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DEPTH FORMULAS, RESTRICTED TOR-DIMENSION UNDER BASE CHANGE

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ABSTRACT. Let R be a commutative Noetherian ring, and let M and N be R-modules. It is shown that

$$\sup\{i \mid \operatorname{Tor}_i^R(M,N)\neq 0\} = \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \mid \mathfrak{p}\in \operatorname{Supp} M \cap \operatorname{Supp} N\}$$

provided that M has finite flat dimension. Assume that R is a complete local ring, M a finitely generated R-module, and N an R-module of finite flat dimension. It is then proved that

$$\sup\{i \mid \operatorname{Ext}_R^i(N,M) \neq 0\} = \operatorname{depth} R - \operatorname{depth} N.$$

Set

$$\operatorname{Td}_R M = \sup\{i \in \mathbf{N}_0 \mid \operatorname{Tor}_i^R(T,M) \neq 0 \text{ for some } T \text{ of finite flat dimension}\}.$$

In addition, some results concerning $\operatorname{Td}_R M$ under base change are given.

1. Introduction. Throughout this paper all rings are assumed to be commutative and Noetherian. It is well known that flat dimension of an R-module M can be computed by the following formula

$$\operatorname{fd}_R M = \sup\{i \in \mathbf{N}_0 \mid \operatorname{Tor}_i^R(T,M) \neq 0 \text{ for some } R\text{-module } T\}.$$

If flat dimension of M is finite, then it can be computed by Chouinard's formula [**5**, Corollary 1.2]; $\operatorname{fd}_R M = \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} | \mathfrak{p} \in \operatorname{Spec} R\}$. Foxby has studied the restricted Tor-dimension: $\operatorname{Td}_R M = \sup\{i \in \mathbf{N}_0 \mid \operatorname{Tor}_i^R(T,M) \neq 0$ for some T with $\operatorname{fd}_R T < \infty\}$. Over a ring of finite Krull dimension it is easy to see that $\operatorname{Td}_R M \leq \dim R < \infty$ for any R-module M. In this case $\operatorname{Td}_R M$

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is equal to $\sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}$, cf. [**6**, Theorem 5.3.6]. The restricted Tor-dimension is a refinement of the flat dimension, i.e., $\operatorname{Td}_R M \leq \operatorname{fd}_R M$, and equality holds if $\operatorname{fd}_R M$ is finite.

In [**11**, Theorem 2.2], Jorgensen has given a generalization of the Auslander-Buchsbaum formula by proving the following result:

$$\begin{aligned}&\sup\{i \in \mathbf{N}_0 \mid \operatorname{Tor}_i^R(M,N) \neq 0\}\\ &\quad = \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp} M \cap \operatorname{Supp} N\}\end{aligned}$$

while M and N are finitely generated modules over a local ring R and $\operatorname{pd}_R M < \infty$. In Section 2 we prove this equality for any R-modules M and N while $\operatorname{fd}_R M < \infty$ without assuming that M and N are finitely generated and R is local. In this section the following result is also given (Theorem 2.5). Let R be a complete local ring, M a finitely generated R-module and N an R-module of finite flat dimension. Then

$$\sup\{i \mid \operatorname{Ext}_R^i(N,M) \neq 0\} = \operatorname{depth} R - \operatorname{depth}_R N.$$

In Section 3, we study the behavior of restricted Tor-dimension under change of rings. One of the main results in this section is the following statement (Proposition 3.4). Let $\varphi : R \to S$ be a ring homomorphism. For any S-module M, $\operatorname{Td}_R M \leq \operatorname{Td}_R S + \operatorname{Td}_S M$. The equality holds for arbitrary S-module if it holds for any S-module M with $\operatorname{Td}_S M \leq 1$.

Assume that $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring, $x_1, x_2, \dots, x_r \in \mathfrak{m}$ an R-regular sequence. Set $S = R/(x_1, x_2, \dots, x_r)$. Furthermore assume that M is an R-module of finite flat dimension and $(x_1, x_2, \dots, x_r)M = 0$, such that any S-regular sequence $y_1, y_2, \dots, y_s \in \mathfrak{m}$ is a weak M-regular. Then Ischebeck has proved that $\operatorname{fd}_R M \leq r$. The following statement, Theorem 3.6 (a), generalizes this result for restricted Tor-dimension. Let $\varphi : R \to S$ be a local homomorphism with $\operatorname{cmd} S := \dim S - \operatorname{depth} S \leq 1$, and let M be an R-module and let $n \geq 0$ be an integer. Suppose that any R-regular sequence of length less than or equal to n is also M-regular. Then, for any R-module N, $\operatorname{Td}_R(M \otimes_R^{\mathbf{L}} N) \leq \operatorname{fd}_R N + \operatorname{depth} S + \operatorname{Td}_R S - n$.

In Section 4 we study the restricted covariant Ext-dimension that is a dual notion of the restricted Tor-dimension. The restricted covariant Ext-dimension of R-module M is denoted by $\operatorname{Ed}_R M$ and is defined by

$$\operatorname{Ed}_R M = \sup\{i \in \mathbf{N}_0 \mid \operatorname{Ext}_R^i(T,M) \neq 0 \text{ for some } T \text{ with } \operatorname{pd}_R T < \infty\}.$$

From the definition it is easy to see that $\mathrm{Ed}_R M \leq \dim R < \infty$.

