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CONTINUED FRACTIONS AND RESTRAINED SEQUENCES OF MÖBIUS MAPS

A.F. BEARDON AND L. LORENTZEN

ABSTRACT. Modified approximants of a continued fraction are designed to increase the rate of convergence, and these led to the notion of restrained sequences of Möbius transformations. Here we give some analytic and geometric characterizations of restrained sequences and related topics. We also give an expository account of the use of geometry, including hyperbolic geometry, in discussing restrained sequences and continued fractions.

1. Introduction. Originally a continued fraction was considered to be an expression of the form

$$\mathbf{K}(a_n|b_n) = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}} \tag{1.1}$$

where $a_n \neq 0$ for every n. This continued fraction was said to be convergent to $\mathbf{K}$ if the sequence of truncated (finite) continued fractions converges to $\mathbf{K}$. It is well known that one can write continued fractions in terms of Möbius transformations, and this is the point of view we take here. Indeed, we define a *continued fraction* to be a *sequence* S_n *of Möbius transformations of the form*

$$S_n = s_1 \circ \cdots \circ s_n, \quad s_n(z) = \frac{a_n}{b_n + z}, \tag{1.2}$$

where $a_n \neq 0$ for every n. The classical concept of convergence is then expressed by saying that the value of the continued fraction (1.1) is the limit (when it exists) of the sequence $S_n(0)$. We denote the complex plane by $\mathbf{C}$, and the extended complex plane by $\mathbf{C}_\infty$. Throughout this paper we shall regard Möbius transformations as acting on $\mathbf{C}_\infty$, and we reserve the notation a_n, b_n, s_n and S_n for the maps in (1.2). Note that

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because the a_n are nonzero, each s_n and each S_n is a (nonconstant) Möbius transformation.

In 1957 Piranian and Thron proved the following result [**9**] and expressed the hope that this might be useful in continued fraction theory.

Theorem A. *Suppose that a sequence T_n of Möbius transformations converges to the same value, say α, at two distinct points of $\mathbf{C}_\infty$. Then with the possible exception of one particular value of z, if $T_n(z)$ converges then it converges to α.*

For the general convergent continued fraction (1.1) we have $S_n(\infty) = S_{n-1}(0)$ so that Theorem A implies that with possibly one exceptional value of z, if $S_n(z)$ *converges*, then it *converges to* $\lim_{n\to\infty} S_n(0)$. At first sight this result seems to justify the definition of convergence, but despite this there is a very good reason why it should be modified. The problem is that the sequence $S_n(z)$ may converge (to a constant value) for many z, but perhaps not when z is 0 or ∞. For example, consider any continued fraction in which the sequence $s_1, s_2, \dots$ has period 3 and in which $s_1 \circ s_2 \circ s_3$ is a loxodromic transformation g. Let α and β be the attracting and repelling fixed point, respectively, of g. Then the sequence $g^n(z)$ of iterates of g converges to α on $\mathbf{C}_\infty \setminus \{\beta\}$, and it follows from this that S_n converges locally uniformly to α on the complement of the finite set $A = \{\beta, s_1^{-1}(\beta), (s_1 \circ s_2)^{-1}(\beta)\}$. It seems beyond question that in this case one should assign the value α to the continued fraction regardless of whether or not 0, or equivalently, ∞, lies in A. For an explicit example of such a continued fraction, see Example 1.1 in [**6**].

There have been several attempts to obtain a satisfactory definition of convergence of continued fractions, and for an account of the evolution of the definition we refer the reader to [**11**], where the following definition was proposed. The continued fraction (1.1) is said to *converge strongly* to a value α in $\mathbf{C}$ if

(i) there are two distinct points u and v in $\mathbf{C}_\infty$ such that

$$\lim_{n\to\infty} S_n(u) = \lim_{n\to\infty} S_n(v) = \alpha,$$

and

(ii) the set $\{S_n^{-1}(\infty) : n = 1, 2, 3, \dots\}$ is not dense in $\mathbf{C}_\infty$.

Theorem A shows that in these circumstances, and possibly with one exceptional value of z, $\lim_n S_n(z) = \alpha$ whenever this limit exists. The following result, [**7**, Theorem 2.6] and [**11**], shows exactly where this limit does exist and also why (ii) is present in this definition.

Theorem B. *Suppose that $\mathbf{K}(a_n|b_n)$ converges strongly to the complex number α, and let L be the set of limit points, in $\mathbf{C}_\infty$, of the sequence $S_n^{-1}(\infty)$. Then $S_n \to \alpha$ on the complement of L.*

Although ∞ is given a special role here, we shall see shortly that it can be replaced by any point other than α. As the definition of convergence evolved it became clear that it is useful to consider the limiting behavior of the so-called *modified approximants* $S_n(w_n)$ where the w_n are chosen to suit a given continued fraction (for example, to accelerate its convergence). However, any definition involving the convergence $S_n(w_n)$ has to overcome the fact that for any given continued fraction the choice of w_n is often not at all obvious and that a particularly bad choice can yield any preassigned limiting behavior of $S_n(w_n)$ whatever. This difficulty was overcome by Jacobsen (now Lorentzen) in 1986 in [**6**] in the following way. First, to remove the special role of ∞ we use the chordal metric

$$\sigma(u, v) = \frac{2|u - v|}{\sqrt{1 + |u|^2}\sqrt{1 + |v|^2}}$$

on $\mathbf{C}_\infty$. We then have the following definition.

Definition 1.1 [**7**, p. 480]. The continued fraction $\mathbf{K}(a_n|b_n)$ in (1.1) *converges generally* or is *generally convergent*, to a value α in $\mathbf{C}_\infty$ if there exist two sequences u_n and v_n in $\mathbf{C}_\infty$, such that

$$(1.3) \quad \lim_{n\to\infty} S_n(u_n) = \lim_{n\to\infty} S_n(v_n) = \alpha \quad \text{and} \quad \liminf_{n\to\infty} \sigma(u_n, v_n) > 0.$$

Note that this definition no longer gives ∞ a special role. Jacobsen then verified the first crucial step in any limiting process, namely that $\mathbf{K}(a_n|b_n)$ converges generally to *at most one value* [**6**, Theorem 3.3];

this can be proved using the standard recurrence relations in continued fraction theory, as in [**6**], or by using cross ratios. Later Jacobsen and Thron extended the idea of a generally convergent continued fraction to an arbitrary sequence of Möbius maps [**7**] and gave the following natural generalization of Definition 1.1.

Definition 1.2 [**7**, p. 142]. A sequence T_n of Möbius maps is *restrained* if there are sequences u_n and v_n in $\mathbf{C}_\infty$ such that

$$\lim_{n\to\infty} \sigma(T_n(u_n), T_n(v_n)) = 0 \quad \text{and} \quad \liminf_{n\to\infty} \sigma(u_n, v_n) > 0.$$

It is known that if $\mathbf{K}(a_n|b_n)$ converges in the classical sense then it converges strongly, and if $\mathbf{K}(a_n|b_n)$ converges strongly then it converges generally (in each case, to the same value). Moreover, if the continued fraction $\mathbf{K}(a_n|b_n)$ is (generally, strongly, or classically) convergent, then the sequence S_n is restrained so that any result on restrained sequences is automatically applicable to convergent continued fractions. The main reason for the interest in restrained sequences was that in this case, S_n^{-1} is also a restrained sequence.

