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NONDEGENERATE IDEALS IN FORMAL POWER SERIES RINGS

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ABSTRACT. We give a characterization of the ideals of finite colength of $\mathbf{C}[[x_1, \dots, x_n]]$ which have monomial integral closure using concepts from local algebra, thus leading to an algebraic proof of a result of Saia. We also construct a canonical minimal reduction of any ideal in the class of monomial ideals of $\mathbf{C}[[x_1, \dots, x_n]]$ of finite colength. Therefore, we give an effective method for the computation of the multiplicity of any ideal in this class.

1. Introduction. We denote by $\mathcal{A}$ the ring of formal power series $\mathbf{C}[[x_1, \dots, x_n]]$. By a result of Teissier [**15**], it is known that, when an ideal $I \subseteq \mathcal{A}$ of finite colength (that is, $\dim_{\mathbf{C}} \mathcal{A}/I < \infty$) is generated by monomials, then its multiplicity $e(I)$ is equal to $n!v(I)$, where $v(I)$ is the n-dimensional volume of the complementary in $\mathbf{R}^n_+$ of the Newton polyhedron of I. The equality $e(I) = n!v(I)$ is, in turn, equivalent to say that the integral closure of I is generated by those monomials x^k such that k belongs to the Newton polyhedron of I. In this work we characterize the ideals of $\mathbf{C}[[x_1, \dots, x_n]]$ satisfying this expression for the multiplicity. Then we give an algebraic proof of the main result of Saia in [**13**] that might be reproduced in more general contexts.

Motivated by the work of Yoshinaga [**17**] on the characterization of Newton nondegenerate functions, in the sense of Kouchnirenko [**7**], Saia established in [**13**] the definition of Newton nondegeneracy for ideals in the ring of convergent power series $\mathcal{O}_n$ and proved that, if $I \subseteq \mathcal{O}_n$ has finite colength, then I is Newton nondegenerate (or simply, *nondegenerate*, for short) if and only if the integral closure of I is determined by all the monomials with exponents in the Newton polyhedron of I. The proof that Saia gives of this result deals with the notion of toroidal embedding and the *growth condition* for the integral

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closure, which is a characterization of the integral closure of ideals in $\mathcal{O}_n$ through an analytic condition, see Theorem 2.4. Our approach to determine the integral closure of nondegenerate ideals in $\mathcal{A}$ is based on results from multiplicity theory in local rings and it can also be reproduced for ideals in $\mathcal{O}_n$ of *finite relative colength*, which is a notion defined in this work. The techniques introduced by Kouchnirenko in [**7**], especially in the proof of [**7**, Theorem 2.8], are essential in our work. Further developments of these techniques can be found in [**1**]. We remark that, using [**13**] and Rees's theorem, see Theorem 2.6, the equivalence between the equality $e(I) = n!v(I)$ and the nondegeneracy of I arises quite easily, see [**1**] for details.

To make the exposition clearer, we consider ideals of finite colength in Sections 2 and 3. In Section 4 we show how to apply the ideas of the previous sections to ideals of finite relative colength. In the last section we use the notion of nondegeneracy to obtain a canonical minimal reduction of any monomial ideal of $\mathcal{A}$ of finite colength, thus giving an easy way to compute the multiplicity of a monomial ideal.

2. Nondegenerate ideals. We start by giving some elementary definitions. As before, let $\mathcal{A}$ denote the ring of formal power series $\mathbf{C}[[x_1, \dots, x_n]]$. We set $\mathbf{R}_+ = [0, +\infty[$, $\mathbf{Z}_+ = \mathbf{Z} \cap \mathbf{R}_+$ and $\mathbf{Q}_+ = \mathbf{Q} \cap \mathbf{R}_+$. If $k = (k_1, \dots, k_n) \in \mathbf{Z}_+^n$, we denote the monomial $x_1^{k_1} \cdots x_n^{k_n}$ by x^k.

Given an element $g = \sum_k a_k x^k \in \mathcal{A}$, the *support* of g is defined as $\operatorname{supp}(g) = \{k \in \mathbf{Z}_+^n : a_k \neq 0\}$. If $I \subseteq \mathcal{A}$ is an ideal, we define the *support* of I as the union of the supports of the elements of I and we denote this subset of $\mathbf{Z}_+^n$ by $\operatorname{supp}(I)$.

Definition 2.1. We say that a subset Γ_+ of $\mathbf{R}_+^n$ is a *Newton polyhedron* when there exists some $A \subseteq \mathbf{Q}_+^n$ such that Γ_+ is equal to the convex hull in $\mathbf{R}_+^n$ of the set $\{k + v : k \in A, v \in \mathbf{R}_+^n\}$. In this case we say that Γ_+ is the *Newton polyhedron determined* by A and it is also denoted by $\Gamma_+(A)$. If $g \in \mathcal{A}$, the *Newton polyhedron* of g is $\Gamma_+(g) = \Gamma_+(\operatorname{supp}(g))$. Let I be an ideal of $\mathcal{A}$; then the *Newton polyhedron* of I is $\Gamma_+(I) = \Gamma_+(\operatorname{supp}(I))$.

We observe that if $g_1, \dots, g_s$ is a generating system of $I \subseteq \mathcal{A}$, then $\Gamma_+(I)$ is equal to the convex hull of $\Gamma_+(g_1) \cup \cdots \cup \Gamma_+(g_s)$. It is also obvious that, if $J \subseteq I$ are two ideals of $\mathcal{A}$, then $\Gamma_+(J) \subseteq \Gamma_+(I)$.

