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# A NEW INTEGRAL REPRESENTATION OF THE RIEMANN ZETA FUNCTION

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ABSTRACT. The series $\sum_{n=1}^{\infty}(1/n^{l+1})e^{-z^k/n^k}$, $k$ is any positive integer, $l$ is a positive odd number and $l \leq 2k-1$, is studied, and for each pair $(k,l)$, an integral representation of the Riemann zeta function is given. For small pairs, this provides known representations.

**1. Introduction.** In [**2**], Tennenbaum discussed the series $\sum_{n=1}^{\infty}(1/n^2)e^{-z/n}$ and mainly obtained a proof of the functional equation of the Riemann zeta function. In [**6**] Zhang studied the series $\sum_{n=1}^{\infty}(1/n^2)e^{-z^2/n^2}$ and gave two integral representations and three different proofs of the functional equation of the Riemann zeta function. In [**4**], Wu researched the series $\sum_{n=1}^{\infty}(1/n^{k+1})e^{-z^{2k}/n^{2k}}$ and generalized all results in [**6**]. In [**5**], Wu discussed the series $\sum_{n=1}^{\infty} n^{2t}/(n^{2k}+x^{2k})$ and deduced integral representations for the Riemann zeta function which hold for $\mathrm{Re}\,(s) > 1$. Now in this paper we study the series $\sum_{n=1}^{\infty}(1/n^{l+1})e^{-z^k/n^k}$, where $k$ is any positive integer, $l$ is a positive odd number and $l \leq 2k-1$ and imply a new integral representation for the Riemann zeta function which holds for $-l < \mathrm{Re}\,(s) < 0$ or $\mathrm{Re}\,(s) > 0$, that is, we prove the following theorem

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**Theorem.** *For each pair $(k,l)$ and $\sigma>0$ or $-l<\sigma<0$, we have*

(1)
$$\Gamma(s)\zeta(s)=\frac{\sin(\pi(s+l)/2k)}{2\cos(\pi s/2)}\times\int_0^{+\infty}\Bigg[\sum_{m=0}^{k-1}(-1)^m\frac{v_{k,m,l}\sinh(x\lambda_{k,m})-u_{k,m,l}\sin(x\tau_{k,m})}{\cosh(x\lambda_{k,m})-\cos(x\tau_{k,m})}-\frac{\delta(s)}{\sin(\pi l/2k)}\Bigg]x^{s-1}\,dx,$$

*where*

$$\varphi_{k,m}=\frac{(2m+1)\pi}{2k},\quad \theta_{k,m,l}=\frac{(2m+1)(k-l)\pi}{2k},\quad \delta(s)=\begin{cases}0 & -l<\sigma<0,\\ 1 & \sigma>0,\end{cases}$$

$$\lambda_{k,m}=\sin\varphi_{k,m},\qquad \tau_{k,m}=\cos\varphi_{k,m},$$
$$u_{k,m,l}=\sin\theta_{k,m,l},\quad v_{k,m,l}=\cos\theta_{k,m,l},$$

*and $k$ is any positive integer, $l$ is a positive odd number and $l\le 2k-1$.*

From the theorem we can see that all results in [**2, 4**–**6**] are included as our special cases. In (1), setting $(k,l)=(1,1)$, we obtain the well-known integral representation (see [**7**] and [**3**])

$$\Gamma(s)\zeta(s)=\int_0^{+\infty}\frac{e^{-x}x^{s-1}}{1-e^{-x}}\,dx,\quad \sigma>1;$$

setting $(k,l)=(2,1)$ or $(k,l)=(2,3)$, we give all results in [**6**]; setting $k=2l$ or $3k=2l$, we deduce all results in [**4**]; setting $\sigma>1$, we achieve the integral representations in [**5**].