In this paper definitions and results are formulated within the framework of the derived category of the category of modules. Thus, we outline some definitions on complexes that we use in the rest of the paper. The reader is referred to [**8**] for details of the following brief summary of the homological theory of complexes of modules.

An *R-complex* X is a sequence of R-modules X_l and R-linear maps ∂_l^X, $l \in \mathbf{Z}$,

$$X = \cdots \longrightarrow X_{l+1} \xrightarrow{\partial_{l+1}^X} X_l \xrightarrow{\partial_l^X} X_{l-1} \longrightarrow \cdots$$

such that $\partial_l^X \partial_{l+1}^X = 0$ for all $l \in \mathbf{Z}$. We call X_l and ∂_l^X the module in degree l and the lth differential of X, respectively. We identify any module M with a complex of R-modules, which has M in degree zero and is trivial elsewhere.

A homology isomorphism is a morphism $\alpha : X \to Y$ such that $H(\alpha)$ is an isomorphism; homology isomorphisms are marked by the sign $\simeq$, while $\cong$ is used for isomorphisms. The equivalence relation generated by the homology isomorphisms is also denoted by $\simeq$.

The *supremum* and *infimum* of X are defined by

$$\sup X = \sup\{l \in \mathbf{Z} \mid H_l(X) \neq 0\}$$
$$\inf X = \inf\{l \in \mathbf{Z} \mid H_l(X) \neq 0\}.$$

The symbol $\mathcal{D}(R)$ denotes the category of R-complexes and morphisms of R-complexes.

The full subcategories $\mathcal{D}_-(R)$, $\mathcal{D}_+(R)$, $\mathcal{D}_b(R)$ and $\mathcal{D}_0(R)$ of $\mathcal{D}(R)$ consist of complexes X with $X_l = 0$, for respectively $l \gg 0$, $l \ll 0$, $|l| \gg 0$ and $l \neq 0$. By $\mathcal{D}^f(R)$ we denote the full subcategory consisting of complexes X with all homology modules $H_l(X)$ finitely generated over R.

The right derived functor of the homomorphism functor of R-complexes and the left derived functor of the tensor product of R-complexes are denoted by $\mathbf{R}\mathrm{Hom}_R(-,-)$ and $- \otimes_R^{\mathbf{L}} -$, respectively.

The following results hold, cf. [**9**, Lemma 2.1].

Theorem 1.1. *Let* $X \in \mathcal{D}_+(R)$ *and* $Y \in \mathcal{D}_-(R)$. *Then* $\mathbf{R}\mathrm{Hom}_R(X,Y) \in \mathcal{D}_-(R)$ *and there is an inequality:*

$$\sup(\mathbf{R}\mathrm{Hom}_R(X,Y)) \leq \sup Y - \inf X.$$

With $i = \inf X$ *and* $s = \sup Y$, *we have*

$$H_{s-i}(\mathbf{R}\mathrm{Hom}_R(X,Y)) = \mathrm{Hom}_R(H_i(X), H_s(Y)).$$

In particular,

$$\sup(\mathbf{R}\mathrm{Hom}_R(X,Y)) = \sup Y - \inf X \Longleftrightarrow \mathrm{Hom}_R(H_i(X), H_s(Y)) \neq 0.$$

Theorem 1.2. *Let* $X, Y \in \mathcal{D}_+(R)$. *Then* $X \otimes^{\mathbf{L}}_R Y \in \mathcal{D}_+(R)$ *and there is an inequality*

$$\inf(X \otimes^{\mathbf{L}}_R Y) \geq \inf X + \inf Y.$$

With $i = \inf X$ *and* $j = \inf Y$ *we have*

$$H_{i+j}(X \otimes^{\mathbf{L}}_R Y) = H_i(X) \otimes_R H_j(Y).$$

In particular,

$$\inf(X \otimes^{\mathbf{L}}_R Y) = \inf X + \inf Y \Longleftrightarrow H_i(X) \otimes_R H_j(Y) \neq 0.$$

The injective R-module E is called faithfully injective if the functor $\mathrm{Hom}_R(-, E)$ is faithful. Every ring R admits a faithfully injective module E, e.g.,

$$E = \prod_{m \in \mathrm{Max}\, R} E(R/\mathfrak{m}),$$

where $E(R/\mathfrak{m})$ is the injective hull of $R/\mathfrak{m}$. For any faithful injective module E we use the notation $-^{\vee}$ for $\mathbf{R}\mathrm{Hom}_R(-, E)$. If E is a faithfully injective R-module, then for any $X \in \mathcal{D}(R)$ we have $\sup(X^{\vee}) = -\inf(X)$ and $\inf(X^{\vee}) = -\sup(X)$.

A complex $X \in \mathcal{D}_b(R)$ is said to be of finite *projective* (*injective* or *flat*, respectively) *dimension* if $X \simeq U$, where U is a complex of projective (injective or flat, respectively) modules and $U_l = 0$ for $|l| \gg 0$.

The full subcategories of $\mathcal{D}_b(R)$ consisting of complexes of finite projective, injective or flat dimension are denoted by $\mathcal{P}(R)$, $\mathcal{I}(R)$ and $\mathcal{F}(R)$, respectively.

Theorem 1.3. *Let $\varphi : R \to S$ be a ring homomorphism. Then the following identities hold, cf.* [**6**, pp. 179–180].