In order to explore further what it means for a sequence T_n to be restrained, Jacobsen and Thron went on to consider what they called admissible sequences and strongly exceptional sequences. Here we use a slightly different terminology, but our definitions agree with those in [**7**]. Given two sequences z_n and w_n, we shall say that z_n and w_n are

(i) *asymptotic* if $\lim_{n\to\infty} \sigma(z_n, w_n) = 0$,

and

(ii) *separated* if $\liminf_{n\to\infty} \sigma(z_n, w_n) > 0$.

In this terminology, T_n is restrained if and only if there are sequences u_n and v_n that are separated but that $T_n(u_n)$ and $T_n(v_n)$ are asymptotic. There are two other definitions in [**7**] that are relevant in this discussion, and these are as follows.

(iii) A sequence u_n, in $\mathbf{C}_\infty$, is T_n-*admissible* if there is a sequence v_n such that u_n and v_n are separated, and $T_n(u_n)$ and $T_n(v_n)$ are asymptotic (notice that T_n is restrained if and only if there exists a T_n-admissible sequence).

(iv) A sequence z_n is T_n-*strongly exceptional* if there is an admissible sequence u_n such that $T_n(z_n)$ and $T_n(u_n)$ are separated (it is proved in Theorem 2.1 [**7**] that this property does not depend on the choice of the u_n).

Jacobsen and Thron observed that the sequence T_n is restrained if and only if T_n^{-1} is restrained [**7**, Theorem 2.6], and it follows from this that if T_n converges generally to a value α, then the set of limit points of $T_n^{-1}(w)$ is independent of w in $\mathbf{C}_\infty \setminus \{\alpha\}$. These ideas provide the following generalization of Theorem B.

Theorem C. *A sequence T_n is restrained if and only if there exists a sequence z_n such that $\sigma(T_n(u_n), T_n(v_n)) \to 0$ whenever the sequences u_n and v_n are separated from the sequence z_n.*

Our aim in this paper is to study *restrained sequences of Möbius maps, admissible sequences, and strongly exceptional sequences.* The three definitions of convergence of a continued fraction, the definition of a restrained sequence, and the definitions of admissible and strongly exceptional sequences in [**7**] are all based on a comparison of the two chordal distances $\sigma(u_n, v_n)$ and $\sigma(T_n(u_n), T_n(v_n))$ for certain sequences u_n and v_n. Here we shall obtain new, and more detailed, results by replacing the consideration of sequences with a study of the chordal distortion of a Möbius map. Explicitly, we shall study the maximum and minimum values of the function

$$(u, v) \longmapsto \frac{\sigma(T(u), T(v))}{\sigma(u, v)},$$

where T is a Möbius transformation, on the compact product space $\mathbf{C}_\infty \times \mathbf{C}_\infty$, where this is defined when $u = v$ as the chordal distortion

$$T^{\#}(v) = \lim_{u \to v} \frac{\sigma(T(u), T(v))}{\sigma(u, v)}.$$

We now introduce the functions $\delta(T)$ and $\Delta(T)$ by

$$\delta(T) = \inf_{u \neq v} \frac{\sigma(T(u), T(v))}{\sigma(u, v)}, \quad \Delta(T) = \sup_{u \neq v} \frac{\sigma(T(u), T(v))}{\sigma(u, v)}. \tag{1.4}$$

It is easy to see that $\Delta(T)$ is finite; thus, T is a Lipschitz map of $(\mathbf{C}_\infty, \sigma)$ onto itself with Lipschitz constant $\Delta(T)$. Later we shall obtain a lot more information about the functions $\delta(T)$ and $\Delta(T)$, and our proofs will be based on this information (which controls the geometrical action of T).

Next, for any Möbius map

$$T(z) = \frac{az+b}{cz+d}, \quad ad-bc=1, \tag{1.5}$$

we define the *norm* $\|T\|$ of T to be the norm of an associated unimodular matrix for T; thus,

$$\|T\| = \sqrt{|a|^2+|b|^2+|c|^2+|d|^2} \tag{1.6}$$

(it is essential to have $ad-bc=1$ here). Again there is much to be said about how the norm $\|T\|$ constrains the geometrical action of T; for example, the conditions (i) T is unitary, (ii) T is a chordal isometry and (iii) $\|T\|=2$ are equivalent to each other. Later we shall discuss these ideas in detail.

It will be convenient for us to use $\mu(E)$ to denote the chordal area of a subset E of $\mathbf{C}_\infty$, so that $\mu(E) \le \mu(\mathbf{C}_\infty) = 4\pi$. By a *chordal disc* we mean a disc in the metric space $(\mathbf{C}_\infty, \sigma)$ and the set of these discs coincides with the set of Euclidean discs and half-planes in $\mathbf{C}_\infty$.

Finally when we speak of the convergence of Möbius maps, say $T_n \to T$, we always mean that $T_n \to T$ *uniformly on* $\mathbf{C}_\infty$ *with respect to the chordal metric*. It is well known that $T_n \to T$ in this sense if and only if T_n converges pointwise at three distinct points to three distinct values, and also if and only if there exists a choice of matrices, say A_n for T_n and A for T such that $A_n \to A$ in the usual sense.

We now have all we need to state our main result (although we shall need much more for its proof). After they introduced the notion of restrained sequences, the authors of [**7**] remark, at the end of [**7**], that "it is difficult to state a simple necessary condition for T_n to be restrained." We shall now characterize restrained sequences in a variety of ways, and without the use of admissible or strongly exceptional sequences.

Theorem 1.3. *For any sequence T_n of Möbius maps the following are equivalent*:

(1) T_n *is restrained*;

(2) $\delta(T_n) \to 0$ *as* $n \to \infty$;

(3) $\Delta(T_n) \to +\infty$ *as* $n \to \infty$;

(4) $\|T_n\| \to +\infty$ *as* $n \to \infty$;

(5) *there is no subsequence of* T_n *that converges to a Möbius map on* $\mathbf{C}_\infty$;

(6) *there exist unitary Möbius transformations* A_i *and* B_j, $i, j = 1, 2, \ldots,$ *and a sequence* λ_n *where* $\lambda_n \to 0$ *such that for all* n, $A_n T_n B_n(z) = \lambda_n z$;

(7) *there exists a sequence* $\mathcal{D}_n$ *of chordal discs such that* $\mu(\mathcal{D}_n) \to 4\pi$ *and* $\mu(T_n(\mathcal{D}_n)) \to 0$;

(8) *there exists a sequence* $\mathcal{D}'_n$ *of chordal discs such that* $\mu(\mathcal{D}'_n) \to 0$ *and* $\mu(T_n(\mathcal{D}'_n)) \to 4\pi$;

(9) *there exists a sequence* $\mathcal{D}''_n$ *of chordal discs that* $\mu(\mathcal{D}''_n) \to 4\pi$ *and*

$$\sup\{T_n^{\#}(z) : z \in \mathcal{D}''_n\} \longrightarrow 0.$$

Theorem 1.3 gives us many ways of looking at restrained sequences of Möbius maps without any reference to admissible or strongly exceptional sequences, although we shall return to the relationship between these ideas in Sections 5 and 6. Theorem 1.3 gives us a clear, informal description of the behavior of a restrained sequence T_n. As $\|T_n\|$ increases, an increasingly large proportion $\mathcal{D}_n$ of the Riemann sphere $\mathbf{C}_\infty$ is mapped, and 'collapsed,' by the T_n into a decreasingly small proportion of the sphere and, by taking complements, an increasingly small part is expanded onto an increasingly large part. This behavior characterizes restrained sequences of Möbius maps; indeed, up to rotations of the Riemann sphere (before and after applying T_n), a restrained sequence is nothing more than a sequence of maps $z \mapsto \lambda_n z$, where $\lambda_n \to 0$. We also note that the fact that T_n is restrained if and only if T_n^{-1} is restrained is a direct consequence of Theorem 1.3 because $\|T_n^{-1}\| = \|T_n\|$. Moreover, as $\|ST\| \leq \|S\|\|T\|$, it is evident that the T_n is restrained if and only if any sequence ST_nS^{-1} of conjugates of T_n is restrained.