**2. Proof of theorem.** For convenience, first we show four lemmas. Finally we give the proof of the theorem.

**Lemma 1.** *Let* $\operatorname{Re}(s)=\sigma>0$, *$l$ a positive real number, and*

(2)
$$f_{k,l}(z)=\sum_{n=1}^{\infty}\frac{1}{n^{l+1}}\,e^{-z^k/n^k},$$

*we have*

$$\Gamma\left(\frac{s+l}{k}\right)\zeta(1-s) = k\int_0^{+\infty}\left[t^l f_{k,l}(t) - \frac{1}{k}\,\Gamma\left(\frac{l}{k}\right)\right]t^{s-1}\,dt. \tag{3}$$

*Proof*. For $\sigma > -l$, we see that

$$\Gamma\left(\frac{s+l}{k}\right) = \int_0^{+\infty} e^{-x}x^{(s+l)/k-1}\,dx.$$

Replacing $x$ by $(t/n)^k$, we obtain

$$\frac{1}{n^{1-s}}\,\Gamma\left(\frac{s+l}{k}\right) = k\int_0^{+\infty}\frac{1}{n^{l+1}}\,e^{-t^k/n^k}t^{s+l-1}\,dt.$$

Summing over all $n \geq 1$, we get

$$\Gamma\left(\frac{s+l}{k}\right)\zeta(1-s) = k\sum_{n=1}^{\infty}\int_0^{+\infty}\frac{1}{n^{l+1}}\,e^{-t^k/n^k}t^{s+l-1}\,dt.$$

Since

$$\begin{aligned} k\sum_{n=1}^{\infty}\int_0^{+\infty}&\left|\frac{1}{n^{l+1}}\,e^{-t^k/n^k}t^{s+l-1}\right|dt \\ &\leq k\sum_{n=1}^{\infty}\int_0^{+\infty}\frac{1}{n^{l+1}}\,e^{-t^k/n^k}t^{\sigma+l-1}\,dt \\ &= \Gamma\left(\frac{\sigma+l}{k}\right)\zeta(1-\sigma), \end{aligned}$$

we can interchange the order of summation and integration and obtain

$$\Gamma\left(\frac{s+l}{k}\right)\zeta(1-s) = k\int_0^{+\infty} f_{k,l}(t)t^{s+l-1}\,dt, \quad -l < \sigma < 0. \tag{4}$$

It is clear that the series (2) converges absolutely and uniformly in any bounded domain; therefore, $f_{k,l}(z)$ is an entire function.

Now we estimate approximate property on the $f_{k,l}(t)$ for $t \to +\infty$. Let $g_{k,l}(x) = 1/x^{l+1}e^{-t^k/x^k}$ and, by the Euler-Maclaurin formula, we deduce

$$\sum_{n=2}^{\infty} g_{k,l}(n) = \int_1^{+\infty} \frac{1}{x^{l+1}}\, e^{-t^k/x^k}\, dx + \sum_{m=1}^{q} \frac{(-1)^m B_m}{m!}\, g_{k,l}^{(m-1)}(x)\Big|_1^{+\infty} + R_q,$$

where

$$R_q = \frac{(-1)^{q+1}}{q!} \int_1^{+\infty} B_q(x-[x]) g_{k,l}^{(q)}(x)\, dx,$$

$B_q(x)$ is a Bernoulli polynomial, and $B_m$ is a Bernoulli number. Obviously, we have

$$g_{k,l}^{(m)}(x) = \frac{1}{x^{m+l+1}}\, P_m\left(\frac{t^k}{x^k}\right) e^{-t^k/x^k},$$

where $P_m$ is a polynomial of degree $m$. Because

$$g_{k,l}^{(m)}(x)|_1^{+\infty} = O(t^{km}e^{-t^k}), \quad t \to +\infty,$$

$$\begin{aligned} |R_q| &\le c \int_1^{+\infty} \frac{1}{x^{q+l+1}} \left|P_q\left(\frac{t^k}{x^k}\right)\right| e^{-t^k/x^k}\, dx \\ &\le \frac{c_1}{t^{q+l}} \int_0^{t^k} u^{(q+l)/k-1} P_q(u) e^{-u}\, du = O\left(\frac{1}{t^{q+l}}\right) \end{aligned}$$