Let $\Gamma_+ \subseteq \mathbf{R}^n_+$ be a Newton polyhedron and let $v \in \mathbf{R}^n_+$; then we define

$$l(v, \Gamma_+) = \min\{\langle v, k\rangle : k \in \Gamma_+\};$$
$$\Delta(v, \Gamma_+) = \{k \in \Gamma_+ : \langle v, k\rangle = l(v, \Gamma_+)\}.$$

Then a *face* of Γ_+ is any set of the form $\Delta(v, \Gamma_+)$ for some $v \in \mathbf{R}^n_+$. The *dimension* of a face Δ of Γ_+, denoted by $\dim \Delta$, is defined as the dimension of the smallest affine subspace of $\mathbf{R}^n_+$ containing Δ. We denote by Γ the union of the compact faces of Γ_+ and by Γ_- the union of all segments joining the origin and a point of Γ. If I is an ideal of $\mathcal{A}$, the sets $\Gamma(I)$ and $\Gamma_-(I)$ are defined analogously.

We observe that, when I is an ideal of $\mathcal{A}$ of finite colength, then its Newton polyhedron intersects all coordinate axes, since I contains a power of the maximal ideal of $\mathcal{A}$. This implies that the n-dimensional volume of $\Gamma_-(I)$ is not zero. We will denote this volume by $v(I)$, and we observe that $v(I) = \mathrm{Vol}_n(\mathbf{R}^n_+ \setminus \Gamma_+(I))$, where Vol_n denotes n-dimensional volume.

Let Δ be a compact face of a Newton polyhedron Γ_+; then $C(\Delta)$ shall denote the cone obtained by all half lines starting at the origin and passing through a point of Δ. Then we denote by $\mathcal{A}_\Delta$ the subring of $\mathcal{A}$ given by all $h \in \mathcal{A}$ such that $\mathrm{supp}\,(h) \subseteq C(\Delta)$. Moreover, the ring $\mathcal{A}_\Delta$ can be expressed as $\mathcal{A}_\Delta = \mathcal{A}/P_\Delta$, where P_Δ is the ideal of $\mathcal{A}$ generated by all the monomials x^k such that $k \notin C(\Delta)$. Hence we have that $\mathcal{A}_\Delta$ is a complete local ring of dimension $\dim \Delta + 1$, see [**2**, p. 210].

If $g = \sum_k a_k x^k \in \mathcal{A}$, we denote by g_Δ the polynomial obtained by the sum of all terms $a_k x^k$ such that $k \in \mathrm{supp}\,(g) \cap \Delta$.

Definition 2.2. Let $I \subseteq \mathcal{A}$ be an ideal; we say that I is *Newton nondegenerate* or simply, *nondegenerate* if there exists a system of generators $g_1, \dots, g_s$ of I such that, for every compact face Δ of $\Gamma_+(I)$, the ideal of $\mathcal{A}_\Delta$ generated by $(g_1)_\Delta, \dots, (g_s)_\Delta$ has finite colength in $\mathcal{A}_\Delta$. In this case we say that $\{g_1, \dots, g_s\}$ is a *nondegenerate system of generators* of I.

It is easy to check that if $I \subseteq \mathcal{A}$ is nondegenerate, then every system of generators of I is nondegenerate. As can be seen in [**7**, Theorem 6.2], the fact that $(g_1)_\Delta, \dots, (g_s)_\Delta$ has finite colength in $\mathcal{A}_\Delta$ is equivalent to say that, for all face $\Delta' \subseteq \Delta$, the set of common zeros of $(g_1)_{\Delta'}, \dots, (g_s)_{\Delta'}$ is contained in $\{x \in \mathbf{C}^n : x_1 \cdots x_n = 0\}$. It is obvious that Definition 2.2 can be stated analogously for ideals in the ring $\mathcal{O}_n$ of germs of analytic functions $f : (\mathbf{C}^n, 0) \to \mathbf{C}$.

We say that an ideal of $\mathcal{A}$ is *monomial* when it is generated by monomials. Observe that every monomial ideal of $\mathcal{A}$ is nondegenerate. Let us denote the multiplicity, in the Samuel sense, of an ideal $I \subseteq \mathcal{A}$ of finite colength by $e(I)$, see [**9**] or [**14**] for elementary facts about multiplicity theory in local rings. Now we state a result from [**15**] on the multiplicity of a monomial ideal, see also [**15**, p. 340].

Lemma 2.3. *Let $I \subseteq \mathcal{A}$ be a monomial ideal of finite colength*; *then $e(I) = n!v(I)$.*

As we already mentioned in the introduction, the relation $e(I) = n!v(I)$ characterizes the class of nondegenerate ideals I in $\mathcal{A}$ of finite colength. To show this, we need to introduce some concepts from commutative algebra.

Let I be an ideal in an arbitrary ring R and $h \in R$. We say that h is *integral* over I when there is a monic polynomial of the form $p(x) = x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m$, with $a_i \in I^i$, $m \geq 1$, such that $p(h) = 0$. The subset of R of those elements which are integral over I is an ideal called the *integral closure* of I and it is denoted by $\bar{I}$. The ideal $\bar{I}$ contains I and, if $I = \bar{I}$, we say that I is *integrally closed*.

For the sake of completeness, we state a result of Lejeune-Teissier [**8**] characterizing the integral closure of an ideal in $\mathcal{O}_n$ by an analytic condition known as the *growth condition for the integral closure*, see also [**16**, p. 326].

Theorem 2.4. *Let I be an ideal of $\mathcal{O}_n$ and $h \in \mathcal{O}_n$, then h is integral over I if and only if there exists a constant $C > 0$ and a neighborhood U of the origin in $\mathbf{C}^n$ such that*

$$|h(x)| \leq C \sup\{|g_i(x)| : i = 1, \dots, s\}, \quad \textit{for all } x \in U,$$

where $g_1, \dots, g_s$ is any generating system of I.

Remark 2.5. Given an arbitrary ideal $I \subseteq \mathcal{A}$, we denote by I_0 the ideal generated by all x^k such that $k \in \Gamma_+(I)$. It is clear that $I \subseteq I_0$. When I is a monomial ideal, then its integral closure is equal to I_0, see [**4**, p. 141]. Then for any ideal $I \subseteq \mathcal{A}$, we have that $\Gamma_+(\bar{I}) \subseteq \Gamma_+(\bar{I}_0) = \Gamma_+(I_0) = \Gamma_+(I)$ and this gives the equality $\Gamma_+(I) = \Gamma_+(\bar{I})$.