An important point to notice about Theorem 1.3 is that it characterizes restrained sequences in terms of the T_n *without reference to points* *where these maps act*. By contrast, Definition 1.2 of a restrained sequence is a sequential description of the *action* of the Möbius maps. This situation may be compared with the definition of a discrete group (which is given in terms of the topology on the group) and the definition of a discontinuous action of the group (which is given in terms of its action on some particular space). This distinction is important because Möbius maps act on different spaces (for example, they act on both $\mathbf{C}_\infty$ and on hyperbolic 3-space) and any statement involving their action is dependent on which particular space the group action is assumed to take place.

Our principal aim in this paper is to exploit the link between geometry (in particular, hyperbolic geometry) and Möbius transformations in the hope that this will encourage further applications of these ideas to the theory of continued fractions. With this in mind, this paper is partly written in an expository manner, and we are not particularly interested in the shortest, or the most elementary proofs, unless these serve our primary aim of illustrating the geometry. We shall develop the ideas about hyperbolic geometry as we need them, but in a brief, expository manner, as they are all well known in the theory of discrete Möbius groups. Finally we emphasize that the action of Möbius maps on hyperbolic 3-space, which we discuss below, is crucial to a proper understanding of Möbius maps *even if one is only interested in their action on* $\mathbf{C}_\infty$ (which is the boundary of hyperbolic 3-space).

In Section 2 we state some basic facts about the Euclidean geometry of the action of a Möbius transformation, and we use these to give our proof of Theorem 1.3. In Sections 3 and 4 we give proofs of these basic facts from both an Euclidean and a hyperbolic perspective. In Sections 5 and 6 we review admissible sequences, and strongly exceptional sequences, respectively, from this new point of view. Finally, in Section 7, we consider an example taken from [**7**].

2. The proof of Theorem 1.3. Theorem 1.3 is easily derived from some basic geometric facts about Möbius maps which we now discuss. First we define the *chordal derivative* $T^\#(z)$ of the Möbius map T given in (1.5) by each of the three equivalent expressions:

$$T^\#(z) = \lim_{w \to z} \frac{\sigma(T(w), T(z))}{\sigma(w, z)} \tag{2.1}$$

$$= \frac{(1+|z|^2)|T'(z)|}{1+|T(z)|^2}$$
$$= \frac{1+|z|^2}{|az+b|^2+|cz+d|^2}.$$

Using these one obtains the formula

$$\frac{\sigma(T(z),T(w))}{\sigma(z,w)} = \sqrt{T^{\#}(z)T^{\#}(w)} \tag{2.2}$$

which follows from the definition of the chordal metric by simple algebra, and which is the chordal analogue of the familiar Euclidean formula

$$\frac{|T(z)-T(w)|}{|z-w|} = \sqrt{|T'(z)||T'(w)|}.$$

As $T^{\#}(z)$ is a continuous function of z with respect to the chordal metric, the next result follows immediately from (2.2).

Lemma 2.1. *For any Möbius map T,*

$$\Delta(T) = \max_{z\in\mathbf{C}_\infty} T^{\#}(z), \quad \delta(T) = \min_{z\in\mathbf{C}_\infty} T^{\#}(z). \tag{2.3}$$

Thus $\delta(T)$ and $\Delta(T)$ are attained and the best Lipschitz constant $\Delta(T)$ of T coincides with the maximum infinitesimal distortion of T, namely, $\max_z T^{\#}(z)$.

Next we recall from (1.6) the matrix norm $\|T\|$ of the Möbius map T in (1.5). It is easy to see that $\|T\|^2 \geq 2$ and that $\|ST\| \leq \|S\|\|T\|$. In fact, the norm of T controls a certain amount of the geometric action of T, and we can gain some insight into this by looking at unitary maps (which are the maps T for which $\|T\|$ is minimal). A 2×2 complex matrix A is *unitary* if and only if $A^{-1} = \overline{A}^T$, and as A is unitary if and only if $-A$ is, we can talk of a Möbius map being unitary. We now quote the following result, see [**3**, pp. 17–18 and p. 63].

Lemma 2.2. *For each Möbius map A the following are equivalent*:

(1) *A is unitary*:

(2) $\|A\|^2 = 2$;

(3) *A is an isometry of* $\mathbf{C}_\infty$ *with the chordal metric* σ;

(4) *A corresponds under stereographic projection to a rotation of the unit sphere in* $\mathbf{R}^3$*, the Riemann sphere*;

(5) *if* T *is any Möbius map, then* $\|AT\| = \|T\| = \|TA\|$.

Later we shall see that the values $\delta(T)$ and $\Delta(T)$ are attained by $T^{\#}$ at *unique points*, say η_T and ξ_T, respectively (Theorem 2.5), and the role of these fundamental points is crucial to this paper. To illustrate this, and to bring these ideas together, we mention the following result (which will be proved later and which shows where the chordal distortion of T is small).

Theorem 2.3. *Let* T *be any Möbius transformation that is not unitary, so that* $\|T\|^2 \neq 2$. *Then*

$$T^{\#}(z) \leq \frac{4}{(\|T\|^2 - 2)\sigma(z, \zeta_T)^2}.$$

Finally we mention three more results, not all of which are needed to prove Theorem 1.3.

Theorem 2.4. *Let* T *be any Möbius map. Then*

(1) $\Delta(T) = (\|T\|^2 + \sqrt{\|T\|^4 - 4})/2$;

(2) $\delta(T)\Delta(T) = 1$;

(3) $\Delta(T^{-1}) = \Delta(T)$ and $\delta(T^{-1}) = \delta(T)$.

We have already remarked that each Möbius map T is a Lipschitz map with respect to the chordal metric, and (1) gives the best Lipschitz constant (which is slightly less than $\|T\|^2$) explicitly in terms of the coefficients of T.

We shall say that z and w in $\mathbf{C}_\infty$ are *diametrically opposite* if they correspond under stereographic projection to the endpoints of a diameter of the Riemann sphere. It is well known that a necessary and sufficient condition for z and w to be diametrically opposite is that $z\bar{w} = -1$, and this leads to a simple description of circles in the chordal metric. First, for any z and a in $\mathbf{C}$ we have the identity

$$|z-a|^2 + |\bar{a}z+1|^2 = (1+|a|^2)(1+|z|^2).$$

This implies that

$$\sigma(z,a)^2 = \frac{4|z-a|^2}{(1+|a|^2)(1+|z|^2)} = \frac{4}{1+|(\bar{a}z+1)/(z-a)|^2},$$

so that the chordal circle with chordal centre a and chordal radius R, where $R \leq 2$, is given by

$$\left|\frac{z-a}{\bar{a}z+1}\right| = \sqrt{\frac{R^2}{4-R^2}}.$$

Another way to see this is to note that the circle with chordal centre a and chordal radius R is given by either (and both) of the equivalent equations

$$\sigma(z,a) = R, \quad \sigma(z,-1/\bar{a}) = \sqrt{4-R^2}.$$

In particular, as each great circle has chordal radius $\sqrt{2}$, the stereographic projection of a great circle back into the extended complex plane is given by

$$\left|\frac{z-a}{\bar{a}z+1}\right| = 1.$$

Theorem 2.5. *Let T be any Möbius map. Then $\delta(T)$ and $\Delta(T)$ defined by* (2.3) *are attained at unique points η_T and ζ_T, respectively, where η_T and ζ_T are diametrically opposite each other. Further, the level curves of $T^{\#}(z)$ are circles in the chordal metric whose chordal centres are at η_T and ζ_T or, equivalently, the circles in $\mathbf{C}_\infty$ which have η_T and ζ_T as inverse points.*

Theorem 2.6. *Given any Möbius transformation T, there are diametrically opposite points z_1 and z_2 such that $T(z_1)$ and $T(z_2)$ are also diametrically opposite points. Further, $\{z_1, z_2\}$ is uniquely* *determined by T unless T is unitary* (*a rotation of the Riemann sphere*) *and* $\{z_1, z_2\} = \{\eta_T, \zeta_T\}$.