(a) *For all $Y \in \mathcal{D}_b(R)$ and $X, Z \in \mathcal{D}_b(S)$*

$$\mathbf{R}\mathrm{Hom}_R(Z \otimes^{\mathbf{L}}_S X, Y) \simeq \mathbf{R}\mathrm{Hom}_S(Z, \mathbf{R}\mathrm{Hom}_R(X, Y)).$$

(b) *Let $X, Y \in \mathcal{D}_b(S)$ and $Z \in \mathcal{D}^f_b(S)$. If $Y \in \mathcal{I}(R)$ or $Z \in \mathcal{P}(S)$, then*

$$Z \otimes^{\mathbf{L}}_S \mathbf{R}\mathrm{Hom}_R(X, Y) \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_S(Z, X), Y).$$

The *support* of the complex X, $\mathrm{Supp}\, X$, consists of all $\mathfrak{p} \in \mathrm{Spec}\, R$ with the localization $X_\mathfrak{p}$ not homologically trivial. Thus $\mathrm{Supp}\, X = \{\mathfrak{p} \in \mathrm{Spec}\, R \mid X_\mathfrak{p} \not\simeq 0\}$.

The (*Krull*) *dimension* of a complex $X \in \mathcal{D}_+(R)$ is defined in terms of the (Krull) dimensions of its homology modules by the formula $\dim_R X = \sup\{\dim_R H_l(X) - l \mid l \in \mathbf{Z}\}$, with the convention that the dimension of the zero module is equal to $-\infty$.

Over a local ring $(R, \mathfrak{m}, k)$ the *depth* of a complex $X \in \mathcal{D}_-(R)$ is defined as

$$\mathrm{depth}_R X = -\sup(\mathbf{R}\mathrm{Hom}_R(k, X)).$$

The following result is immediate by Theorem 1.1.

Theorem 1.4. *For $X \in \mathcal{D}_-(R)$,*

$$\mathrm{depth}_{R_\mathfrak{p}} X_\mathfrak{p} \geq -\sup X_\mathfrak{p} \geq -\sup X,$$

and for X not homologically isomorphic to 0;

$$\mathfrak{p} \in \mathrm{Ass}_R(H_{\sup X}(X)) \Longleftrightarrow \mathrm{depth}_{R_\mathfrak{p}} X_\mathfrak{p} = -\sup X.$$

For the homological dimensions of the complexes of modules the following results hold, cf. [**6**, pp. 181–182].

Theorem 1.5. *For any* $X \in \mathcal{D}_b(R)$,

(a) $\mathrm{pd}_R X = \sup\{\inf W - \inf(\mathbf{R}\mathrm{Hom}_R(X, W)) \mid W \in \mathcal{D}_b(R) \wedge H(W) \neq 0\}$;

(b) $\mathrm{id}_R X = \sup\{-\sup U - \inf(\mathbf{R}\mathrm{Hom}_R(U, X)) \mid U \in \mathcal{D}_b(R) \wedge H(U) \neq 0\}$; *and*

(c) $\mathrm{fd}_R X = \sup\{\sup(U \otimes^{\mathbf{L}}_R X) - \sup U \mid U \in \mathcal{D}_b(R) \wedge H(U) \neq 0\}$.

2. The Auslander-Buchsbaum formula. In this section a generalization of the Auslander-Buchsbaum formula is given. First we bring the general version of the Auslander-Buchsbaum formula [**10**, Theorem 4.1].

Theorem 2.1. *Let* R *be local. If* $X \in \mathcal{F}(R)$ *and* $Y \in \mathcal{D}_b(R)$, *then*

$$\mathrm{depth}_R(X \otimes^{\mathbf{L}}_R Y) = \mathrm{depth}_R X + \mathrm{depth}_R Y - \mathrm{depth} R.$$

Theorem 2.2. *Let* $X, Y \in \mathcal{D}_b(R)$ *with* $\mathit{fd}\, X < \infty$. *Then*

$$\sup(X \otimes^{\mathbf{L}}_R Y) = \sup\{\mathrm{depth} R_{\mathfrak{p}} - \mathrm{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} - \mathrm{depth}_{R_{\mathfrak{p}}} Y_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Supp}\,(X) \cap \mathrm{Supp}\,(Y)\}.$$

Proof. For $\mathfrak{p} \in \mathrm{Spec}\, R$, using Theorems 1.4 and 2.1, we have

$$\begin{aligned}\sup(X \otimes^{\mathbf{L}}_R Y) &\geq -\mathrm{depth}_{R_{\mathfrak{p}}}(X \otimes^{\mathbf{L}}_R Y)_{\mathfrak{p}} \\ &= -\mathrm{depth}_{R_{\mathfrak{p}}}(X_{\mathfrak{p}} \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} Y_{\mathfrak{p}}) \\ &= \mathrm{depth} R_{\mathfrak{p}} - \mathrm{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} - \mathrm{depth}_{R_{\mathfrak{p}}} Y_{\mathfrak{p}}.\end{aligned}$$

The equality holds if $\mathfrak{p} \in \mathrm{Ass}_R H_s(X \otimes^{\mathbf{L}}_R Y)$ where $s = \sup(X \otimes^{\mathbf{L}}_R Y)$. □

The following corollary is a generalization of Jorgensen's result, cf. [**11**, Theorem 2.2].