and

$$\begin{aligned} \int_1^{+\infty} \frac{1}{x^{l+1}}\, e^{-t^k/x^k}\, dx &= \frac{1}{kt^l} \int_0^{t^k} y^{1/k-1} e^{-y}\, dy \\ &= \frac{1}{kt^l} \left( \int_0^{+\infty} - \int_{t^k}^{+\infty} \right) \\ &= \frac{1}{kt^l} \Gamma\left(\frac{l}{k}\right) + O\left(\frac{1}{t^k} e^{-t^k}\right), \end{aligned}$$

we deduce

$$f_{k,l}(t) = \frac{1}{kt^l}\, \Gamma\left(\frac{l}{k}\right) + O\left(\frac{1}{t^{q+l}}\right), \quad t \to +\infty, \tag{5}$$

where $q$ is any positive integer.

For $-l < \sigma < 0$, by (4) and (5) we have

(6)
$$\Gamma\left(\frac{s+l}{k}\right)\zeta(1-s) = k\int_0^1 f_{k,l}(t)t^{s+l-1}\,dt + k\int_1^{+\infty}\left[f_{k,l}(t) - \frac{1}{kt^l}\,\Gamma\left(\frac{l}{k}\right)\right]t^{s+l-1}\,dt - \frac{1}{s}\,\Gamma\left(\frac{l}{k}\right).$$

The first integral of the right side in (6) is an analytic function of $s$ in the half-plane $\sigma > -l$. The second integral of right side in (6) is an entire function of $s$. Therefore, (6) provides an analytic continuation, that is, $\Gamma((s+l)/k)\zeta(1-s)$ is analytic for all $\sigma > -l$ except for a simple pole at $s = 0$ with residue $\Gamma(l/k)$. Noting that

$$\int_0^1 x^{s-1}\,dx = \frac{1}{s}, \quad \sigma > 0,$$

and by (6) we deduce (3).

**Lemma 2.** *Let*

(7)
$$h_{k,l}(x) = k\int_0^{+\infty}\left[t^{k-1}f_{k,l}(t) - \frac{1}{k}\,\Gamma\left(\frac{l}{k}\right)t^{k-l-1}\right]\sin\left(\frac{t}{x}\right)^k dt, \quad x > 0,$$

*we have*

$$h_{k,l}(x) = x^k\sum_{n=1}^{\infty}\frac{n^{2k-l-1}}{n^{2k}+x^{2k}} - \frac{\pi}{2k\sin(\pi l/2k)}\,x^{k-l}. \tag{8}$$

*Proof.* By (7), we have

$$h_{k,l}(x) = k\int_0^{+\infty} t^{k-1}f_{k,l}(t)\sin\left(\frac{t}{x}\right)^k dt - \Gamma\left(\frac{l}{k}\right)\int_0^{+\infty} t^{k-l-1}\sin\left(\frac{t}{x}\right)^k dt$$

$$
\begin{aligned}
&= -x^k \int_0^{+\infty} f_{k,l}(t)\, d\cos\left(\frac{t}{x}\right)^k \\
&\quad - \Gamma\left(\frac{l}{k}\right) x^{k-l} \int_0^{+\infty} t^{k-l-1} \sin t^k\, dt \\
&= x^k f_{k,l}(0) - kx^k \int_0^{+\infty} \sum_{n=1}^{\infty} \frac{1}{n^{k+l+1}}\, e^{-t^k/n^k} t^{k-1} \cos\left(\frac{t}{k}\right)^k dt \\
&\quad - \frac{1}{k}\, x^{k-l} \Gamma\left(\frac{l}{k}\right) \Gamma\left(\frac{k-l}{k}\right) \sin\frac{\pi(k-l)}{2k} \\
&= x^k f_{k,l}(0) - kx^k \int_0^{+\infty} \sum_{n=1}^{\infty} \frac{1}{n^{k+l+1}}\, e^{-t^k/n^k} t^{k-1} \cos\left(\frac{t}{k}\right)^k dt \\
&\quad - \frac{\pi x^{k-l}}{2k \sin(\pi l/2k)}.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \int_0^{+\infty} \left| \frac{1}{n^{k+l+1}}\, e^{-t^k/n^k} k t^{k-1} \cos\left(\frac{t}{k}\right)^k \right| dt \\
&\qquad \le \sum_{n=1}^{\infty} \frac{1}{n^{k+l+1}} \int_0^{+\infty} e^{-t^k/n^k} k t^{k-1}\, dt = \zeta(l+1),
\end{aligned}
$$

