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INFINITESIMAL NEIGHBORHOODS OF INFINITE-DIMENSIONAL COMPLEX PROJECTIVE SPACES

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ABSTRACT. Let V be an infinite dimensional complex Banach space and Y a complex Banach manifold containing $X := \mathbf{P}(V)$ as a codimension r closed split submanifold. Assume that X admits smooth partitions of unity and that there is $\mathcal{O}_Y(1) \in \mathrm{Pic}\,(Y)$ such that $\mathcal{O}_Y(1)|X \cong \mathcal{O}_X(1)$. Fix an integer $n \geq 1$ and a finite rank holomorphic vector bundle E on the order n infinitesimal neighborhood $X^{(n)}$ of X in V. Set $s := \mathrm{rank}\,(E)$. Then $H^1(X^{(n)}, E) = 0$ and there are uniquely determined integers a_i, $1 \leq i \leq s$, such that $a_1 \geq \cdots \geq a_s$ and $E \cong \mathcal{O}_{X^{(n)}}(a_1) \oplus \cdots \oplus \mathcal{O}_{X^{(n)}}(a_s)$.

1. Introduction. Let V be a complex Banach space and $\mathbf{P}(V)$ the projective space of all its one-dimensional subspaces. We assume that $\mathbf{P}(V)$ admits smooth partitions of unity. For instance this is the case if V is a separable Hilbert space. Set $X := \mathbf{P}(V)$ and let Y be a complex Banach manifold containing X as a closed split submanifold; we recall that X is a split submanifold of Y if for every $P \in X$ there is an open neighborhood U of P in Y and a holomorphic submersion $f : U \to W$, W open neighborhood of 0 in $\mathbf{C}^r$ such that $U \cap X = f^{-1}(0)$. The integer r is the codimension of X in Y. For every integer $n \geq 0$ let $X^{(n)}$ be the infinitesimal neighborhood of order n of X in Y, i.e., the unreduced complex analytic subspace of Y with $\mathcal{I}_X^{n+1}$ as ideal sheaf; in the chart (U, f) with $f = (z_1, \dots, z_r)$ the complex space $U \cap X^{(n)}$ is defined by all monomials of degree $n+1$ in the variables $z_1, \dots, z_r$. We prove the following result.

Theorem 1. *Let V be an infinite dimensional complex Banach space such that $X := \mathbf{P}(V)$ admits smooth partitions of unity and Y a complex Banach manifold containing X as a codimension r closed*

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split submanifold. Assume the existence of $\mathcal{O}_Y(1) \in Pic(Y)$ *such that* $\mathcal{O}_Y(1)|X \cong \mathcal{O}_X(1)$. *Fix an integer* $n \geq 1$ *and a holomorphic vector bundle* E *on* $X^{(n)}$ *with finite rank. Set* $s := rank(E)$. *Then* $H^1(X^{(n)}, E) = 0$ *and there are uniquely determined integers* a_i, $1 \leq i \leq s$, *such that* $a_1 \geq \cdots \geq a_s$ *and* $E \cong \mathcal{O}_{X^{(n)}}(a_1) \oplus \cdots \oplus \mathcal{O}_{X^{(n)}}(a_s)$.

2. The proof. For any Banach space V the projective space $\mathbf{P}(V)$ is metrizable (use the Fubini-Study metric) and hence it is paracompact. $\mathbf{P}(V)$ is covered by charts biholomorphic to closed hyperplanes, H, of V. Thus $\mathbf{P}(V)$ admits smooth partitions of unity if H admits smooth partitions of unity. For instance this is the case if V is a separable Hilbert space. For many more examples, see [**2**, Section 8].

Lemma 1. *Fix* V, X, Y *and* n *as in the statement of Theorem* 1. *For every holomorphic line bundle* L *on* $X^{(n)}$ *there is a unique integer* t *such that* L *is the unique line bundle whose restriction to* X *is isomorphic to the degree* t *line bundle* $\mathcal{O}_X(t)$; *set* $L := \mathcal{O}_{X^{(n)}}(t)$. *Conversely, for every integer* t *there is a holomorphic line bundle* L *on* $X^{(n)}$ *such that* $L|X \cong \mathcal{O}_X(t)$.