Corollary 2.3. *Let M and N be R-modules and $\mathrm{fd}_R M < \infty$. Then*

$$\sup\{i | \mathrm{Tor}_i^R(M,N) \neq 0\} = \sup\{\mathrm{depth} R_{\mathfrak{p}} - \mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} - \mathrm{depth}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Supp}\, M \cap \mathrm{Supp}\, N\}.$$

Proof. Put $X = M$ and $Y = N$ in Theorem 2.2. □

Corollary 2.4. *If R is a ring (not necessarily local) and $X \in \mathcal{F}(R)$, then*

$$\mathrm{fd}_R X = \sup\{\mathrm{depth} R_{\mathfrak{p}} - \mathrm{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Supp}\,(X)\}.$$

Proof. The assertion follows from [**2**, Proposition 2.2] and Theorem 2.2. □

Theorem 2.5. *Let $(R, \mathfrak{m})$ be a complete local ring, $X \in \mathcal{D}_b^f(R)$ and $Y \in \mathcal{F}(R)$. Then*

$$-\inf(\mathbf{R}\mathrm{Hom}_R(Y,X)) = \mathrm{depth} R - \inf X - \mathrm{depth}_R Y.$$

In particular, for any finitely generated R-module M and any R-module N with $\mathrm{fd}\, N < \infty$,

$$\sup\{i \mid \mathrm{Ext}_R^i(N,M) \neq 0\} = \mathrm{depth} R - \mathrm{depth}_R N.$$

Proof. Let $E = E(R/\mathfrak{m})$ be the injective hull of $R/\mathfrak{m}$. Using [**12**, Theorem 18.6] and Theorem 1.3(b), we have

$$\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X,E),E) = X \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(E,E) = X,$$

and hence,

$$\begin{aligned} -\inf(\mathbf{R}\mathrm{Hom}_R(Y,X)) &= -\inf(\mathbf{R}\mathrm{Hom}_R(Y, \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X,E),E))) \\ &= -\inf(\mathbf{R}\mathrm{Hom}_R(Y \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(X,E),E)) \quad \text{(by using 1.3(a))} \\ &= \sup(Y \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(X,E)). \end{aligned}$$

Now the assertion follows from Theorem 2.2. ◻

3. Restricted Tor-dimension under base change. In this section we assume that $\varphi : R \to S$ is a ring homomorphism of commutative Noetherian rings. For $X \in \mathcal{D}_+(R)$ the restricted Tor-dimension of X is defined as

$$\mathrm{Td}_R X = \sup\{\sup(T \otimes^{\mathbf{L}}_R X) \mid \text{for some } R\text{-module } T \text{ of finite flat dimension}\}.$$

The following results will also be used.

Theorem 3.1 [**6**, Theorem 5.3.8]. *If $X \in \mathcal{D}_b(R)$, then*

$$\mathrm{Td}_R X = \sup\{\sup(U \otimes^{\mathbf{L}}_R X) - \sup U \mid U \in \mathcal{F}(R) \wedge H(U) \neq 0\}.$$

Theorem 3.2 [**6**, Theorem 5.3.10]. *If R is a Cohen-Macaulay local ring, and $X \in \mathcal{D}^f_b(R)$, then*

$$\mathrm{Td}_R X = \mathrm{depth} R - \mathrm{depth}_R X.$$

Proposition 3.3. *The following hold*:

(a) *If $X \in \mathcal{D}_b(S)$ and $Y \in \mathcal{F}(R)$, then*

$$\mathrm{Td}_R(X \otimes^{\mathbf{L}}_R Y) \leq \mathrm{Td}_S X + \mathrm{Td}_R S + \mathrm{fd}_R Y.$$

(b) *If $Y \in \mathcal{D}_b(R)$, $X \in \mathcal{F}(S)$, and $\mathrm{fd}\varphi < \infty$, then*

$$\mathrm{Td}_S(X \otimes^{\mathbf{L}}_R Y) \leq \mathrm{fd}_S X + \mathrm{Td}_R Y.$$

(c) *If $X \in \mathcal{D}_b(S)$ and $Y \in \mathcal{F}(S)$, then*

$$\mathrm{Td}_R(X \otimes^{\mathbf{L}}_S Y) \leq \mathrm{Td}_R X + \mathrm{fd}_S Y.$$

Proof. (a) Choose the R-module T with finite flat dimension such that

$$\mathrm{Td}_R(X \otimes^{\mathbf{L}}_R Y) = \sup(T \otimes^{\mathbf{L}}_R (X \otimes^{\mathbf{L}}_R (Y)).$$

Since $T \otimes_R^{\mathbf{L}} Y \in \mathcal{F}(R)$ we have that $(T \otimes_R^{\mathbf{L}} Y) \otimes_R^{\mathbf{L}} S \in \mathcal{F}(S)$ and thus

$$\begin{aligned}\mathrm{Td}_R(X \otimes_R^{\mathbf{L}} Y) &= \sup(((T \otimes_R^{\mathbf{L}} Y) \otimes_R^{\mathbf{L}} S) \otimes_S^{\mathbf{L}} X)\\ &\leq \mathrm{Td}_S X + \sup((T \otimes_R^{\mathbf{L}} Y) \otimes_R^{\mathbf{L}} S)\\ &\leq \mathrm{Td}_S X + \mathrm{Td}_R S + \sup(T \otimes_R^{\mathbf{L}} Y)\end{aligned}$$

where the last two inequalities follow from Theorem 3.1. Now the assertion follows from Theorem 1.5(c).