We shall temporarily assume the validity of Theorems 2.3, 2.4, 2.5 and 2.6 (all of which are known) and show how Theorem 1.3 follows directly from these.

Proof of Theorem 1.3. The equivalence of (2), (3) and (4) in Theorem 1.3 follows trivially from (1) and (2) of Theorem 2.4. Next it is easy to see that (4) and (5) in Theorem 1.3 are equivalent. Indeed, suppose that (4) fails. Then there is a subsequence of the T_n with bounded norms, and hence there is a further subsequence, say T_{n_j}, which converges uniformly to some Möbius map T on $\mathbf{C}_\infty$. Thus (5) fails. Conversely, suppose that (5) fails. Then there is a subsequence, say T_{n_j}, that converges uniformly on $\mathbf{C}_\infty$ to some Möbius map T. This implies that we can choose matrices A_n representing T_n and a matrix A representing T such that $A_{n_j} \to A$ and $\|T_{n_j}\| \to \|T\|$; thus (4) fails. So far, we have proved the equivalence of (2), (3), (4) and (5) in Theorem 1.3.

Now suppose that (6) holds. Then, from Lemma 2.2,

$$\|T_n\| = \|A_n T_n B_n\| = |\lambda_n| + 1/|\lambda_n| \longrightarrow +\infty$$

because $\lambda_n \to 0$ so that (4) holds. Conversely suppose that (4) holds. Then, by Theorem 2.6, there are unitary maps A_n and B_n such that $A_n T_n B_n$ fixes 0 and ∞ and so is of the form $z \mapsto \lambda_n z$. It is clear that A_n and B_n may be chosen so that $|\lambda_n| \le 1$, and now (4) implies that $\lambda_n \to 0$ so that (6) holds.

It is clear from the definition of a restrained sequence and (1.4) that (1) implies (2). Next, suppose that (6) holds and let $u_n = B_n(1)$ and $v_n = B_n(-1)$. Then $\sigma(u_n, v_n) = \sigma(1, -1) = 2$ and

$$\sigma(T_n(u_n), T_n(v_n)) = \sigma(A_n T_n B_n(1), A_n T_n B_n(-1)) = \sigma(\lambda_n, -\lambda_n) \to 0$$

as $n \to \infty$. This shows that (6) implies (1) so that now (1)–(6) in Theorem 1.4 are equivalent to each other.

By taking $\mathcal{D}_n = \mathcal{D}''_n$, it is clear that (9) implies (7). It is also clear that (7) and (8) are equivalent to each other, with $\mathcal{D}'_n = \mathbf{C}_\infty \setminus \mathcal{D}_n$.

Next, as

$$\mu(T_n(\mathcal{D}'_n)) \le \Delta(T_n)\mu(\mathcal{D}'_n),$$

we see that (8) implies (3).

To complete the proof it is now sufficient to show that (6) implies (9). Suppose then that (6) holds. Let $H_n = A_n T_n B_n$. Then, from (2.1),

$$|H_n^{\#}(z)| = \frac{(1+|z|^2)|\lambda_n|}{1+|\lambda_n z|^2} \le (1+|z|^2)|\lambda_n|.$$

Now let Σ_n be the disc $\{z : |z| < |\lambda_n|^{-1/3}\}$ and put $\mathcal{D}''_n = B_n(\Sigma_n)$. Then, as A_n and B_n are chordal isometries, it is clear that $\mu(\mathcal{D}''_n) \to 4\pi$ and

$$\sup\{|H_n^{\#}(z) : z \in \Sigma_n\} \longrightarrow 0$$

as $n \to \infty$. Finally, as $T_n^{\#}(B_n(z)) = H_n^{\#}(z)$, (9) follows. We have now completed our proof of Theorem 1.3 subject to proving Theorems 2.3–2.5, for we have not used Theorem 2.6 yet.

3. The Euclidean geometry of Möbius maps. The best explanation of Theorems 2.3–2.6 comes from studying the action of Möbius maps on hyperbolic space, but first we need to study their action in $\mathbf{R}^3$. This section contains such a study and includes proofs of Theorems 2.3–2.6. For more details on Möbius maps, in all dimensions, we recommend [**1, 2, 8**] and [**10**].

So far, we have been considering Möbius maps of the form $z \mapsto (az+b)/(cz+d)$ acting on $\mathbf{C}_\infty$. Now it is well known that the group of Möbius maps is a subgroup, of index two, of the group generated by reflections in Euclidean lines and circles in $\mathbf{C}^\infty$. We can replace reflection in a line L by the reflection in the plane (in $\mathbf{R}^3$) through L and orthogonal to $\mathbf{C}$, and reflection in a circle C by the reflection in a sphere in $\mathbf{R}^3$ of which C is the equator. In this way we see that the group $\mathcal{M}_{\mathbf{C}}$ of complex Möbius maps is a subgroup of the Möbius group $\mathcal{M}$ consisting of an even number of reflections in planes or spheres in $\mathbf{R}^3 \cup \{\infty\}$.

Next we recall the stereographic projection Φ of $\mathbf{C}_\infty$ onto the boundary $\partial\mathbf{B}$ of the unit ball $\mathbf{B}$ in $\mathbf{R}^3$. The first important fact is that Φ is an *isometry between* $(\mathbf{C}_\infty, \sigma)$ *and* $\partial\mathbf{B}$ *with its Euclidean metric*, as a subset of $\mathbf{R}^3$; of course, this is nothing more than the definition of the chordal metric. The next crucial fact to note is that Φ is actually the restriction to $\mathbf{C}_\infty$ of inversion in the sphere, in $\mathbf{R}^3$, with centre $(0,0,1)$ and radius $\sqrt{2}$. This means that, instead of restricting our attention to the action of T on $\mathbf{C}_\infty$, we can first extend the action to obtain the map $T:\mathbf{R}^3_\infty \to \mathbf{R}^3_\infty$, and then consider the conjugate element $\Phi T \Phi^{-1}$ (this is the composition of maps) which also acts on $\mathbf{R}^3_\infty$, but which now preserves the unit ball $\mathbf{B}$ and its boundary $\partial\mathbf{B}$. We shall denote the map $\Phi T\Phi^{-1}$ by T^*; we shall use this notation consistently throughout the paper where, in general, a $*$ refers to the action of a map after it has been transformed by stereographic projection.