we can interchange the order of summation and integration, and obtain

$$
\begin{aligned}
h_{k,l}(x) &= x^k f_{k,l}(0) - x^k \sum_{n=1}^{\infty} \frac{1}{n^{k+l+1}} \int_0^{+\infty} e^{-u/n^k} \cos\frac{u}{x^k}\, du - \frac{\pi x^{k-l}}{2k \sin(\pi l/2k)} \\
&= x^k f_{k,l}(0) - x^k \sum_{n=1}^{\infty} \frac{x^{2k}}{n^{l+1}(n^{2k}+x^{2k})} - \frac{\pi x^{k-l}}{2k \sin(\pi l/2k)} \\
&= x^k \sum_{n=1}^{\infty} \frac{n^{2k-l-1}}{n^{2k}+x^{2k}} - \frac{\pi x^{k-l}}{2k \sin(\pi l/2k)}.
\end{aligned}
$$

This proves Lemma 2. □

**Lemma 3.** *Let* $S_{k,a}(x) = \sum_{n=1}^{\infty}(n^{2a})/(n^{2k}+x^{2k})$, $2a = 2k-l-1$, $l$ *is a positive odd number and* $l \leq 2k-1$, *we have*

(9)
$$S_{k,a}(x) = \frac{\pi}{2kx^l\lambda_{k,a}} + \frac{\pi}{kx^l}\sum_{m=0}^{k-1}(-1)^m\sum_{n=1}^{\infty}e^{-2n\pi x\lambda_{k,m}}\cos(2n\pi x\tau_{k,m}+\theta_{k,m,l}), \quad 1 \leq a \leq k-1,$$

(10)
$$S_{k,0}(x) = -\frac{1}{2x^{2k}} + \frac{\pi}{2kx^{2k-1}\lambda_{k,0}} + \frac{\pi}{kx^{2k-1}}\sum_{m=0}^{k-1}(-1)^m\sum_{n=1}^{\infty}e^{-2n\pi x\lambda_{k,m}}\cos(2n\pi x\tau_{k,m}+\theta_{k,m,2k-1})$$

and

(11)
$$S_{k,a}(x) = \frac{\pi}{2kx^l}\sum_{m=0}^{k-1}(-1)^m\frac{v_{k,m,l}\sinh(2\pi x\lambda_{k,m}) - u_{k,m,l}\sin(2\pi x\tau_{k,m})}{\cosh(2\pi x\lambda_{k,m}) - \cos(2\pi x\tau_{k,m})}, \quad 1 \leq a \leq k-1,$$

(12)
$$S_{k,0}(x) = -\frac{1}{2x^{2k}} + \frac{\pi}{2kx^{2k-1}} \times \sum_{m=0}^{k-1}(-1)^m\frac{v_{k,m,2k-1}\sinh(2\pi x\lambda_{k,m}) - u_{k,m,2k-1}\sin(2\pi x\tau_{k,m})}{\cosh(2\pi x\lambda_{k,m}) - \cos(2\pi x\tau_{k,m})}.$$

*Proof.* Let
$$F_{k,a}(z) = \frac{z^{2a}}{z^{2k}+x^{2k}}\cot\pi z;$$

we consider the contour integral $\int_{|z|=R} F_{k,a}(z)\,dz$, where $R$ is no integer. By the residue theorem, and letting $R \to +\infty$, we achieve

