* $\Phi(I)$ *is invertible.*

(3) *Either there exist invertible operators* $A \in \mathcal{B}(X,Y)$ *and* $B \in \mathcal{B}(Y,X)$ *such that* $\Phi(T) = ATB$ *for every* $T \in \mathcal{A}$ *or there exist invertible operators* $A \in \mathcal{B}(X^*,Y)$ *and* $B \in \mathcal{B}(Y,X^*)$ *such that* $\Phi(T) = AT^*B$ *for every* $T \in \mathcal{A}$*. In the last case,* X *and* Y *must be reflexive.*

Proof. We need only to prove (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3). Assume (2) and let $\Psi = \Phi(I)^{-1}\Phi$. Then Ψ preserves semi-invertibility and $\Psi(I) = I$. It is easily checked that $\sigma_l(\Psi(T)) \cap \sigma_r(\Psi(T)) \subseteq \sigma_l(T) \cap \sigma_r(T)$ for every $T \in \mathcal{A}$. Now (3) follows by applying Theorem 4.4 with $\Delta(\cdot) = \sigma_l(\cdot) \cap \sigma_r(\cdot)$. That (1) $\Rightarrow$ (3) can be proved similarly. $\square$

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