ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 34, Number 3, Fall 2004

POLYNOMIAL CHARACTERIZATION OF THE COMPACT RANGE PROPERTY

RAFFAELLA CILIA AND JOAQUÍN M. GUTIÉRREZ

ABSTRACT. Among other results it is proved that, for a Banach space F and an integer m, the following assertions are equivalent:

(a) F has the compact range property;

(b) for every Banach space E, each m-homogeneous Pietsch integral polynomial from E into F is compact;

(c) every m-homogeneous 1-dominated polynomial from $C([0,1])$ into F is compact;

(d) every m-homogeneous polynomial from $L_1([0,1])$ into F is completely continuous.

A Banach space F is said to have the *compact range property* (CRP, for short) if every F-valued countably additive measure of bounded variation has compact range [**15**]. Every Banach space with the weak Radon-Nikodým property has the CRP. A dual Banach space has the CRP if and only if its predual contains no copy of l_1. We refer to [**9, 10, 15, 17**] for more about the CRP.

We recall the following characterizations of the CRP in terms of (linear bounded) operators:

Theorem 1. *For a Banach space F the following facts are equivalent*:

(a) *F has the* CRP;

(b) *for any compact Hausdorff space K, every absolutely summing operator from $C(K)$ into F is compact*;

(c) *every absolutely summing operator from $C([0,1])$ into F is compact*;

2000 AMS *Mathematics Subject Classification.* Primary 46G25, Secondary 46B20, 47H60.
Key words and phrases. 1-dominated polynomial, Pietsch integral polynomial, compact range property.
The first named author was supported in part by GNAFA (Italy).
The second named author was supported in part by Dirección General de Investigación, BFM 2000-0609 (Spain).
Received by the editors on January 4, 2002.

(d) *if* (Ω, Σ, μ) *is a finite measure space then every operator from* $L_1(\mu)$ *into* F *is completely continuous*;

(e) *every operator from* $L_1([0,1])$ *into* F *is completely continuous*;

(f) *for any Banach space* E, *every Pietsch integral operator from* E *into* F *is compact.*

The equivalence (a) $\Longleftrightarrow$ (e) is stated in [**17**,7]. The other implications may be seen in [**9, 10**].

Here we extend this result to the polynomial setting.

Throughout, E and F denote Banach spaces, and B_E stands for the closed unit ball of E. By $\mathbf{N}$ we represent the set of all natural numbers. Given $m \in \mathbf{N}$, we denote by $\mathcal{P}(^m E, F)$ the space of all m-homogeneous (continuous) polynomials from E into F. Recall that to each $P \in \mathcal{P}(^m E, F)$ we can associate a unique symmetric m-linear $\widehat{P} : E \times \overset{(m)}{\cdots} \times E \to F$ so that

$$P(x) = \widehat{P}\left(x, \overset{(m)}{\ldots}, x\right), \quad x \in E.$$

For the general theory of polynomials on Banach spaces, we refer to [**8**] and [**14**].

We use the notation $\otimes^m E := E \otimes \overset{(m)}{\cdots} \otimes E$ for the m-fold tensor product of E, $\otimes_\varepsilon^m E := E \otimes_\varepsilon \overset{(m)}{\cdots} \otimes_\varepsilon E$ for the m-fold injective tensor product of E, and $\otimes_\pi^m E$ for the m-fold projective tensor product of E (see [**7**] for the theory of tensor products). By $\otimes_s^m E := E \otimes_s \overset{(m)}{\cdots} \otimes_s E$ we denote the m-fold symmetric tensor product of E, i.e., the set of all elements $u \in \otimes^m E$ of the form

$$u = \sum_{j=1}^{n} \lambda_j x_j \otimes \overset{(m)}{\cdots} \otimes x_j, \quad n \in \mathbf{N},\ \lambda_j \in \mathbf{K},\ x_j \in E,\ 1 \leq j \leq n.$$

By $\otimes_{\pi,s}^m E$ we denote the closure of $\otimes_s^m E$ in $\otimes_\pi^m E$. For symmetric tensor products, we refer to [**11**]. For simplicity, we write $\otimes^m x := x \otimes \overset{(m)}{\cdots} \otimes x$.