(13)
$$\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{n^{2a}}{n^{2k}+x^{2k}}+\sum_{m=0}^{k-1}[\mathrm{Res}\,(F_{k,a}(z),z_m)+\mathrm{Res}\,(F_{k,a}(z),\bar{z}_m)]=0,$$

where $z_m = xe^{(2m+1)\pi i/2k}$, $m = 0,1,\ldots,k-1$. Writing $z_{\delta_m} = z_m e^{\delta\pi i/k}$, $0<|\delta|<1$, noting that

$$\begin{aligned}\prod_{n=0}^{k-1}&(z_{\delta_m}-z_n)(z_{\delta_m}-\bar{z}_n)\\ &=2^{2k}x^k z_{\delta_m}^k\prod_{n=0}^{k-1}\sin\left(\frac{n}{2k}-\frac{m+\delta}{2k}\right)\pi\sin\left(\frac{n+1}{2k}+\frac{m+\delta}{2k}\right)\pi\\ &=2^{2k}x^k z_{\delta_m}^k\prod_{n=0}^{2k-1}\sin\left(\frac{n}{2k}-\frac{m+\delta}{2k}\right)\pi\\ &=-2x^k z_{\delta_m}^k\sin(m+\delta)\pi,\end{aligned}$$

we obtain

$$\begin{aligned}\mathrm{Res}\,(F_{k,a}(z),z_m)&=\lim_{\delta\to 0}\frac{(z_{\delta_m}-z_m)z_{\delta_m}^{2a}}{(-1)^{m+1}2x^k z_{\delta_m}^k\sin\delta\pi}\cot(\pi z_{\delta_m})\\ &=\lim_{\delta\to 0}\frac{(1-e^{-\delta\pi i/k})z_{\delta_m}^{2a-k+1}}{(-1)^{m+1}2x^k\sin\delta\pi}\cot(\pi z_{\delta_m})\\ &=\frac{(-1)^m z_m^{2a-k+1}}{2kix^k}\cot(\pi z_m)\end{aligned}$$

and

$$\mathrm{Res}\,(F_{k,a}(z),\bar{z}_m)=\frac{(-1)^{m+1}\bar{z}_m^{2a-k+1}}{2kix^k}\cot(\pi\bar{z}_m).$$

Therefore, (13) gives us

$$S_{k,a}(x)=-\frac{\pi}{2kx^k}\mathrm{Re}\left\{\frac{1}{i}\sum_{m=0}^{k-1}(-1)^m z_m^{2a-k+1}\cot(\pi z_m)\right\},\quad a\neq 0.$$

Recalling the formula

$$\cot \pi z = i\biggl(1+\frac{2}{e^{2\pi i z}-1}\biggr),$$

we have

$$(14)\quad S_{k,a}(x) = \frac{-\pi}{2kx^l}\,\mathrm{Re}\biggl\{\sum_{m=0}^{k-1}(-1)^m \times e^{i\theta_{k,m,l}}\biggl(1+\frac{2}{e^{-2\pi x\lambda_{k,m}}e^{2\pi i x\tau_{k,m}}-1}\biggr)\biggr\},\quad a\neq 0.$$

Since

$$\begin{aligned}\mathrm{Re}\biggl\{\sum_{m=0}^{k-1}(-1)^m e^{i\theta_{k,m,l}}\biggr\} &= \mathrm{Re}\biggl\{e^{(2a-k+1)\pi i/2k}\sum_{m=0}^{k-1}(-1)^m e^{2m(2a-k+1)\pi i/2k}\biggr\}\\ &= \frac{1}{\sin(2a+1)\pi/(2k)} = \frac{1}{\lambda_{k,a}} = \frac{1}{\sin(\pi l/2k)},\end{aligned}$$

we can expand (14) into the power series

$$S_{k,a}(x) = \frac{\pi}{kx^l}\biggl[\frac{1}{2\lambda_{k,a}}+\mathrm{Re}\biggl\{\sum_{m=0}^{k-1}(-1)^m e^{i\theta_{k,m,l}}\sum_{n=0}^{\infty}e^{-2n\pi x\lambda_{k,m}+2n\pi i x\tau_{k,m}}\biggr\}\biggr],\quad a\neq 0.$$

This implies (9). On the other hand, by (14), we deduce

$$S_{k,a}(x) = \frac{\pi}{2kx^l}\mathrm{Re}\biggl\{\sum_{m=0}^{k-1}(-1)^m e^{i\theta_{k,m,l}}\frac{\sinh(2\pi x\lambda_{k,m})+i\sin(2\pi x\tau_{k,m})}{\cosh(2\pi x\lambda_{k,m})-\cos(2\pi x\tau_{k,m})}\biggr\},\quad a\neq 0,$$

which implies (11). Similar to (9) and (11), we achieve (10) and (12).