(b) Choose the S-module T of finite flat dimension such that

$$\mathrm{Td}_S(X \otimes_R^{\mathbf{L}} Y) = \sup(T \otimes_S^{\mathbf{L}} (X \otimes_R^{\mathbf{L}} Y)).$$

Therefore, $\mathrm{Td}_S(X \otimes_R^{\mathbf{L}} Y) = \sup((T \otimes_S^{\mathbf{L}} X) \otimes_R^{\mathbf{L}} Y)$. Since $T \otimes_S^{\mathbf{L}} X \in \mathcal{F}(S) \subseteq \mathcal{F}(R)$, we have that $\mathrm{Td}_S(X \otimes_R^{\mathbf{L}} Y) \leq \sup((T \otimes_S^{\mathbf{L}} X)) + \mathrm{Td}_R Y$ by Theorem 3.1. Now the assertion follows from Theorem 1.5(c).

(c) Choose the R-module T of finite flat dimension such that

$$\mathrm{Td}_R(X \otimes_S^{\mathbf{L}} Y) = \sup(T \otimes_R^{\mathbf{L}} (X \otimes_S^{\mathbf{L}} Y)).$$

Then we have

$$\begin{aligned}\mathrm{Td}_R(X \otimes_S^{\mathbf{L}} Y) &= \sup((T \otimes_R^{\mathbf{L}} X) \otimes_S^{\mathbf{L}} Y)\\ &\leq \sup(T \otimes_R^{\mathbf{L}} X) + \mathrm{fd}_S Y\end{aligned}$$

where the last inequality follows from Theorem 1.5(c). Now the assertion holds. ◻

Corollary 3.4. *If* $X \in \mathcal{D}_b(S)$, *then* $\mathrm{Td}_R X \leq \mathrm{Td}_S X + \mathrm{Td}_R S$.

Proof. Set $Y = R$ in Theorem 3.3(a). ◻

Proposition 3.5. *For any S-module M,*

$$\mathrm{Td}_R M \leq \mathrm{Td}_R S + \mathrm{Td}_S M.$$

The equality holds for arbitrary S-module if it holds for any S-module M with $\mathrm{Td}_S M \leq 1$.

Proof. The first inequality follows from Corollary 3.4. Now assume that the equality holds for all S-modules M with $\mathrm{Td}_S M \leq 1$. Let $\mathrm{Td}_S M = n \geq 2$ and set $\mathrm{Td}_R S = s$. Consider the exact sequence $0 \to K \to F \to M \to 0$ of S-modules and S-homomorphisms where F is free. Clearly we have $\mathrm{Td}_R F = s$ and $\mathrm{Td}_S K = n-1$. By the induction hypothesis $\mathrm{Td}_R K = \mathrm{Td}_S K + \mathrm{Td}_R S = s+n-1$. For any R-module T of finite flat dimension the following long exact sequence is induced

$$\cdots \to \mathrm{Tor}_i^R(T,F) \to \mathrm{Tor}_i^R(T,M) \to \mathrm{Tor}_{i-1}^R(T,K) \to \mathrm{Tor}_{i-1}^R(T,F) \to \cdots .$$

Since $\mathrm{Td}_R K = s+n-1$, there exists $L \in \mathcal{F}_0(R)$ such that $\mathrm{Tor}_{s+n-1}^R(L,K) \neq 0$. If $i > s+n$, then $\mathrm{Tor}_i^R(L,M) = 0$. Since $\mathrm{Tor}_{s+n-1}^R(L,F) = 0$, we have $\mathrm{Tor}_{s+n}^R(L,M) \neq 0$ and, hence, $\mathrm{Td}_R M = s+n$. □

Let a be an R-regular element, $S = R/aR$ and M an S-module with $\mathrm{fd}_S M < \infty$. Then it is a well-known result that $\mathrm{fd}_R M = \mathrm{fd}_S M + 1$. In the following theorem we generalize this result for restricted Tor-dimension.

Theorem 3.6. *Let $x = x_1, x_2, \ldots, x_r$ be an R-regular sequence, $S = R/(x)$ and $X \in \mathcal{D}_b(S)$. Then $\mathrm{Td}_R X = \mathrm{Td}_S X + r$.*

Proof. It is sufficient to prove the equality for $r = 1$. Assume that $\mathrm{Td}_S X = n$. There exists an S-module T with finite flat dimension such that $H_n(T \otimes_S^{\mathbf{L}} X) \neq 0$. Let F be a flat resolution of S-complex X. Then $H_n(T \otimes_S F) \neq 0$. Since $T \otimes_R^{\mathbf{L}} X = T \otimes_R^{\mathbf{L}} (S \otimes_S^{\mathbf{L}} X) = (T \otimes_R^{\mathbf{L}} S) \otimes_S^{\mathbf{L}} X$, the complex $T \otimes_R^{\mathbf{L}} X$ is represented by $(T \otimes_R K(x)) \otimes_S F$, where $K(x)$ is the koszul complex for x. Set $L = T \otimes_R K(x) : 0 \to T \xrightarrow{x} T \to 0$ which is a complex concentrated in degrees 0 and 1. We have $(L \otimes_S F)_l = (T \otimes_S F_l) \oplus (T \otimes_S F_{l-1})$. Since $H_n(T \otimes_S F) \neq 0$ we have $H_{n+1}(L \otimes_S F) \neq 0$ and hence $H_{n+1}(T \otimes_R^{\mathbf{L}} X) \neq 0$. Therefore, $\mathrm{Td}_R X \geq n+1 = \mathrm{Td}_S X + 1$. Now the assertion follows from Corollary 3.3. □

For an R-module M and an integer $n \geq 0$, we say that M is (a_n) R-module if any R-regular sequence of length at most n is also M-regular (for $n = 1$ this simply means that M is torsion free).