Given a pair of ideals I and J in a ring R such that $J \subseteq I$, we say that J is a *reduction* of I when there exists an integer $r > 0$ such that $I^{r+1} = JI^r$. As we shall see in the following lemma, the notions of reduction, integral closure and multiplicity in a formally equidimensional local ring (R, m) are closely related. Let $e(I)$ denote the multiplicity of an m-primary ideal I of R.

Theorem 2.6 [**11**]. *Let (R, m) be a formally equidimensional local ring, and let J and I be a pair of m-primary ideals of R such that $J \subseteq I$. Then the following conditions are equivalent*:

(1) *J is a reduction of I*;

(2) $e(I) = e(J)$;

(3) $\bar{I} = \bar{J}$.

If $I \subseteq \mathcal{A} = \mathbf{C}[[x_1, \dots, x_n]]$ has finite colength and J is a reduction of I, then $\Gamma_+(I) = \Gamma_+(J)$, as a consequence of the above theorem and Remark 2.5.

Next we give a result from [**10**, p. 153] on the existence of reductions of an ideal I in a local ring, see also [**9**, p. 112]. As we shall see, reductions of an ideal can be obtained through generic linear combinations of a generating system of I.

Theorem 2.7. *Let R be an n-dimensional local ring whose residual field is equal to $\mathbf{C}$. Let $I \subseteq R$ be an ideal of finite colength. Suppose that I is generated by $g_1, \dots, g_s$. Then there exists an analytic set $S(I) \subseteq \mathbf{C}^s \times \mathbf{C}^n$ such that, if $(a_{11}, \dots, a_{1s}, \dots, a_{n1}, \dots, a_{ns}) \in \mathbf{C}^s \times \mathbf{C}^n \setminus S(I)$ and we define $h_i = \sum_j a_{ij} g_j$, $i = 1, \dots, n$, then the ideal of R generated by $h_1, \dots, h_n$ is a reduction of I.*

Now we show a pair of propositions, the proof of which are given in

Section 3 using the notion of Newton filtration.

Proposition 2.8. *Let $I \subseteq \mathcal{A}$ be a nondegenerate ideal of finite colength generated by n elements. Then*

$$\dim_{\mathbf{C}} \frac{\mathcal{A}}{I} = n!v(I).$$

Proposition 2.9. *Let $I \subseteq \mathcal{A}$ be a nondegenerate ideal of finite codimension and let J be a reduction of I. Then J is also nondegenerate.*

Theorem 2.10. *Let I be an ideal of finite colength in $\mathcal{A}$. Then I is nondegenerate if and only if $e(I) = n!v(I)$.*

Proof. Suppose that I is nondegenerate. Let us fix a system of generators $g_1, \dots, g_s$ of I. If Δ is any compact face of $\Gamma_+(I)$, we denote by I_Δ the ideal of $\mathcal{A}_\Delta$ generated by $(g_1)_\Delta, \dots, (g_s)_\Delta$.

By Theorem 2.7, we can find an analytic subset $S \subseteq \mathbf{C}^s \times \mathbf{C}^n$ such that whenever $(a_{11}, \dots, a_{1s}, \dots, a_{n1}, \dots, a_{ns}) \in \mathbf{C}^s \times \mathbf{C}^n \setminus S$, then the ideal J generated by the series $h_1 = \sum_j a_{1j} g_j, \dots, h_n = \sum_j a_{nj} g_j$ is a reduction of I and the ideal J_Δ of $\mathcal{A}_\Delta$ generated by $(h_1)_\Delta, \dots, (h_n)_\Delta$ is a reduction of I_Δ for any compact face Δ of $\Gamma_+(I)$. We observe that, keeping the notation introduced in Theorem 2.7, the set S can be taken as the union of $S(I)$ and the subsets $S(I_\Delta)$, where Δ varies over the compact faces of $\Gamma_+(I)$.

Let $\Delta \subseteq \Gamma(I)$ be a compact face, then J_Δ has finite colength since J_Δ is a reduction of I_Δ. Therefore, J is a nondegenerate ideal generated by n elements. But this implies that $e(I) = e(J) = \dim_{\mathbf{C}} \mathcal{A}/J = n!v(J) = n!v(I)$, by Theorem 2.6, Proposition 2.8 and the fact that $\Gamma_+(I) = \Gamma_+(J)$.

Suppose that $e(I) = n!v(I)$; then by Lemma 2.3 and Theorem 2.6, the ideal I is a reduction of I_0. Then, by Proposition 2.9 we have that I is also nondegenerate. □

Given an ideal $I \subseteq \mathcal{A}$, we denote by K_I the ideal generated by all monomials x^k belonging to $\bar{I}$. The Newton polyhedron of K_I is also denoted by $C(\bar{I})$ in [**13**].

Corollary 2.11. *Let I be an ideal of finite colength in $\mathcal{A}$. Then the equality $e(I) = n!v(I)$ holds if and only if $\Gamma_+(I) = \Gamma_+(K_I)$.*

Proof. If $e(I) = n!v(I)$, then I is a reduction of I_0 by Lemma 2.3 and Theorem 2.6. Then, $\bar{I} = \bar{I}_0$ and it follows that $\Gamma_+(I) = \Gamma_+(K_I)$.