**Lemma 4.** *Let*

$$I_{k,l}(s)=\int_0^{+\infty} h_{k,l}(x)x^{s-k+l-1}\,dx, \tag{15}$$

*and, for $\sigma>0$ or $-l<\sigma<0$, we have*

$$\begin{aligned} I_{k,l}(s)=\frac{\pi}{2k}\int_0^{+\infty}\Bigg[\sum_{m=0}^{k-1}(-1)^m \\ \times\frac{v_{k,m,l}\sinh(2\pi x\lambda_{k,m})-u_{k,m,l}\sin(2\pi x\tau_{k,m})}{\cosh(2\pi x\lambda_{k,m})-\cos(2\pi x\tau_{k,m})}-\frac{\delta(s)}{\lambda_{k,a}}\Bigg]x^{s-1}\,dx \end{aligned} \tag{16}$$

*and*

$$I_{k,l}(s)=\frac{\pi}{(2\pi)^s k}\,\Gamma(s)\zeta(s)\sum_{m=0}^{k-1}(-1)^m\cos\left[s\left(\frac{\pi}{2}-\varphi_{k,m}\right)+\theta_{k,m,l}\right], \tag{17}$$

*where*

$$\delta(s)=\begin{cases}0 & -l<\sigma<0,\\ 1 & \sigma>0.\end{cases}$$

*Proof.* First we give that (16) and (17) hold for $l<2k-1$. Combining (15) and (8), we have

$$I_{k,l}(s)=\int_0^{+\infty}\left[S_{k,a}(x)-\frac{\pi}{2kx^l\sin(\pi l/2k)}\right]x^{s+l-1}\,dx. \tag{18}$$

Obviously, (9) gives us

$$S_{k,a}(x)-\frac{\pi}{2kx^l\sin(\pi l/2k)}=O\left(\frac{1}{x^l}\,e^{-2nx\lambda_0}\right),\quad x\to+\infty;$$

therefore, $I_{k,l}(s)$ is an analytic function of $s$ for $\sigma>0$. For $\sigma>0$ and by (18), we have

$$\begin{aligned} I_{k,l}(s)=\int_0^1+\int_1^{+\infty}=\int_0^1 S_{k,a}(x)x^{s+l-1}\,dx-\frac{\pi}{2sk\lambda_{k,a}} \\ +\int_1^{+\infty}\left[S_{k,a}(x)-\frac{\pi}{2k\lambda_{k,a}x^l}\right]x^{s+l-1}\,dx. \end{aligned} \tag{19}$$

Similar to (6), we can see that $I_{k,l}(s)$ is analytic for all $\sigma > -l$ except for a simple pole at $s = 0$ with residue $\pi/2k\lambda_{k,a}$. Noting that

$$\frac{\pi}{2k\lambda_{k,a}} \int_1^{+\infty} x^{s-1}\, dx = -\frac{\pi}{2sk\lambda_{k,a}}, \quad -l < \sigma < 0,$$

we have (16). On the other hand, combining (18) and (9), we have

(20)

$$I_{k,l}(s) = \frac{\pi}{k} \int_0^{+\infty} \sum_{m=0}^{k-1} (-1)^m \sum_{n=1}^{\infty} e^{-2n\pi x \lambda_{k,m}} \cos(2n\pi x \tau_{k,m} + \theta_{k,m,l}) x^{s-1}\, dx.$$