Assume that $(R,\mathfrak{m})$ is a Cohen-Macaulay ring $x_1, x_2, \ldots, x_r \in \mathfrak{m}$ an R-regular sequence. Set $S = R/(x_1, x_2, \ldots, x_r)$. Furthermore, assume that M is an R-module with $(x_1, x_2, \ldots, x_r)M = 0$ and $\mathrm{fd}_R M < \infty$ such that any S-regular sequence $y_1, y_2, \ldots, y_s \in \mathfrak{m}$ is weak M-regular. Then Ischebeck has proved that $\mathrm{fd}_R M \leq r$. In the proof of this result the assumption "R is a Cohen-Macaulay ring" is not necessary and we only need that "for all $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} R$ if $\mathfrak{p} \subseteq \mathfrak{q}$ then $\mathrm{depth}_{R_\mathfrak{p}} R_\mathfrak{p} \leq \mathrm{depth}_{R_\mathfrak{q}} R_\mathfrak{q}$." With this remark, Foxby noted that if $\mathrm{cmd}\, R$ (Cohen-Macaulay defect of R) is at most 1, $\mathrm{fd}_R M$ is finite, and any R-regular sequence is weak M-regular, then M is flat. In the following theorem we generalize this result for restricted Tor-dimension.

Theorem 3.7. *Let $\varphi : R \to S$ be a local homomorphism. Then the following hold.*

(a) *If $\mathrm{cmd}\, S \leq 1$ and M is (a_n) as an S-module, then for any R-module N,*

$$\mathrm{Td}_R(M \otimes^{\mathbf{L}}_R N) \leq \mathrm{fd}_R N + \mathrm{depth} S + \mathrm{Td}_R S - n.$$

(b) *If $\mathrm{cmd}\, R \leq 1$ and M is (a_n) as an R-module, then for any S-module N,*

$$\mathrm{Td}_R(M \otimes^{\mathbf{L}}_S N) \leq \mathrm{depth} R + \mathrm{fd}_S N - n.$$

(c) *Let φ be a flat homomorphism, i.e., S is a flat R-module such that the closed fiber of φ, $S/\mathfrak{m}S$, is (a_n) as an R-module, and let M be (a_n) as an S-module. If $\mathrm{cmd}\, S \leq 1$, then for any R-module N, $\mathrm{Td}_R(M \otimes^{\mathbf{L}}_S N) \leq \mathrm{depth} R + \mathrm{fd}_S N - n$.*

Proof. First we show that for any local ring Q with $\mathrm{cmd}\, Q \leq 1$ and any Q-module M with (a_n) property we have $\mathrm{Td}_Q M \leq \mathrm{depth} Q - n$. There exists $\mathfrak{p} \in \operatorname{Supp} M$ such that $\mathrm{Td}_Q M = \mathrm{depth} Q_\mathfrak{p} - \mathrm{depth}_{Q_\mathfrak{p}} M_\mathfrak{p}$. Since $\mathrm{cmd}\, Q \leq 1$ we have depth $Q_\mathfrak{p} = \mathrm{grade}\,(\mathfrak{p}, Q)$ (the length of maximal Q-sequence in $\mathfrak{p}$), cf., [**7**, Theorem 5.1]. Set $\mathrm{grade}\,(\mathfrak{p}, Q) = t$. If $t \leq n$, then it is an M-regular sequence and hence an $M_\mathfrak{p}$-regular. Therefore, $\mathrm{depth}_{Q_\mathfrak{p}} M_\mathfrak{p} \geq t$. Thus, $\mathrm{Td}_Q M = 0$.

If $t > n$ then there exists a Q-regular sequence of length n in $\mathfrak{p}$. Therefore we have $\mathrm{depth}_{Q_\mathfrak{p}} M_\mathfrak{p} \geq n$ and thus $\mathrm{Td}_Q M \leq \mathrm{depth} Q_\mathfrak{p} - n$. Therefore, $\mathrm{Td}_Q M \leq \mathrm{depth} Q - n$.

(a) Suppose that $\mathrm{fd}_R N < \infty$. Since $\mathrm{cmd}\, S \leq 1$ and M is (a_n) S-module, the assertion follows from Proposition 3.3(a).

(b) Suppose that $\mathrm{fd}_S N < \infty$. Since $\mathrm{cmd}\, R \leq 1$; the assertion follows from Proposition 3.3(c).