Given $P \in \mathcal{P}(^m E, F)$, let

$$\overline{\widehat{P}} : \otimes^m E \longrightarrow F$$

be the linearization of $\widehat{P}$, defined by

$$\overline{\widehat{P}}\left(\sum_{j=1}^{n} x_{1j} \otimes \cdots \otimes x_{mj}\right) = \sum_{j=1}^{n} \widehat{P}(x_{1j}, \ldots, x_{mj})$$

where $x_{kj} \in E$ $(1 \leq k \leq m, 1 \leq j \leq n)$; and let

$$\overline{P} : \otimes_s^m E \longrightarrow F$$

be the linearization of P, given by

$$\overline{P}\left(\sum_{j=1}^{n} \lambda_j x_j \otimes \overset{(m)}{\cdots} \otimes x_j\right) = \sum_{j=1}^{n} \lambda_j P(x_j)$$

where $x_j \in E$ $(1 \leq j \leq n)$.

Recall that $P \in \mathcal{P}(^m E, F)$ is *completely continuous* if, for every sequence $(x_n) \subset E$ weakly convergent to x, we have that $(P(x_n))$ converges in norm to $P(x)$; P is *compact* if $P(B_E)$ is relatively compact in F.

Given $1 \leq r < \infty$, a polynomial $P \in \mathcal{P}(^m E, F)$ is *r-dominated* (see, e.g., [**12, 13**]) if there exists a constant $k > 0$ such that, for all $n \in \mathbf{N}$ and $(x_i)_{i=1}^n \subset E$, we have

$$\left(\sum_{i=1}^{n} \|P(x_i)\|^{r/m}\right)^{m/r} \leq k \sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^{n} |x^*(x_i)|^r\right)^{m/r}.$$

For $m = 1$ we obtain the absolutely r-summing operators.

A polynomial $P \in \mathcal{P}(^m E, F)$ is *Pietsch integral* if it can be written in the form

$$P(x) = \int_{B_{E^*}} [x^*(x)]^m \, d\mathcal{G}(x^*), \quad x \in E$$

where $\mathcal{G}$ is an F-valued regular countably additive Borel measure, of bounded variation, defined on B_{E^*}, where B_{E^*} is endowed with the weak-star topology. A similar definition may be given for the Pietsch integral multilinear mappings (see [**1**]).

We refer to [**6, 7**] for the theory of absolutely summing and Pietsch integral operators between Banach spaces.

We first give a characterization of the CRP in terms of polynomials on $L_1(\mu)$ spaces.

Theorem 2. *Given a Banach space F, the following assertions are equivalent*:

(a) *F has the CRP*;

(b) *for all $m \in \mathbf{N}$ and any finite measure μ, every m-homogeneous polynomial from $L_1(\mu)$ into F is completely continuous*;

(c) *there is $m \in \mathbf{N}$ such that for any finite measure μ, every m-homogeneous polynomial from $L_1(\mu)$ into F is completely continuous*;

(d) *there is $m \in \mathbf{N}$ such that every m-homogeneous polynomial from $L_1([0,1])$ into F is completely continuous.*

Proof. (a) $\Rightarrow$ (b). Let $P \in \mathcal{P}(^m L_1(\mu), F)$. Choose a sequence $(f_n) \subset L_1(\mu)$ weakly convergent to some f. By the Dunford-Pettis property of $L_1(\mu)$, the sequence $(\otimes^m f_n)_n$ converges weakly to $\otimes^m f$ in $\otimes_\pi^m L_1(\mu)$ [**5**, Theorem 16]. Since $\otimes_\pi^m L_1(\mu)$ is an $L_1(\nu)$ space with ν finite, the operator

$$\overline{\widehat{P}} : \otimes_\pi^m L_1(\mu) \longrightarrow F$$

is completely continuous, Theorem 1. Therefore, we have

$$P(f_n) = \overline{\widehat{P}}(\otimes^m f_n) \overset{\text{norm}}{\longrightarrow} \overline{\widehat{P}}(\otimes^m f) = P(f),$$

so P is completely continuous.