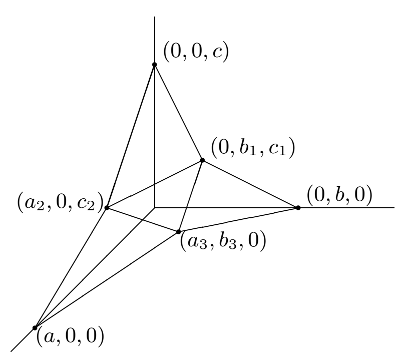
Suppose that $\Gamma_+(I) = \Gamma_+(K_I)$, then $v(I) = \mathrm{Vol}_n(\mathbf{R}^n_+ \setminus \Gamma_+(K_I))$. But we observe that, in general,

$$\begin{aligned} n!\mathrm{Vol}_n(\mathbf{R}^n_+ \setminus \Gamma_+(K_I)) &= n!v(K_I) = e(K_I) \geq e(\bar{I}) \\ &= e(I) \geq e(I_0) = n!v(I). \end{aligned}$$

Therefore, we obtain the equality $e(I) = n!v(I)$. □

From Theorem 2.10 and Corollary 2.11 we have that an ideal $I \subseteq \mathcal{A}$ of finite colength is nondegenerate if and only if $\Gamma_+(I) = \Gamma_+(K_I)$. The same equivalence was obtained by Saia in [**13**] for ideals of finite colength in the ring $\mathcal{O}_n$ of convergent power series using a different approach based on the growth condition for the integral closure, see Theorem 2.4. Therefore, we see that the membership test for integral closure of monomial ideals shown in [**3**] can obviously also be applied for the integral closure of nondegenerate ideals.

Example 2.12. Consider the ideal I of $\mathcal{A} = \mathbf{C}[[x, y, z]]$ generated by $f_1 = y^b + y^{b_1}z^{c_1} + z^c$, $f_2 = x^a + x^{a_2}z^{c_2} + z^c$, $f_3 = x^a + x^{a_3}y^{b_3} + y^b$ and $f_4 = y^{b_1}z^{c_1} + x^{a_2}z^{c_2} + x^{a_3}y^{b_3}$, where $a, b, c, a_i, b_i, c_i > 0$. We suppose that the points $(0, b_1, c_1)$, $(a_2, 0, c_2)$ and $(a_3, b_3, 0)$ do not belong to the half upper plane determined by the points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$. That is, the Newton polyhedron of I has the following form



Then, by Theorem 2.10, the multiplicity of I is equal to $e(I) = 3!v(I) = ab_3c_2 + a_3bc_1 + a_2b_1c + a_3b_1c_2 + a_2b_3c_1$.

3. The Newton filtration. As can be seen in Section 2 of [**7**], we can associate a filtration in $\mathcal{A}$ to each Newton polyhedron $\Gamma_+ \subseteq \mathbf{R}^n_+$ intersecting the coordinate axis. The family of cones $C(\Delta)$, where Δ varies in the set of compact faces of Γ_+, gives a decomposition of $\mathbf{R}^n_+$. Then we can consider the map $h : \mathbf{R}^n_+ \to \mathbf{R}_+$ which is piecewise linear on each cone $C(\Delta)$ and whose restriction to Γ is identically 1. We observe that it is possible to choose an integer $M > 0$ such that $Mh(\mathbf{N}^n) \subseteq \mathbf{N}$. We take the minimum number M with this condition and we consider the map $\phi : \mathbf{R}^n_+ \to \mathbf{R}_+$ given by $\phi = Mh$. Now for any $r \in \mathbf{N}$, we define the ideals

$$\mathcal{A}_r = \{g \in \mathcal{A} : \operatorname{supp}(g) \subseteq \phi^{-1}(r + \mathbf{N})\} \cup \{0\}.$$

The collection $\{A_r\}_{r\geq 0}$ is a filtration of $\mathcal{A}$, that is, $\mathcal{A}_r\mathcal{A}_s \subseteq \mathcal{A}_{r+s}$, for all $r, s \in \mathbf{N}$. Moreover, each ideal $\mathcal{A}_r$ is integrally closed and $\cap_{r\geq 0}\mathcal{A}_r = \{0\}$. This is known as the *Newton filtration of* $\mathcal{A}$ *associated to* Γ_+. From this filtration, we can define the map $d : \mathcal{A} \to \mathbf{N} \cup \{+\infty\}$ given by $d(g) = \max\{r \in \mathbf{N} : g \in \mathcal{A}_r\} = \min\{\phi(k) : k \in \operatorname{supp}(g)\}$ when $g \neq 0$ and $d(0) = +\infty$. Then we have that $\mathcal{A}_r = \{g \in \mathcal{A} : d(g) \geq r\}$, for all $r \in \mathbf{N}$. If no confusion arises, we shall also refer to the map $d : \mathcal{A} \to \mathbf{N} \cup \{+\infty\}$ as the Newton filtration associated to $\Gamma_+(I)$.

From this filtration we can construct the graded ring $A = \operatorname{gr}\mathcal{A} = \oplus_{r\geq 0}A_r$, where $A_r = \mathcal{A}_r/\mathcal{A}_{r+1}$, for all $r \geq 0$. We define the *Rees ring of the filtration* d as $\mathcal{R}(\lceil) = \oplus_{\nabla\in\mathbf{Z}}\mathcal{A}_\nabla\sqcup^\nabla \subseteq \mathcal{A}[\sqcup, \sqcup^{-\infty}]$ where we set $\mathcal{A}_r = \mathcal{A}$ if $r < 0$. Then we have that $A = \mathcal{R}(\lceil)/\sqcup^{-\infty}\mathcal{R}(\lceil)$.

Definition 3.1. Given a series $g \in \mathcal{A}$, we define the *initial form* of g $\operatorname{in}(g)$, as the image of g in $A = \operatorname{gr}\mathcal{A}$, that is, $\operatorname{in}(g) = g + \mathcal{A}_{d(g)+1} \in A_{d(g)}$. If I is any ideal of $\mathcal{A}$, then we define the *initial ideal* of I, denoted by $\operatorname{in}(I)$, as the ideal of A generated by all $\operatorname{in}(f)$, where $f \in I$. If $g = \sum_k a_k x^k \in \mathcal{A}$, we denote by $p(g)$ the polynomial obtained by the sum of those $a_k x^k$ such that $d(x^k) = d(g)$. This is called the *principal part of* g *with respect to* Γ_+. Obviously, we have that $\operatorname{in}(g) = \operatorname{in}(p(g))$.