(c) Let $r_1, r_2, \ldots, r_n \in \mathfrak{m}$ be an R-sequence. Set $\varphi(r_i) = s_i$ for any $1 \leq i \leq n$. Since $S/\mathfrak{m}S$ is (a_n) R-module, thus for any $1 \leq i \leq n$, s_i is $S/\mathfrak{m}S$-regular and hence S-regular, cf. [**12**, p. 177]. Therefore, S is (a_n) R-module and hence M is (a_n) R-module. Now the assertion follows from [**4**, Proposition 1.2.16]. □

Corollary 3.8. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay ring. If $x_1, x_2, \ldots, x_r \in \mathfrak{m}$ is an R-regular sequence, $(x_1, x_2, \ldots, x_r)M = 0$, and any $R/(x_1, x_2, \ldots, x_r)$-regular sequence is also M-regular, then $\mathrm{Td}_R M \leq r$.*

Proof. Put $N = R$ and $n = \mathrm{depth}_{R}/(x_1, x_2, \ldots, x_r)R$ in Theorem 3.7(a) and use 3.6. □

Theorem 3.9. *Let $\varphi : R \to S$ be a local homomorphism, S a finite R-module. Let R be a Cohen-Macaulay ring. Then S is Cohen-Macaulay if and only if $\mathrm{Td}_R X = \mathrm{Td}_S X + \mathrm{Td}_R S$ for all $X \in \mathcal{D}_b^f(S)$.*

Proof. Assume that S is a Cohen-Macaulay ring. Then from Theorem 3.2 we have the following equalities:

(a) $\mathrm{Td}_S X + \mathrm{depth}_S X = \mathrm{depth} S$;

(b) $\mathrm{Td}_R X + \mathrm{depth}_R X = \mathrm{depth} R$; and

(c) $\mathrm{Td}_R S + \mathrm{depth}_R S = \mathrm{depth} R$.

Thus the "only if" part follows from the equality $\mathrm{depth}_R X = \mathrm{depth}_S X$, cf. [**1**, Corollary 2].

Now assume that $\mathrm{Td}_R X = \mathrm{Td}_S X + \mathrm{Td}_R S$ for all $X \in \mathcal{D}_b^f(S)$. Using Theorem 3.2 we have $\mathrm{Td}_R S = \mathrm{depth} R - \mathrm{depth}_R S$. Therefore, $\mathrm{Td}_R X = \mathrm{Td}_S X + \mathrm{depth} R - \mathrm{depth}_R S$ and, hence, $\mathrm{Td}_R X = \mathrm{Td}_S X + (\mathrm{depth}_R X + \mathrm{Td}_R X) - \mathrm{depth} S$, since R is Cohen-Macaulay. Thus, $\mathrm{Td}_S X = \mathrm{depth} S - \mathrm{depth}_S X$. Now the assertion holds from Theorem 3.2. □

The small Tor-dimension of $X \in \mathcal{D}_+(R)$, $\operatorname{td}_R X$, is defined as follows, cf. [**7**, Definition 4.1]:

$$\operatorname{td}_R X = \sup\left\{ \sup\left(T \otimes^{\mathbf{L}}_R X\right) \mid \text{for some finitely generated } R\text{-module } T \text{ with } \operatorname{pd} T < \infty \right\}.$$

From the definition it is easy to see that $\sup X \leq \operatorname{td}_R X \leq \operatorname{Td}_R X \leq \sup X + \dim R$.

Corollary 3.10. *The following hold*:

(a) *If* $X \in \mathcal{D}_b(S)$ *and* $Y \in \mathcal{F}(R)$, *then*

$$\operatorname{td}_R(X \otimes^{\mathbf{L}}_R Y) \leq \operatorname{Td}_S X + \operatorname{Td}_R S + \operatorname{fd}_R Y.$$

(b) *If* $X \in \mathcal{D}_b(S)$, $Y \in \mathcal{F}(R)$, $\operatorname{fd}\varphi < \infty$ *and* $\operatorname{cmd} S \leq 1$, *then*

$$\operatorname{td}_R\left(X \otimes^{\mathbf{L}}_R Y\right) \leq \operatorname{td}_S X + \operatorname{td}_R S + \operatorname{fd}_R Y.$$

(c) *If* $Y \in \mathcal{D}_b(R)$, $X \in \mathcal{F}(S)$ *and* $\operatorname{fd}\varphi < \infty$, *then*

$$\operatorname{td}_S(X \otimes^{\mathbf{L}}_R Y) \leq \operatorname{fd}_S X + \operatorname{Td}_R Y.$$

(d) *If* $Y \in \mathcal{D}_b(R)$, $X \in \mathcal{F}(S)$, $\operatorname{fd}\varphi < \infty$ *and* $\operatorname{cmd} S \leq 1$, *then*

$$\operatorname{td}_S\left(X \otimes^{\mathbf{L}}_R Y\right) \leq \operatorname{fd}_S X + \operatorname{td}_R Y.$$

Proof. Since $\operatorname{td}_R Z \leq \operatorname{Td}_R Z$ for any $Z \in \mathcal{D}_+(R)$, the statements (a) and (c) hold by Theorem 3.3.

If $\operatorname{fd}\varphi < \infty$, then $\operatorname{cmd} R \leq \operatorname{cmd} S \leq 1$, cf. [**3**, Corollary 4.3]. Now the assertions (b) and (d) follow from [**7**, Corollary 5.3] and Theorem 3.3. □

4. Covariant Ext-dimension. The restricted covariant Ext-dimension, that is, a dual notion of the restricted Tor-dimension of $X \in \mathcal{D}_-(R)$ is denoted by $\operatorname{Ed}_R X$ and is defined by

$$\operatorname{Ed}_R X = \sup\{-\inf(\mathbf{R}\operatorname{Hom}_R(T, X)) \mid \text{for some } R\text{-module } T \text{ of finite projective dimension}\}.$$

From the definition it is easy to see that $-\inf X \leq \mathrm{Ed}_R X \leq \dim R - \inf X$.