We now recall the geometric description of the action of a Möbius map in terms of its isometric circle. Isometric circles were introduced by Ford in [**5**] for the study of discrete Möbius groups, and since their introduction they have played a central role in that theory. However, so far they have been used only a little in the theory of continued fractions. Let $T(z) = (az+b)/(cz+d)$, where $ad-bc=1$ and $c \neq 0$. Then the *isometric circle* $C(T)$ of T is the circle given by

$$C(T) = \{z : |T'(z)| = 1\} = \{z : |cz+d| = 1\}.$$

It is known that the action of T is the inversion in $C(T)$ followed by a Euclidean isometry; thus the *Euclidean distortion that arises by an application of T arises only by virtue of the inversion in $C(T)$*, and this is the real significance of isometric circles.

It is known that Ford's theory of isometric circles is valid in all dimensions and it is this that we turn to next. We assume that T^* is a Möbius map acting on $\mathbf{R}^3_\infty$ and that T^* preserves the closed unit ball $\overline{\mathbf{B}}$. We also assume that $T^*(\infty) \neq \infty$ (as otherwise, T^* will not have an isometric sphere). The isometric sphere $\Sigma(T^*)$ of T^* is the unique sphere in $\mathbf{R}^3$ on which T^* acts as an (infinitesimal) Euclidean isometry (this is the analogous statement to $|T'(z)| = 1$ in $\mathbf{C}$) and it is known that the action of T^* on $\mathbf{R}^3_\infty$ is an inversion, say I_T, in $\Sigma(T^*)$ followed by a Euclidean isometry, say E_T, of $\mathbf{R}^3$ that fixes the origin (and which therefore is given by an orthogonal matrix). It follows that for any u and v in $\mathbf{C}_\infty$,

$$\begin{aligned}
\sigma(T(u),T(v)) &= |\Phi(T(u)) - \Phi(T(v))| \\
&= |T^*(\Phi(u)) - T^*(\Phi(v))| \\
&= |E_T I_T(\Phi(u)) - E_T I_T(\Phi(v))| \\
&= |I_T(\Phi(u)) - I_T(\Phi(v))|.
\end{aligned}$$

Explicitly, this gives the *chordal distortion* of T in $\mathbf{C}_\infty$ in terms of the *Euclidean distortion of the inversion* I_T on $\partial\mathbf{B}$. Specifically, it shows that the change of scale in the chordal metric induced by T at z is the same as the change of scale in the Euclidean metric induced by I_T at $\Phi(z)$. We remark that the Euclidean isometry E_T is known (explicitly) but as it is irrelevant here, we shall not discuss it further.

Next it is known that $\Sigma(T^*)$ has centre $(T^*)^{-1}(\infty)$ and is orthogonal to $\partial\mathbf{B}$. With this available, the following lemma will tell us all we need to know about the chordal distortion of T on $\mathbf{C}_\infty$.

Lemma 3.1. *Let Σ be a sphere in $\mathbf{R}^3$, with centre y^* and radius R, that is orthogonal to $\partial\mathbf{B}$, and let I be inversion in Σ. Then for all u^* and v^* on $\partial\mathbf{B}$,*

$$\frac{|I(u^*) - I(v^*)|}{|u^* - v^*|} = \frac{R^2}{|u^* - y^*||v^* - y^*|}. \tag{3.1}$$

In particular, if T is a Möbius map, and if R_T is the radius of the isometric sphere of T^*, then

$$T^\#(z) = \frac{R_T^2}{|z^* - (T^*)^{-1}(\infty)|^2}. \tag{3.2}$$

The proof of (3.1) is trivial because the triangle with ordered triple of vertices y^*, u^*, v^* is similar to the triangle with ordered triple of vertices y^*, $I(v^*)$, $I(u^*)$. Moreover, (3.2) is just (3.1) written in terms of the action of T in $\mathbf{C}_\infty$. Although (3.2) is elementary, it is important because it tells us that the chordal distortion $T^\#(z)$ of T is a quantity which is dependent on the action of T^* in *three-dimensional space*, for example, $(T^*)^{-1}(\infty)$ lies outside the unit ball in $\mathbf{R}^3$. It follows from this that if we work with $T^\#(z)$ entirely within the context of $\mathbf{C}_\infty$ we can at best only obtain estimates of $T^\#(z)$; if we want precise formulae, then we have to discuss the action on 3-space.

It is apparent from (3.1) that the maximum distortion of I with respect to the Euclidean metric on $\partial\mathbf{B}$ occurs at the point ζ^* of $\partial\mathbf{B}$ that is nearest to the centre y^* of Σ and that the minimum distortion of I occurs at the point η^* of $\partial\mathbf{B}$ that is farthest from the centre y^* of

Σ. If we apply Lemma 3.1 to the case in which $\Sigma = \Sigma(T)$, the isometric sphere of T^*, then there are several important deductions that can be made from th is observation.

First it follows that, for any complex Möbius map T, there is a unique point, say ζ_T, in $\mathbf{C}_\infty$ which maximizes the chordal derivative $T^\#$ and a unique point, say η_T, in $\mathbf{C}_\infty$ which minimizes $T^\#$ and, moreover, the images ζ_T^* and η_T^* of these points under stereographic projection lie at the ends of the diameter of $\mathbf{B}$ that lies on a ray from the centre of $\Sigma(T)$. As the description of the level curves of $T^\#$ given in Theorem 2.5 also follow from this argument, this proves Theorem 2.5.

Next the geometric description of ζ_T^* and η_T^* shows that, for any T, the inversion I_T interchanges the two points ζ_T^* and η_T^*. This means that $|y^* - \zeta_T^*||y^* - \eta_T^*| = R^2$, and as (3.1) implies that

$$\Delta(T) = \frac{R^2}{|y^* - \zeta_T^*|^2}, \quad \delta(T) = \frac{R^2}{|y^* - \eta_T^*|^2},$$

we see that $\delta(T)\Delta(T) = 1$; thus, Theorem 2.4 (2) holds.

Theorem 2.4 (3) also follows easily. First in the notation above, we have $T^* = E_T I_T$. Now this shows that

$$(T^*)^{-1} = (I_T)^{-1}(E_T)^{-1} = I_T(E_T)^{-1},$$

and it follows from this that $\Delta(T) = \Delta(T^{-1})$ because both of these terms are given by the maximal Euclidean distortion of I_T on $\partial\mathbf{B}$. Alternatively, given the Möbius map T, write $u = T(z)$ and $v = T(w)$. Then

$$\frac{\sigma(T(z), T(w))}{\sigma(z,w)} = \left(\frac{\sigma(T^{-1}(u), T^{-1}(v))}{\sigma(u,v)}\right)^{-1},$$

so that in the notation above, $\Delta(T)\delta(T^{-1}) = 1$, or equivalently, $\delta(T)\Delta(T^{-1}) = 1$. As $\delta(T)\Delta(T) = 1$, Theorem 2.4 (3) again follows. We shall defer the proof of Theorem 2.4 (1) and Theorem 2.3 until later.

Suppose now that we are given a Möbius transformation T for which T^* is not a rotation of $\mathbf{R}^3$, and let ζ_T^* and η_T^* be the images of ζ_T and η_T under stereographic projection from $\partial\mathbf{B}$ to $\mathbf{C}_\infty$. Then, as we have just seen, ζ_T^* and η_T^* are at the extremities of a diameter of $\mathbf{B}$, they are the points on $\partial\mathbf{B}$ that are nearest to, and farthest from, the centre of $\sigma(T^*)$, and they are interchanged by I_T. As $T^* = E_T I_T$, and as E_T is a Euclidean isometry (which therefore maps each diameter of $\mathbf{B}$ onto another diameter of $\mathbf{B}$) we have now made the following important observation.