If Δ is a compact face of Γ_+, then the Newton filtration associated to Γ_+ induces in a natural way a filtration in $\mathcal{A}_\Delta$. We denote by $\mathcal{R}_\Delta(\lceil)$

the Rees ring associated to this filtration and by A_Δ the graded ring $\mathcal{R}_\Delta(\lceil)/\sqcup^{-\infty}\mathcal{R}_\Delta(\lceil)$. Now let $\pi_\Delta : A \to A_\Delta$ be the epimorphism defined by $\pi_\Delta(f+\mathcal{A}_r) = f_{C(\Delta)} + (\mathcal{A}_\Delta)_r$, where $(\mathcal{A}_\Delta)_r = \mathcal{A}_\Delta \cap \mathcal{A}_r$, for all $r \geq 0$, and $f_{C(\Delta)}$ is the sum of all terms $a_k x^k$ such that $k \in \operatorname{supp}(f) \cap C(\Delta)$ whenever $f = \sum_k a_k x^k$.

Definition 3.2. If $g \in \mathcal{A}$, we call the *initial form of g over Δ* to $\pi_\Delta(\operatorname{in}(g))$ and we denote this element of A_Δ by $\operatorname{in}_\Delta(g)$. If $I \subseteq \mathcal{A}$ is any ideal, the *initial ideal of I over Δ* is the ideal of A_Δ generated by all $\operatorname{in}_\Delta(g)$ where $g \in I$. This ideal is denoted by $\operatorname{in}_\Delta(I)$.

Remark 3.3. By the construction of the filtration, it follows that $d(x^\alpha x^\beta) \geq d(x^\alpha)+d(x^\beta)$ for any monomials x^α and x^β, and the equality holds if and only if there exists a compact fact $\Delta \subseteq \Gamma_+$ such that $\alpha, \beta \in C(\Delta)$. This fact implies that either $\operatorname{in}(f)\operatorname{in}(g) = \operatorname{in}(fg)$ or $\operatorname{in}(f)\operatorname{in}(g) = 0$, for every $f, g \in \mathcal{A}$, and that the last occurs if and only if there exists no compact face $\Delta \subseteq \Gamma_+$ such that $\operatorname{supp}(p(f)) \cap C(\Delta) \neq \varnothing$ and $\operatorname{supp}(p(g)) \cap C(\Delta) \neq \varnothing$.

On the other hand, it is not true that the ideal $\operatorname{in}_\Delta(I)$ is generated by the initial forms over Δ of a generating system of I.

Proposition 3.4. *Let I be an ideal of $\mathcal{A}$, and let $g_1, \dots, g_s$ be a system of generators of I. Let Δ be a compact face of $\Gamma_+(I)$. Then the ideal of $\mathcal{A}_\Delta$ generated by $(g_1)_\Delta, \dots, (g_s)_\Delta$ has finite colength if and only if the ideal generated by $\operatorname{in}_\Delta(g_1), \dots, \operatorname{in}_\Delta(g_s)$ in A_Δ has finite colength.*

Proof. Let M be the value of the Newton filtration induced by $\Gamma_+(I)$ along the monomials with exponents in the boundary $\Gamma(I)$. Suppose that Δ is a compact fact of $\Gamma(I)$ and that the ideal I_Δ of $\mathcal{A}_\Delta$ generated by $(g_1)_\Delta, \dots, (g_s)_\Delta$ has finite colength in $\mathcal{A}_\Delta$. Let $r = \dim \mathcal{A}_\Delta$, since the ring $\mathcal{A}_\Delta$ is local with residual field equal to $\mathbf{C}$, there is a sequence $h_1, \dots, h_r$ of linear combinations of $(g_1)_\Delta, \dots, (g_s)_\Delta$ that is a system of parameters in $\mathcal{A}_\Delta$. Moreover, by [**6**], we know that the ring $\mathcal{A}_\Delta$ is Cohen-Macaulay, then $h_1, \dots, h_r$ is a regular sequence in $\mathcal{A}_\Delta$. In particular, $h_1 t^M, \dots, h_r t^M$ is a regular sequence in $\mathcal{R}_\Delta(d)$, which implies that $\operatorname{in}_\Delta(h_1), \dots, \operatorname{in}_\Delta(h_r)$ is a regular sequence in $A_\Delta = \mathcal{R}_\Delta(d)/t^{-1}\mathcal{R}_\Delta(d)$. Since the ring

A_Δ is also Cohen-Macaulay of dimension r, it follows that this sequence must generate an ideal of finite colength. We have defined the elements h_i as linear combinations of $(g_1)_\Delta, \dots, (g_s)_\Delta$, thus $\{\mathrm{in}_\Delta(h_1), \dots, \mathrm{in}_\Delta(h_r)\}A_\Delta \subseteq \{\mathrm{in}_\Delta(g_1), \dots, \mathrm{in}_\Delta(g_s)\}A_\Delta$, and this implies that the ideal $\{\mathrm{in}_\Delta(g_1), \dots, \mathrm{in}_\Delta(g_s)\}A_\Delta$ has also finite colength in A_Δ.

Suppose that the ideal in A_Δ generated by $\mathrm{in}_\Delta(g_1), \dots, \mathrm{in}_\Delta(g_s)$ has finite colength. As above, consider a system of parameters $H_1, \dots, H_r$ of A_Δ obtained as linear combinations of the initial forms $\mathrm{in}_\Delta(g_1), \dots, \mathrm{in}_\Delta(g_s)$. Suppose that $H_i = \sum_j a_{ij}\mathrm{in}_\Delta(g_j)$ where $a_{ij} \in \mathbf{C}$ for all $i = 1, \dots, r$. Since A_Δ is an r-dimensional Cohen-Macaulay ring, we find that $H_1, \dots, H_r$ is a regular sequence. In particular, the Koszul complex associated to these elements is acyclic. Then we can apply Theorem 4.1 of [**7**] to say that the graded rings $\mathrm{gr}\,(\mathcal{A}_\Delta/\langle h_1, \dots, h_r\rangle)$ and $A_\Delta/\langle H_1, \dots, H_r\rangle$ are isomorphic as graded complex vector spaces, where $h_i = \sum_j a_{ij}(g_j)_\Delta$, $i = 1, \dots, r$. Then the ideal in $\mathcal{A}_\Delta$ generated by $h_1, \dots, h_r$ has finite colength in $\mathcal{A}_\Delta$ and I_Δ has also finite colength. $\square$