Proof. We will follow [**3**, Section 8]. For any complex space T (even not reduced) the group $\mathrm{Pic}\,(T)$ of isomorphism classes of holomorphic line bundles on T is isomorphic to $H^1(T, \mathcal{O}_T^*)$. For any reduced complex space B there is an exponential sequence

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathcal{O}_B \rightarrow \mathcal{O}_B^* \longrightarrow 0 \tag{1}$$

induced by the exponential sequence of abelian groups

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathbf{C} \longrightarrow \mathbf{C}^* \longrightarrow 0 \tag{2}$$

For every integer $i \geq 0$ we have an exact sequence ([**1**], p. 446):

$$0 \longrightarrow S^i(N^*) \longrightarrow \mathcal{O}_{X^{(i+1)}}^* \longrightarrow \mathcal{O}_{X^{(i)}}^* \longrightarrow 0 \tag{3}$$

Since $H^1(X, S^i(N^*)) = 0$ for every $i \geq 0$, [**1**, Remark 5], we obtain that for all integers $i \geq 0$ the restriction map $\rho : \mathrm{Pic}\,(X^{(i+1)}) \rightarrow \mathrm{Pic}\,(X^{(i)})$ is injective. Our assumption on Y gives the surjectivity of ρ. $\square$

Proof of Theorem 1. Since X is a split closed submanifold of codimension r of Y, the normal bundle N of X in Y is a rank r holomorphic vector bundle on X. For every integer $i \geq 1$ we have $\mathcal{I}_X^i/\mathcal{I}_X^{i+1} \cong S^i(N^*)$ (symmetric product). Hence for every integer i with $0 \leq i < n$ we have an exact sequence of sheaves with X as support:

$$(4) \qquad 0 \longrightarrow S^i(N^*) \longrightarrow \mathcal{O}_{X^{(i+1)}} \longrightarrow \mathcal{O}_{X^{(i)}} \longrightarrow 0$$

In (4) the term $S^i(N^*)$ may be seen as a holomorphic vector bundle with finite rank on X. For every integer i with $0 \leq i \leq n$ set $E_i := E|X^{(i)}$. By tensoring (4) with E we obtain the following exact sequence of sheaves with X as support:

$$(5) \qquad 0 \longrightarrow S^i(N^*) \otimes E_0 \longrightarrow E_{i+1} \longrightarrow E_i \longrightarrow 0.$$

The sheaf $S^i(N^*) \otimes E_0$ is a holomorphic vector bundle on X with finite rank. By [**1**, Remark 5], for every holomorphic vector bundle A on X with finite rank we have $H^1(X, A) = 0$. Hence from (5) for $i = 0$ we obtain $H^1(X^{(n)}, E) = 0$ when $n = 1$. Now assume $n \geq 2$ and that this vanishing is true for the integer $n-1$. By [**1**, Remark 5], we have $H^1(X, S^i(N^*) \otimes E_0) = 0$. Apply the exact sequence (5) for the integer $n-1$ and the inductive assumption. After n steps we obtain $H^1(X^{(n)}, E) = 0$. By [**4**, Theorems 8.5 and 7.1], there exist uniquely determined integers a_i, $1 \leq i \leq s$ such that $a_1 \geq \cdots \geq a_s$ and $E|X \cong \mathcal{O}_X(a_1) \oplus \cdots \oplus \mathcal{O}_X(a_s)$. Set $F := \mathcal{O}_{X^{(n)}}(a_1) \oplus \cdots \oplus \mathcal{O}_{X^{(n)}}(a_s)$. For all integers i with $0 \leq i \leq n$ set $E_i : E|X^{(i)}$ and $F_i : F|X^{(i)}$. By assumption there is an isomorphism $\psi_0 : E_0 \to F_0$. Notice that $\mathrm{Hom}\,(E_i, F_i) \cong \mathrm{Hom}\,(E, F)|X^{(i)}$. By [**1**, Remark 5], for all integers i with $0 \leq i < n$ we have $H^1(X, \mathrm{Hom}\,(E_0, F_0) \otimes S^i(N^*)) = 0$. Hence inductively for each integer i such that $0 \leq i < n$ we find $\psi_{i+1} : E_{i+1} \to F_{i+1}$ such that $\psi_{i+1}|X^{(i)} = \psi_i$, use (5). In particular we have $\psi_n|X = \psi_0$. Thus ψ_n is an isomorphism (Nakayama's Lemma), concluding the proof. □

Remark 1. By the proof of Theorem 1 we have $H^x(X^{(n)}, E) = 0$ for some $x > 0$ for any Banach space V such that $H^x(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)}(t)) = 0$ for all integers t (sheaf cohomology or Čech cohomology). It would be important to prove [**4**, Theorem 7.3], for sheaf cohomology (not just Dolbeaut cohomology) at least if V is a separable Hilbert space.

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