Since, for $\sigma > 1$,

$$\sum_{n=1}^{\infty} \int_0^{+\infty} \left| e^{-2n\pi x \lambda_{k,m}} \cos(2n\pi x \tau_{k,m} + \theta_{k,m,l}) x^{s-1} \right| dx$$
$$\leq \sum_{n=1}^{\infty} \int_0^{+\infty} e^{-2n\pi x \lambda_{k,m}} x^{\sigma-1}\, dx \leq \frac{1}{(2\pi\lambda_{k,m})^{\sigma}} \Gamma(\sigma)\zeta(\sigma),$$

we can calculate the integral in (20) term by term and obtain

$$\begin{aligned} I_{k,l}(s) &= \frac{\pi}{k} \sum_{m=0}^{k-1} (-1)^m \sum_{n=1}^{\infty} \int_0^{+\infty} e^{-2n\pi x \lambda_{k,m}} \cos(2n\pi x \tau_{k,m} + \theta_{k,m,l}) x^{s-1}\, dx \\ &= \frac{\pi}{k}\, \zeta(s) \sum_{m=0}^{k-1} \frac{(-1)^m}{(2\pi\lambda_{k,m})^s} \int_0^{+\infty} e^{-u} \cos(u \cot\varphi_{k,m} + \theta_{k,m,l}) u^{s-1}\, du. \end{aligned}$$

Recalling the formula

$$\int_0^{+\infty} t^{s-1} e^{-(p+iq)t}\, dt = \frac{\Gamma(s)}{(p^2+q^2)^{s/2}}\, e^{-is \arctan(q/p)}, \quad p, \sigma > 0,$$

we have

$$I_{k,l}(s) = \frac{\pi}{k(2\pi)^s} \Gamma(s)\zeta(s) \sum_{m=0}^{k-1} (-1)^m \cos\left[ s\left(\frac{\pi}{2} - \varphi_{k,m}\right) + \theta_{k,m,l} \right], \quad \sigma > 1.$$

Obviously, (17) holds for $\sigma > 0$ or $-l < \sigma < 0$ by analytic continuation. Similarly, we obtain that (16) and (17) hold for $l = 2k-1$. This proves Lemma 4. ◻

*Proof of Theorem.* Combining (7) and (15), we have

(21)

$$\begin{aligned}
&I_{k,l}(s)\\
&= \int_0^{+\infty} x^{s-k+l-1}\,dx \int_0^{+\infty} k\left[t^{k-1} f_{k,l}(t) - \frac{1}{k}\Gamma\left(\frac{l}{k}\right)t^{k-l-1}\right]\sin\left(\frac{t}{x}\right)^k dt\\
&= k\int_0^{+\infty}\left[t^{k-1} f_{k,l}(t) - \frac{1}{k}\Gamma\left(\frac{l}{k}\right)t^{k-l-1}\right]dt \int_0^{+\infty}\sin\left(\frac{t}{x}\right)^k x^{s-k+l-1}\,dx\\
&= \int_0^{+\infty}\left[t^{k-1} f_{k,l}(t) - \frac{1}{k}\Gamma\left(\frac{1}{k}\right)t^{k-l-1}\right]dt \int_0^{+\infty}\frac{\sin x}{x^{(s+l)/k}}\,dx\\
&= \frac{1}{k}\Gamma\left(\frac{s+l}{k}\right)\zeta(1-s)\Gamma\left(\frac{k-s-l}{k}\right)\sin\frac{\pi(k-s-l)}{2k}\\
&= \frac{\pi\zeta(1-s)}{2k\sin(\pi(s+l)/2k)}.
\end{aligned}$$

Recalling the functional equation

$$\zeta(1-s) = \frac{2}{(2\pi)^s}\,\Gamma(s)\zeta(s)\cos\frac{\pi s}{2}, \tag{22}$$

and, combining (17), (21) and (22), we have

$$\cos\frac{\pi s}{2} = \sin\frac{\pi(s+l)}{2k}\sum_{m=1}^{k-1}(-1)^m\cos\left[s\left(\frac{\pi}{2}-\varphi_{k,m}\right)+\theta_{k,m,l}\right]. \tag{23}$$

By combining (16), (17) and (23), and replacing $2\pi x$ by $x$, we deduce (1), and the theorem is complete. ◻

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