Lemma 3.2. *In the notation above, T^* maps the diametrically opposite points η_T^* and ζ_T^* to another pair of diametrically opposite points.*

We are now in a position to prove Theorem 2.6. If T is unitary, then T^* maps every diameter of $\mathbf{B}$ onto another diameter of $\mathbf{B}$. If T is not unitary, then the existence of z_1 and z_2 follows from Lemma 3.2. To prove the uniqueness of the z_j, write $h(z) = -1/\bar{z}$. Then z and w are diametrically opposite if and only if $w = h(z)$; thus, if z_1 and z_2 satisfy the conditions of Theorem 2.6, we must have $T(h(z_1)) = h(T(z_1))$. After a little simplification, we find that this is a quadratic equation with roots z_1 and $h(z_1)$. Of course, it may happen that all three coefficients are zero, but in this case one can show, by using Lemma 2.2, that T is unitary. We omit the details as we do not need them here.

In conclusion, and with the exception of (1) in Theorem 2.4, we have now proved Theorems 2.4, 2.5 and 2.6.

4. Hyperbolic geometry and Möbius maps. The link between restrained sequences and hyperbolic geometry has already been considered in [**1**], but here, with applications to continued fractions in mind, we are more concerned with stating our results in terms of the transformations S_n rather than, as in [**1**], in terms of hyperbolic geometry. Further, the norm of a Möbius map, which is central to our arguments, is not explicitly considered in [**1**], although it does appear implicitly. The results in [**1**] hold in all dimensions. In this paper we shall restrict our attention to maps of the extended complex plane $\mathbf{C}_\infty$ onto itself, and hence to three-dimensional hyperbolic geometry. In fact, the work here is valid, without change, in all dimensions, but the main difficulty in any exposition of this is the interpretation of the norm $\|T\|$ of a Möbius transformation T in higher dimensions. Although this interpretation is known (the definition involves looking at Möbius maps as maps of projective space onto itself; see [**4**]) we shall not consider it here. We now start our exposition of hyperbolic geometry.

We recall that each complex Möbius map T acts naturally on the one-point compactification $\mathbf{R}^3_\infty$ of Euclidean 3-space $\mathbf{R}^3$, and in doing so it leaves the upper half-space

$$\mathbf{H} = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_3 > 0\}$$

invariant. We now give $\mathbf{H}$ the hyperbolic metric $\rho_\mathbf{H}$ derived from the line element $|dx|/x_3$; then $(\mathbf{H}, \rho_\mathbf{H})$ is a model of hyperbolic 3-space, and we have the added information that the group $\mathcal{M}_\mathbf{C}$ of complex Möbius maps is the *group of all conformal isometries of* $(\mathbf{H}, \rho_\mathbf{H})$. Of course, $\mathbf{C}_\infty$ is the boundary of $\mathbf{H}$, and the action of the complex Möbius maps on $\mathbf{C}_\infty$ is simply the action of the group of hyperbolic isometries of hyperbolic 3-space on the boundary of that space. The advantage that stems from this approach is that the complex Möbius maps now *act as isometries of a complete metric space* and the action within this space has a far richer structure than the action on its boundary. In general, we obtain much more information about Möbius maps by considering their action on hyperbolic space than we do by considering their action on $\mathbf{C}_\infty$.

There is one basic formula which links the coefficients of the complex Möbius map T and the action of T as a hyperbolic isometry, and this is vital in what follows. Let $\mathbf{j} = (0, 0, 1)$ (this is the point in $\mathbf{H}$ that corresponds to the point i in the upper half-plane); then

$$\|T\|^2 = 2\cosh \rho_\mathbf{H}(\mathbf{j}, T(\mathbf{j})), \tag{4.1}$$

see [**3**, p. 61].

We are now ready for the final part of this geometric discussion. We have seen that stereographic projection Φ is an isometry of $(\mathbf{C}_\infty, \sigma)$ onto $\partial\mathbf{B}$ equipped with the Euclidean metric. However, as Φ is in the full Möbius group acting on $\mathbf{R}^3_\infty$, it also acts on $\mathbf{H}$ and it is not difficult to see that Φ maps $\mathbf{H}$ onto the open unit ball $\mathbf{B}$ with $\Phi(\mathbf{j}) = 0$, and that

$$\Phi : (\mathbf{H}, \rho_\mathbf{H}) \longrightarrow (\mathbf{B}, \rho_\mathbf{B})$$

is an *isometry*, where $\rho_\mathbf{B}$ is the standard hyperbolic metric on $\mathbf{B}$ is derived in the usual way from the line element $2|dx|/(1 - |x|^2)$. In particular, (4.1) shows that if T is any complex Möbius map, then

$$\|T\|^2 = 2\cosh\rho_{\mathbf{B}}(0, T^*(0)), \tag{4.2}$$

where, as before, T^* represents the action of T on $\partial\mathbf{B}$ and, by extension, on $\mathbf{R}^3_\infty$. As an illustration of the use of this formula, we note that it shows that $\|T\|^2 \geq 2$ with equality if and only if $T^*(0) = 0$. In particular, the group of Möbius maps T for which T^* fixes 0 in $\mathbf{B}$ is the group of Möbius maps T for which $\|T\|^2 = 2$. This is essentially Lemma 2.2.

We shall now give two proofs of Theorem 2.4 (1), namely that

$$\Delta(T) = (\|T\|^2 + \sqrt{\|T\|^4 - 4}\,)/2. \tag{4.3}$$

First, this follows directly from (4.2) and the formula

$$\Delta(T) = \sup_{z,w\in\mathbf{C}_\infty} \frac{\sigma(T(z), T(w))}{\sigma(z,w)} = \exp\rho_{\mathbf{B}}(0, T^*(0)) \tag{4.4}$$

which can be found in [**3**, p. 43]. However, as our intention here is to illustrate how to use the geometry, and as we have all of the ideas to hand, we shall give the details of the proof which is an attractive interplay between the various geometries.

We have seen that the maximal chordal distortion of T with respect to σ is the same as the maximal Euclidean distortion of the inversion I_T, and that this occurs at the point ζ_T^* of the unit sphere $\partial\mathbf{B}$ that is closest to the centre y^* of the isometric sphere $\Sigma(T^*)$ of T^*. This means that

$$\Delta(T) = \frac{R^2}{(|y^*| - |\zeta_T^*|)^2} = \frac{R^2}{(|y^*| - 1)^2},$$

where R is the radius of $\Sigma(T^*)$. Now $\Sigma(T^*)$ is orthogonal to $\partial\mathbf{B}$, so that $1 + R^2 = |y^*|^2$, and this shows that

$$\Delta(T) = \frac{|y^*| + 1}{|y^*| - 1}.$$

We know that $y^* = (T^*)^{-1}(\infty)$ and, because Möbius maps preserve inverse points, we also have

$$|y^*|\,|(T^*)^{-1}(0)| = |(T^*)^{-1}(\infty)|\,|(T^*)^{-1}(0)| = 1;$$

thus,

$$\Delta(T) = \frac{1 + |(T^*)^{-1}(0)|}{1 - |(T^*)^{-1}(0)|}.$$

On the other hand, as T^* is a hyperbolic isometry,

$$\rho_{\mathbf{B}}(0, T^*(0)) = \rho_{\mathbf{B}}(0, (T^*)^{-1}(0)),$$

so that $|T^*(0)| = |(T^*)^{-1}(0)|$, and hence

$$\Delta(T) = \frac{1 + |T^*(0)|}{1 - |T^*(0)|}.$$

Now it is well known that

$$\rho_{\mathbf{B}}(0, T^*(0)) = \log \frac{1 + |T^*(0)|}{1 - |T^*(0)|} = \log \Delta(T), \tag{4.5}$$

so finally, we arrive at the formula (4.4) which, with (4.2), is all we need to prove (4.3).