Remark 4.1. Let $X \in \mathcal{D}_-(R)$, and let E be a faithfully injective R-module. Choose the R-module T with finite projective dimension such that $\mathrm{td}_R X^\vee = \sup(\mathbf{R}\mathrm{Hom}_R(X,E) \otimes^{\mathbf{L}}_R T)$. Using Theorem 1.3(b) and the property of faithfully injective modules we have

$$\begin{aligned}\mathrm{td}_R X^\vee &= \sup(\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(T,X),E))\\ &= -\inf(\mathbf{R}\mathrm{Hom}_R(T,X)) \leq \mathrm{Ed}_R X.\end{aligned}$$

Now if $X \in \mathcal{D}^f_b(R)$ and $T = R/(x_1,x_2,\dots,x_r)R$ where $x_1,x_2,\dots,x_r \in \mathfrak{m}$ is a maximal R-regular sequence, then $\mathrm{td}_R X^\vee \geq \sup(T \otimes^{\mathbf{L}}_R X^\vee)$ and hence, using Theorem 2.2, we have $\mathrm{td}_R X^\vee \geq \mathrm{depth} R - \mathrm{depth} T - \mathrm{depth} X^\vee = \mathrm{depth} R - \inf X$. Therefore, for any $X \in \mathcal{D}^f_b(R)$;

$$\mathrm{depth} R - \inf X \leq \mathrm{td}_R X^\vee \leq \mathrm{Ed}_R X.$$

Proposition 4.2. *Let $\varphi : R \to S$ be a ring homomorphism and let $X \in \mathcal{D}_b(S)$. Then the following hold*:

(a) *If $X \in \mathcal{I}(S)$ and $Y \in \mathcal{D}_b(S)$, then*

$$\mathrm{Ed}_R(\mathbf{R}\mathrm{Hom}_S(Y,X)) \leq \mathrm{id}_S X + \mathrm{Td}_R Y.$$

(b) *If S admits a dualizing complex, then*

$$\mathrm{Ed}_R X \leq \mathrm{Td}_R S + \dim S - \inf X.$$

In particular, if $X \in \mathcal{D}^f_b(S)$ and S is a local ring with a dualizing module, then $\mathrm{Ed}_R X \leq \mathrm{Td}_R S + \mathrm{Ed}_S X$.

Proof. (a) Choose the R-module T with finite projective dimension such that

$$\mathrm{Ed}_R(\mathbf{R}\mathrm{Hom}_S(Y,X)) = -\inf(\mathbf{R}\mathrm{Hom}_R(T,\mathbf{R}\mathrm{Hom}_S(Y,X))).$$

Then

$$\begin{aligned}\mathrm{Ed}_R(\mathbf{R}\mathrm{Hom}_S(Y,X)) &= -\inf(\mathbf{R}\mathrm{Hom}_R(T\otimes^{\mathbf{L}}_R Y,X)) \text{ (by Theorem 1.3(a))}\\ &\le \mathrm{id}_S X + \sup(Y\otimes^{\mathbf{L}}_R T) \quad \text{(by Theorem 1.5(b))}\\ &\le \mathrm{id}_S X + \mathrm{Td}_R Y.\end{aligned}$$

(b) Choose the R-module T with finite projective dimension such that

$$\mathrm{Ed}_R X = -\inf(\mathbf{R}\mathrm{Hom}_R(T,X)).$$

We have

$$\mathbf{R}\mathrm{Hom}_R(T,X) = \mathbf{R}\mathrm{Hom}_R(T,(\mathbf{R}\mathrm{Hom}_S(S,X)) = \mathbf{R}\mathrm{Hom}_S(T\otimes^{\mathbf{L}}_R S,X).$$

On the other hand, $T\in\mathcal{P}_0(R)\subset\mathcal{F}_0(R)$, thus $S\otimes^{\mathbf{L}}_R T\in\mathcal{F}(S)$. Now since S has dualizing complex we have $\mathrm{pd}_S(S\otimes^{\mathbf{L}}_R T)\le \dim S + \sup(S\otimes^{\mathbf{L}}_R T)$, cf. [**8**, Theorem 21.8]. Furthermore, $\mathrm{Ed}_R X\le \mathrm{pd}_S(S\otimes^{\mathbf{L}}_R T)-\inf X$. Thus,

$$\begin{aligned}\mathrm{Ed}_R X &\le \dim S-\inf X+\sup(S\otimes^{\mathbf{L}}_R T)\\ &\le \dim S-\inf X+\mathrm{Td}_R S.\end{aligned}$$

Since S has a canonical module, S is a Cohen-Macaulay ring. Now the assertion follows from Remark 4.1. □

Corollary 4.3. *Let R be a complex local ring, $X\in\mathcal{D}^f_b(R)$. Then*

$$\mathrm{depth}R-\inf X=\mathrm{td}_R X^{\vee}=\mathrm{Ed}_R X.$$

Proof. Choose the R-module T with finite projective dimension such that

$$\mathrm{Ed}_R X = -\inf(\mathbf{R}\mathrm{Hom}_R(T,X)).$$

Since T has finite flat dimension, by Theorem 2.5 we have $-\inf(\mathbf{R}\mathrm{Hom}_R(T,X))=\mathrm{depth}R-\inf X-\mathrm{depth}_R T\le\mathrm{depth}R-\inf X$. Now the assertion follows from Remark 4.1. □

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