Proof of Proposition 2.8. Let $g_1, \dots, g_n$ be a nondegenerate system of generators of I. Let us consider the Koszul complex associated to $g_1, \dots, g_n$

$$0 \longrightarrow \mathcal{A}^{\binom{n}{n}} \longrightarrow \mathcal{A}^{\binom{n}{n-1}} \longrightarrow \cdots \longrightarrow \mathcal{A}^{\binom{n}{1}} \longrightarrow \mathcal{A}. \tag{1}$$

Let A be the graded ring associated to the filtration d induced by $\Gamma_+(I)$. The above complex induces the graded complex of free A-modules

$$0 \longrightarrow A^{\binom{n}{n}} \longrightarrow A^{\binom{n}{n-1}} \longrightarrow \cdots \longrightarrow A^{\binom{n}{1}} \longrightarrow A. \tag{2}$$

We claim that, under the hypothesis of nondegeneracy, the complex (2) is exact. To see this we observe that, from [**7**, Proposition 2.6], there exists an exact graded complex with grade preserving morphisms of the form

$$0 \longrightarrow A \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0, \tag{3}$$

where C_l is the direct sum of those A_Δ such that Δ is a compact l-dimensional face of $\Gamma_+(I)$, for all $l = 0, 1, \dots, n-1$.

Given a compact face Δ of $\Gamma_+(I)$, we can consider the Koszul complex $\mathcal{K}_\Delta$ with coefficients in A_Δ of $\mathrm{in}_\Delta(g_1), \ldots, \mathrm{in}_\Delta(g_n)$. If we take the direct sum of the complexes $\mathcal{K}_\Delta$ over all compact l-dimensional faces of $\Gamma_+(I)$, we obtain the complex

$$0 \longrightarrow C_l^{\binom{n}{n}} \longrightarrow C_l^{\binom{n}{n-1}} \longrightarrow \cdots \longrightarrow C_l^{\binom{n}{1}} \longrightarrow C_l, \tag{4}$$

for all $l = 0, 1, \ldots, n-1$. Then if we look at the diagram in [**7**, p. 15], we find that the complex (2) is exact if the above complexes are exact in dimensions $\geq n-l$, for all $l = 0, 1, \ldots, n-1$, by a simple diagram chase argument. But this is equivalent to say that the complex $\mathcal{K}_\Delta$ is exact in dimensions $\geq n-l$ for all l-dimensional faces Δ and all $l = 0, 1, \ldots, n-1$.

Let $l \in \{0, 1, \ldots, n-1\}$, and let Δ be a compact l-dimensional face of $\Gamma_+(I)$. Then the ideal of A_Δ generated by $\mathrm{in}_\Delta(g_1), \ldots, \mathrm{in}_\Delta(g_n)$ has finite colength, by Proposition 3.4. This implies that the depth in A_Δ of this ideal is equal to $l+1$, since A_Δ is Cohen-Macaulay [**6**] of dimension $l+1$. Now by [**9**, Theorem 16.8], also known as the rigidity property of the Koszul complex, the homology of $\mathcal{K}_\Delta$ is zero in dimensions $\geq n-l$. Therefore, the complex (2) is exact.

Let I' be the ideal of A generated by $\mathrm{in}\,(g_1), \ldots, \mathrm{in}\,(g_n)$. By the exactness of (2) we have that the Hilbert series of A/I' can be expressed as $H(t) = (1-t^M)^n H_A(t)$, where $H_A(t)$ is the Hilbert series of A and M is the value of d along the monomials with exponents in the boundary $\Gamma(I)$. Then if we now use [**7**, Lemma 2.9] and [**7**, Theorem 4.1], we find that

$$\dim_{\mathbf{C}} \frac{\mathcal{A}}{I} = \dim_{\mathbf{C}} \frac{A}{I'} = \lim_{t \to 1} (1-t^M)^n H_A(t) = n! \mathrm{Vol}_n(\Gamma_-(I)) = n! v(I). \quad \square$$

Proof of Proposition 2.9. Let us consider the Newton filtration induced by $\Gamma_+(I)$. We fix a nondegenerate system of generators $g_1, \ldots, g_s$ of I. Let J be a reduction of I; then there exists an $r > 0$ such that $I^{r+1} = JI^r$. In particular, we find that $\mathrm{in}\,(I)^{r+1} \subseteq \mathrm{in}\,(I^{r+1}) \subseteq \mathrm{in}\,(J)$. But it is obvious that

$$(\{\mathrm{in}\,(g_1), \ldots, \mathrm{in}\,(g_s)\}A)^{r+1} \subseteq \mathrm{in}\,(I)^{r+1}.$$

Then if Δ is any compact face of $\Gamma_+(I)$ and we apply π_Δ to the above inclusion, we have that $\mathrm{in}_\Delta(J)$ contains the ideal $(\{\mathrm{in}_\Delta(g_1), \dots, \mathrm{in}_\Delta(g_s)\} A_\Delta)^{r+1}$ and therefore $\mathrm{in}_\Delta(J)$ has finite colength, by Proposition 3.4.

Since the graded ring A associated to the Newton filtration is a Noetherian ring, there exist $h_1, \dots, h_l \in J$ such that its initial forms $\mathrm{in}\,(h_1), \dots, \mathrm{in}\,(h_l)$ generate the ideal $\mathrm{in}\,(J)$. Then we have that $h_1, \dots, h_l$ is a generating system of J by [**4**, Proposition 7.12]. To apply this proposition the ring $\mathcal{A}$ must be complete with respect to the filtration induced by $\Gamma_+(I)$. But, as can be seen in [**12**, p. 76], the completeness of $\mathcal{A}$ with respect to a filtration only depends on the radical of the filtration (the common radical of each piece of the filtration), which in this case is equal to the maximal ideal.