Finally, we compute the radius of the isometric sphere of T^* in terms of $\|T\|$, and this is given in the following lemma.

Lemma 4.1. *The radius R of the isometric sphere $\Sigma(T^*)$ of T^* is given by*

$$R^2 = \frac{4}{\|T\|^2 - 2}. \tag{4.6}$$

Proof. From (4.2) and the first part of (4.5), we see that

$$\|T\|^2 = \frac{1 + |T^*(0)|}{1 - |T^*(0)|} + \frac{1 - |T^*(0)|}{1 + |T^*(0)|} = 2\left(\frac{1 + |T^*(0)|^2}{1 - |T^*(0)|^2}\right).$$

As $|y^*||T^*(0)| = 1$, because $(T^*)^{-1}(\infty)$ and $(T^*)^{-1}(0)$ are inverse points with respect to $\partial\mathbf{B}^3$, and $R^2 = |y^*|^2 - 1$, (4.6) follows easily from this. $\square$

Lemma 4.1 gives us further insight into Theorem 1.3. According to Theorem 1.3, a sequence T_n is restrained if and only if $R_n \to 0$ where

R_n is the radius of the isometric sphere of T_n^*. Bearing in mind that the action of T_n^* is inversion in its isometric sphere followed by an Euclidean isometry (of $\mathbf{R}^3$) we now see exactly why (7) and (8) in Theorem 1.3 are equivalent to T_n being restrained, and exactly why Theorem 1.3 (9) holds. If $z^* \in \partial \mathbf{B}^3$, then $|z^* - \zeta_T^*| \leq |z^* - y^*|$ so from (3.1), in its infinitesimal form, and (4.6) we have

$$T^{\#}(v) \leq \frac{4}{(\|T\|^2 - 2)|z^* - \zeta_T^*|^2} = \frac{4}{(\|T\|^2 - 2)\sigma(z, \eta_T)^2}.$$

This proves Theorem 2.3, and it shows where $T^{\#}$ is small. However, as we have remarked above, we lose something by expressing this in terms of complex numbers because, although the maximal distortion of T relative to $\mathbf{C}$ occurs at ζ_T, its maximal distortion in $\mathbf{R}^3$ occurs at a point not on the complex plane.

5. Admissible sequences. Throughout this section T_n is a restrained sequence. By discarding a finite number of the T_n we may assume that $\|T_n\|^2 > 2$ and hence that no T_n is unitary. It follows that there are unique points ζ_n and η_n in $\mathbf{C}_\infty$ such that

$$\inf_z T_n^{\#}(z) = T_n^{\#}(\eta_n), \quad \sup_z T_n^{\#}(z) = T_n^{(\#)}(\zeta_n). \tag{5.1}$$

This notation will be used throughout this section. We recall that a sequence u_n in $\mathbf{C}_\infty$ is *admissible* with respect to T_n (more briefly, T_n-admissible) if and only if there is a sequence v_n such that u_n and v_n are separated, and $T_n(u_n)$ and $T_n(v_n)$ are asymptotic. The following result characterizes T_n-admissible sequences in terms of the two fundamental sequences ζ_n and η_n. Roughly speaking, this shows that the sequence η_n is the canonical admissible sequence, and that a sequence z_n is admissible if and only if it is sufficiently far away from the sequence ζ_n.

Theorem 5.1. *For any sequences u_n in $\mathbf{C}_\infty$, the following are equivalent*:

(1) u_n *is T_n-admissible*;

(2) $\sigma(T_n(u_n), T_n(\eta_n)) \to 0$ *as* $n \to \infty$;

(3) $\|T_n\|^2 \sigma(u_n, \zeta_n) \to +\infty$.

Corollary 5.2. *The sequence ζ_n is T_n-admissible.*

Further,

Corollary 5.3. *If z_n and w_n are any T_n-admissible sequences, then $T_n(z_n)$ and $T_n(w_n)$ are asymptotic.*

Proof of Theorem 5.1. First, by Theorem 1.3 there are unitary maps A_n and B_n such that $A_n T_n B_n = H_n$, where $H_n(z) = \lambda_n z$ and $\lambda_n \to 0$. It is easy to see that u_n is T_n-admissible if and only if $B_n^{-1}(u_n)$ is H_n-admissible. We let $z_n = B_n^{-1}(u_n)$; then (1) is equivalent to

(1′) z_n is H_n-admissible.

Next, a simple calculation shows that $H_n^{\#}$ takes its minimum value $|\lambda_n|$ at 0, and its maximum value $|\lambda_n^{-1}|$ at ∞. Because the chordal derivative satisfies the chain rule, and $X^{\#}(z) = 1$ when X is unitary, we see that $\zeta_n = B_n(\infty)$ and $\eta_n = B_n(0)$. As $H_n(0) = 0$, this means that (2) is equivalent to

(2′) $\sigma(H_n(z_n), 0) \to 0$ as $n \to \infty$.

Finally, as $\|T_n\| = \|H_n\|$, we also see that (3) is equivalent to

(3′) $\|H_n\|^2 \sigma(z_n, \infty) \to +\infty$.

It therefore suffices to show that $(1'), (2')$ and $(3')$ are equivalent to each other. We shall show that each is equivalent to

(4′) $H_n(z_n) \to 0$ (that is, $\lambda_n z_n \to 0$).

First, by a direct calculation, (2′) is seen to be equivalent to

$$\frac{|\lambda_n z_n|^2}{1 + |\lambda_n z_n|^2} \longrightarrow 0,$$

and it is easy to see that this is equivalent to $\lambda_n z_n \to 0$. Another calculation shows that (3′) is equivalent to

$$\left(|\lambda_n| + \frac{1}{|\lambda_n|}\right) \frac{2}{\sqrt{1 + |z_n|^2}} \longrightarrow \infty,$$

and again it is easy to see, after taking reciprocals, that this is equivalent to $\lambda_n z_n \to 0$.

Finally, as $H_n \to 0$ on $\mathbf{C}$, it is clear that if $H_n(z_n) \to 0$ then z_n is H_n-admissible. Conversely, if z_n is H_n-admissible with z_n and w_n separated but $H_n(z_n)$ and $H_n(w_n)$ asymptotic, then, for sufficiently large n, one of z_n and w_n is a fixed chordal distance away from ∞. Thus, the minimum of $\sigma(H_n(z_n), 0)$ and $\sigma(H_n(w_n), 0)$ tends to zero and we have proved that $(1')$ and $(4')$ are equivalent. The proof is complete. $\square$

6. Strongly exceptional sequences. We continue to make the same assumptions as in Section 5 that lead to η_n and ζ_n satisfying (5.1). We recall that a sequence z_n is *strongly exceptional* with respect to a sequence T_n, or is T_n-strongly exceptional, if there is an admissible sequence u_n such that $T_n(z_n)$ and $T_n(u_n)$ are separated. We prove two results here; roughly speaking, the first of these implies that ζ_n is the canonical strongly exceptional sequence and that any sequence z_n is strongly exceptional if and only if it is sufficiently close to the sequence ζ_n.