Then, given any compact face Δ of $\Gamma_+(I)$, if we apply the morphism π_Δ to the equality $\mathrm{in}\,(J) = \{\mathrm{in}\,(h_1), \dots, \mathrm{in}\,(h_l)\}A$, we obtain that $\mathrm{in}_\Delta(J) = \{\mathrm{in}_\Delta(h_1), \dots, \mathrm{in}_\Delta(h_l)\}A_\Delta$. But, as we have seen before, the ideal $\mathrm{in}_\Delta(J)$ has finite colength in A_Δ. Then $h_1, \dots, h_l$ is a nondegenerate system of generators of J by Proposition 3.4. □

4. Ideals of finite relative colength. In this section we take advantage of the technique shown in the previous sections to characterize the class of nondegenerate ideals of *finite relative colength*, which is a wide class of ideals including the ideals of finite colength of $\mathcal{A}$. We recall that, if I is an ideal of $\mathcal{A}$, then $\Gamma(I)$ is the union of all compact faces of the Newton polyhedron $\Gamma_+(I)$.

Definition 4.1. Let I be an ideal of $\mathcal{A}$, we set

$$\begin{aligned} \Lambda_+(I) &= \{\lambda k : \lambda \geq 0, k \in \Gamma(I)\} \subseteq \mathbf{R}^n_+; \\ \mathcal{A}_I &= \{h \in \mathcal{A} : \mathrm{supp}\,(h) \subseteq \Lambda_+(I)\}; \\ c(I) &= \max\{\dim \Delta : \Delta \text{ is a compact face of } \Gamma(I)\}. \end{aligned}$$

We observe that, in general, the set $\Lambda_+(I)$ is not convex in $\mathbf{R}^n_+$. Nevertheless, it is not the case when $n = 2$. If $\Lambda_+(I)$ is convex, then $\Lambda_+(I) \cap \mathbf{Z}^n_+$ is a sub-semigroup of $\mathbf{Z}^n_+$ and $\mathcal{A}_I$ becomes a subring of $\mathcal{A}$. In fact, $\mathcal{A}_I$ is a local ring of dimension $c(I) + 1$ in this case. We say that I has *finite relative colength* when $\Lambda_+(I)$ is convex and I admits a generating system $g_1, \dots, g_s$ such that $\mathrm{supp}\,(g_1) \cup \cdots \cup \mathrm{supp}\,(g_s) \subseteq$

$\Lambda_+(I)$, and the ideal in $\mathcal{A}_I$ generated by $g_1, \dots, g_s$ has finite colength in $\mathcal{A}_I$.

We observe that $\Gamma_-(I)$ is contained in an affine subspace of dimension $c(I)+1$. Then we can consider the $(c(I)+1)$-dimensional volume of $\Gamma_-(I)$; we denote this volume also by $v(I)$. When I is an ideal of $\mathcal{A}$ of finite colength, then I has finite relative colength since the Newton polyhedron of I intersects all the coordinate axes and consequently $\mathcal{A}_I = \mathbf{C}[[x_1, \dots, x_n]]$ in this case.

If $I \subseteq \mathcal{A}$ is an ideal of finite relative colength, we can consider I as an ideal of $\mathcal{A}_I$. Therefore, it makes sense to compute the multiplicity of I in $\mathcal{A}_I$ since $\mathcal{A}_I$ is a local ring. We shall denote this multiplicity by $e(I, \mathcal{A}_I)$. In the sequel, we will denote by I_1 the ideal of $\mathcal{A}_I$ generated by all the monomials x^k, with $k \in \Lambda_+(I) \cap \mathbf{Z}_+^n$ (compare with the definition of I_0 in Remark 2.5).

By analogy with Section 3, we can also consider a filtration on $\mathcal{A}_I$ associated to the Newton polyhedron $\Gamma_+(I)$. We use this tool to prove the following results.

Proposition 4.2. *Let $I \subseteq \mathcal{A}$ be a nondegenerate ideal of finite relative codimension. If $J \subseteq \mathcal{A}$ is a reduction of I, then J is also nondegenerate.*

Proof. Is analogous to the proof of Proposition 2.9. □

Theorem 4.3. *Let $I \subseteq \mathcal{A}$ be an ideal of finite relative colength. Then I is nondegenerate if and only if $e(I, \mathcal{A}_I) = (c(I)+1)!v(I)$.*

Proof. Let $r = c(I)+1$ and suppose that $g_1, \dots, g_r$ is a nondegenerate system of generators of I such that $\operatorname{supp}(g_1) \cup \cdots \cup \operatorname{supp}(g_r) \subseteq \Lambda_+(I)$. Let $d : \mathcal{A}_I \to \mathbf{N} \cup \{+\infty\}$ be the filtration in $\mathcal{A}_I$ induced by $\Gamma_+(I)$. Then, following the same argument as in the proof of Theorem 2.8, we can show the acyclicity of the Koszul complex associated to the initial forms of $g_1, \dots, g_r$ with respect to d. Then we apply [**7**, Lemma 2.9] and [**7**, Theorem 4.1] to state that

$$e(\{g_1, \dots, g_r\}\mathcal{A}_I, \mathcal{A}_I) = \dim_{\mathbf{C}} \frac{\mathcal{A}_I}{\{g_1, \dots, g_r\}\mathcal{A}_I} = r!v(I),$$

where the first equality is a consequence of the fact that $\mathcal{A}_I$ is Cohen-Macaulay [**6**].

Suppose that $g_1, \dots, g_s$ is a nondegenerate system of generators of I where $s \geq r$. Then, following the idea of the proof of Theorem 2.10, we can take a reduction J of $\{g_1, \dots, g_s\}\mathcal{A}_I$ in $\mathcal{A}_I$ generated by a nondegenerate system of r elements and constructed by linear combinations of $g_1, \dots, g_s$. From the monomorphism $\mathcal{A}_I \hookrightarrow \mathcal{A}$, we observe that $J\mathcal{A}$ must also be a reduction of I in $\mathcal{A}$. Thus, by Remark 2.5 we have that $\Lambda_+(I) = \Lambda_+(J)$ and $v(I) = v(J)$.

By the fact that J is a reduction of $\{g_1, \dots, g_s\}\mathcal{A}_I$ in $\mathcal{A}_I$, we have that $e(J, \mathcal{A}_I) = e(\{g_1, \dots, g_s\}\mathcal{A}_I, \mathcal{A}_I)$. But

$$e(J, \mathcal{A}_I) = r!v(J) = r!v(I),$$

by the previous step.

Now we show the converse. Since the ideal $I_1 \subseteq \mathcal{A}_I$ is nondegenerate and of finite relative codimension, we can use the previous implication to state that $e(I_1, \mathcal{A}_I) = r!v(I_1) = r!v(I)$. If we suppose that $e(\{g_1, \dots, g_s\}\mathcal{A}_I, \mathcal{A}_I) = r!v(I)$, then $\{g_1, \dots, g_s\}\mathcal{A}_I$ is a reduction of I_1 in $\mathcal{A}_1$ by Theorem 2.6. Therefore, I is a reduction of I_1 and I is nondegenerate by Proposition 4.2. □

Remark 4.4. By an analogous method used in the proof of Corollary 2.11, we can apply the previous theorem to say that an ideal $I \subseteq \mathcal{A}$ of finite relative codimension is nondegenerate if and only if $\bar{I}$ is generated by those monomials x^k with $k \in \Gamma_+(I)$.

5. An effective method to compute the multiplicity of a monomial ideal. As we shall see in this section, it is possible to canonically construct a minimal reduction J_I of any given monomial ideal $I \subseteq \mathcal{A}$ of finite colength. Then we give a solution for the problem of effectively computing the multiplicity of any monomial ideal $I \subseteq \mathcal{A}$ of finite colength. We also refer the reader to [**3**] for a completely different approach to the problem of computing the multiplicity of a monomial ideal.

Theorem 5.1. *Let $I = \langle x^{k_1}, \dots, x^{k_s} \rangle \subseteq \mathcal{A}$ be a monomial ideal of finite colength and consider the polynomial $f_I = x^{k_1} + \cdots + x^{k_s}$. Let*

J_I denote the ideal of $\mathcal{A}$ generated by $x_1(\partial f_I/\partial x_1), \dots, x_n(\partial f_I/\partial x_n)$. Then

$$e(I) = \dim_{\mathbf{C}} \frac{\mathcal{A}}{J_I}.$$

Proof. By virtue of Theorem 2.6 we have to prove that $\bar{J}_I = \bar{I}$. We observe that $\Gamma_+(J_I) = \Gamma_+(I)$. Then the problem reduces to prove that the ideal J_I is Newton nondegenerate, see Theorem 2.10 and Corollary 2.11. Since the integral closure of I is equal to the integral closure of the ideal generated by the vertexes of $\Gamma_+(I)$, we can assume that $\{k_1, \dots, k_s\}$ is equal to the set of vertexes of $\Gamma_+(I)$.

Let Δ be a face of $\Gamma_+(I)$, suppose that Δ is equal to the convex hull of $\{k_1, \dots, k_r\}$ for some $r \leq s$. We can attach to Δ the structure of a simplicial complex $\mathcal{S}$ such that the set of vertexes of $\mathcal{S}$ coincides with $\{k_1, \dots, k_r\}$, see Caratheodory's theorem in [**4**, p. 139] for details. Therefore, we can suppose that $k_1, \dots, k_r$ are linearly independent, in particular $r = \dim \Delta + 1$. In order to see that the ideal of $\mathcal{A}_\Delta$ generated by $(x_1(\partial f_I/\partial x_1))_\Delta, \dots, (x_n(\partial f_I/\partial x_n))_\Delta$ has finite codimension in $\mathcal{A}_\Delta$ it is enough to check, by [**7**, Theorem 6.2], that

$$\left\{x \in \mathbf{C}^n : \left(x_1 \frac{\partial f_I}{\partial x_1}\right)_{\Delta'}(x) = \cdots = \left(x_n \frac{\partial f_I}{\partial x_n}\right)_{\Delta'}(x) = 0\right\} \subseteq \{x \in \mathbf{C}^n : x_1 \cdots x_n = 0\},$$

for each $\Delta' \in \mathcal{S}$. We observe that

$$\left(x_i \frac{\partial f_I}{\partial x_i}\right)_\Delta = x_i \left(\frac{\partial (f_I)_\Delta}{\partial x_i}\right),$$

for any $i = 1, \dots, n$. Therefore, if we write $k_i = (k_i^1, \dots, k_i^n)$ for $i = 1, \dots, r$, the above system of equations is translated into

$$\begin{aligned} k_1^1 x^{k_1} + \cdots + k_r^1 x^{k_r} &= 0 \\ &\vdots \\ k_1^n x^{k_1} + \cdots + k_r^n x^{k_r} &= 0. \end{aligned}$$

Since $k_1, \dots, k_r$ are linearly independent, then $x^{k_i} = 0$ for all $i \in \{1, \dots, r\}$. But this means that some coordinate of x must be zero. $\square$

Example 5.2. Let $I \subseteq \mathbf{C}[[x_1, \dots, x_9]]$ be the ideal generated by the monomials $x_1^2, x_2^2, x_3^2, x_4^3, x_5^3, x_6^3, x_7^4, x_8^4, x_9^4, x_5x_6, x_7x_8x_9$. Then using Theorem 5.1 and, for instance, the program *Singular* [**5**] for the computation of $\dim_{\mathbf{C}} \mathcal{A}/J_I$, we find that $e(I) = 6912$.

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