Theorem 6.1. *Let T_n be a restrained sequence of Möbius transformations.*

(i) *The sequence ζ_n is T_n-strongly exceptional.*

(ii) *A sequence z_n in $\mathbf{C}_\infty$ is T_n-strongly exceptional if and only if $\sigma(z_n, \zeta_n) = O(\|T_n\|^{-2})$ as $n \to \infty$ or, equivalently,*

$$\limsup_{n\to\infty} \|T_n\|^2 \sigma(z_n, \zeta_n) < +\infty. \tag{6.1}$$

(iii) *Let z_n be a T_n-strongly exceptional sequence. Then u_n is T_n-admissible if and only if*

$$\limsup_{n\to\infty} \|T_n\|^2 \sigma(u_n, z_n) = +\infty. \tag{6.2}$$

Theorem 6.2. *The sequence T_n is restrained if and only if for every T_n-strongly exceptional sequence z_n and every positive δ,*

$$\limsup_{n\to\infty} \{T_n^{\#}(z) : \sigma(z, z_n) \geq \delta\} = 0. \tag{6.3}$$

Proof of Theorem 6.1. As T_n is a restrained sequence T_n, we may discard a finite number of terms and so assume that no T_n is unitary. We may now write $A_nT_nB_n = H_n$ where A_n and B_n are unitary, $H_n(z) = \lambda_n z$ and $\lambda_n \to 0$ as $n \to \infty$. We recall that $\eta_n = B_n(0)$, $\zeta_n = B_n(\infty)$ and $\|T_n\| = \|H_n\|$. It is easy to see that z_n is T_n-strongly exceptional if and only if $B_n^{-1}(z_n)$ is H_n-strongly exceptional.

It is sufficient to prove Theorem 6.1 in the case when $T_n = H_n$. Now a sequence w_n is H_n-strongly exceptional if and only if there exists an H_n-admissible sequence u_n with $H_n(u_n)$ and $H_n(w_n)$ separated. Thus if we use the equivalence of $(1')$ and $(4')$ in the proof of Theorem 5.1, we see that w_n is H_n-strongly exceptional if and only if

$$\liminf_{n\to\infty} |\lambda_n w_n| > 0. \tag{6.4}$$

As $\lambda_n \to 0$, we see that

$$\begin{aligned}\|H_n\|^2\sigma(w_n,\infty) &= \left(|\lambda_n| + \frac{1}{|\lambda_n|}\right)\frac{2}{\sqrt{1+|w_n|^2}}\\ &= \frac{2}{|\lambda_n|\sqrt{(1+|w_n|)^2}} + o(1),\end{aligned}$$

and hence that (6.1) and (6.4) are equivalent. This proves (ii), and (i) follows immediately from this.

Finally we prove (iii). As σ is a metric, we have

$$|\sigma(u_n,\zeta_n) - \sigma(u_n,z_n)| \le \sigma(\zeta_n,z_n),$$

so that

$$|\|T_n\|^2\sigma(u_n,\zeta_n) - \|T_n\|^2\sigma(u_n,z_n) \le \|T_n\|^2\sigma(\zeta_n,z_n).$$

By (6.1), the upper bound here is bounded above by some constant. It follows that if one of the two sequences

$$\|T_n\|^2\sigma(u_n,\zeta_n), \quad \|T_n\|^2\sigma(u_n,z_n)$$

converges to $+\infty$ then so does the other, and (iii) now follows directly from Theorem 5.1. Note that the idea in this proof is simply that a comparison of u_n and z_n is essentially the same as a comparison of u_n

with the canonical sequence ζ_n providing that z_n is sufficiently close to ζ_n. This supports the view that we should regard a strongly exceptional sequence as just a, sufficiently small, perturbation from the canonical sequence ζ_n.

Proof of Theorem 6.2. First if we take a positive δ, and let

$$\Omega_n = \{z \in \mathbf{C}_\infty : \sigma(z, \zeta_n) \geq \delta\},$$

then it is obvious from Theorem 2.3 that

$$\sup\{T_n^{\#}(z) : z \in \Omega_n\} \longrightarrow 0$$

as $n \to \infty$. It follows directly from this that the same conclusion holds if we replace ζ_n by any sequence z_n that is asymptotic to ζ_n, and this is so for any strongly exceptional sequence. In fact, one can do better than (6.3) as we have already indicated in Theorem 1.3 (9). Now let

$$\tilde{\Omega}_n = \{z \in \mathbf{C}_\infty : \sigma(z, \zeta_n) \geq r_n\}.$$

Providing that $r_n\|T_n\| \to +\infty$, we obtain

$$\sup\{T_n^{\#}(z) : z \in \tilde{\Omega}_n\} \longrightarrow 0$$

as $n \to \infty$. The case when $r_n = \delta$ is a special case of this, but of course we could take, say, $r_n = \|T_n\|^{-1/2}$ so that $r_n \to 0$. This now gives a quantitative version of Theorem 1.3 (9) and, of course, we could replace ζ_n by any strongly exceptional sequence in this result too providing that $r_n \to 0$ sufficiently quickly.

7. An example. In (3.1) in [**7**] the authors consider Möbius maps of the form

$$T_n(z) = \begin{cases} c_n + \varepsilon_n/(z + h_n) & \text{if } h_n \neq \infty, \\ \varepsilon_n z + d_n & \text{if } h_n = \infty, \end{cases}$$

and in their Theorem 3.1 they give sufficient conditions for T_n to be restrained. After taking account of the need to normalize the matrix for T_n, so that its determinant is one, we see from Theorem 1.3 that T_n is restrained *if and only if* the sequence

$$\begin{cases} (1+|c_n|^2+|h_n|^2+|c_nh_n+\varepsilon_n|^2)/|\varepsilon_n| & \text{if } h_n \neq \infty \\ (1+|d_n|^2+|\varepsilon_n|^2)/|\varepsilon_n| & \text{if } h_n = \infty \end{cases}$$

converges to $+\infty$ as $n \to \infty$. Note that this holds if $\varepsilon_n \to 0$, and this is stronger than Theorem 3.1 in [**7**].

REFERENCES

1. B. Aebischer, *The limiting behaviour of sequences of Möbius transformations*, Math. Z. **205** (1990), 49–59.

2. L.V. Ahlfors, *Möbius transformations in several dimensions*, Univ. Minnesota, 1981.

3. A.F. Beardon, *The geometry of discrete groups*, Graduate Texts in Math., vol. 91, Springer, New York, 1983.

4. A.F. Beardon and J.B. Wilker, *The norm of a Mobius transformation*, Math. Proc. Cambridge Philos. Soc. **96** (1984), 301–308.

5. L.R. Ford, *Automorphic functions*, Chelsea Publ. Co., 2nd ed., New York, 1951.

6. L. Jacobsen, *General convergence of continued fractions*, Trans. Amer. Math. Soc. **294** (1986), 477–485.

7. L. Jacobsen and W.J. Thron, *Limiting structures for sequences of linear fractional transformations*, Proc. Amer. Math. Soc. **99** (1987), 141–146.

8. P.J. Nicholls, *The ergodic theory of discrete groups*, London Math. Soc. Lecture Notes, vol. 143, Cambridge Univ. Press, Cambridge, 1989.

9. G. Piranian and W.J. Thron, *Convergence properties of sequences of linear fractional transformations*, Michigan Math. J. **4** (1957), 129–135.

10. J. Ratcliffe, *Foundations of hyperbolic manifolds*, Graduate Texts in Math., vol. 149, Springer, New York, 1994.

11. W.J. Thron and H. Waadeland, *Modifications of continued fractions, a survey*, Lecture Notes in Math., vol. 932, Springer, New York, 1982, 38–66.

Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB, England
E-mail address: A.F.Beardon@pmms.cam.ac.uk

Faculty of Physics, Informatics and Mathematics, Norwegian University of Science and Technology, N-7941 Trondheim, Norway
E-mail address: lisa@math.ntnu.no