(b) $\Rightarrow$ (c) $\Rightarrow$ (d) are obvious.

(d) $\Rightarrow$ (a). Let $T : L_1([0,1]) \rightarrow F$ be an operator. Suppose $(f_n) \subset L_1([0,1])$ is weakly convergent to some f, and $\|Tf_n - Tf\| > 4\varepsilon > 0$. Without loss of generality, we can assume $f \neq 0$. Choose $\varphi \in L_\infty([0,1])$ with $\varphi(f) = 1$.

Let $P : L_1([0,1]) \rightarrow F$ be the polynomial given by

$$P(g) := (\varphi(g))^{m-1} Tg \qquad (g \in L_1([0,1])).$$

Then,

$$\begin{aligned}\|P(f_n) - P(f)\| &= \big\|(\varphi(f_n))^{m-1}Tf_n - (\varphi(f))^{m-1}Tf\big\| \\ &\geq \big\|(\varphi(f_n))^{m-1}Tf_n - (\varphi(f_n))^{m-1}Tf\big\| \\ &\quad - \big\|(\varphi(f_n))^{m-1}Tf - (\varphi(f))^{m-1}Tf\big\| \\ &= |\varphi(f_n)|^{m-1} \cdot \|Tf_n - Tf\| \\ &\quad - \Big|(\varphi(f_n))^{m-1} - (\varphi(f))^{m-1}\Big| \cdot \|Tf\| \\ &> \frac{1}{2} \cdot 4\varepsilon - \varepsilon = \varepsilon\end{aligned}$$

for n large enough, which contradicts (d). □

We now give the characterization of the CRP in terms of polynomials on $C(K)$ spaces.

Theorem 3. *Given a Banach space F, the following assertions are equivalent*:

(a) *F has the* CRP;

(b) *for all $m \in \mathbf{N}$ and any Banach space E, every m-homogeneous Pietsch integral polynomial from E into F is compact*;

(c) *for all $m \in \mathbf{N}$, every m-homogeneous Pietsch integral polynomial from a $C(K)$ space into F is compact*;

(d) *for all $m \in \mathbf{N}$, every m-homogeneous* 1*-dominated polynomial from a $C(K)$ space into F is compact*;

(e) *there is $m \in \mathbf{N}$ such that every m-homogeneous* 1*-dominated polynomial from a $C(K)$ space into F is compact*;

(f) *there is $m \in \mathbf{N}$ such that every m-homogeneous* 1*-dominated polynomial from $C([0,1])$ into F is compact.*

Proof. (a) ⇒ (b). Let $P \in \mathcal{P}(^m E, F)$ be Pietsch integral. By [**1**], so is $\widehat{P}$. By [**18**], the operator

$$\overline{\widehat{P}} : \otimes_{\varepsilon}^m E \longrightarrow F$$

is well-defined and Pietsch integral. By Theorem 1, $\overline{\widehat{P}}$ is compact. Letting $i : \otimes_{\pi,s}^m E \to \otimes_{\varepsilon}^m E$ be the natural inclusion, we have that $\overline{\widehat{P}} \circ i$

is compact. Since $\overline{\overline{\widehat{P}}} \circ i$ is the linearization of P, we conclude that P is compact [**16**, Lemma 4.1].

(b) $\Rightarrow$ (c) and (d) $\Rightarrow$ (e) $\Rightarrow$ (f) are obvious.

(c) $\Rightarrow$ (d) is clear, since every 1-dominated polynomial on a $C(K)$ space is Pietsch integral [**4**].

(f) $\Rightarrow$ (a). Let $T : C([0,1]) \to F$ be an absolutely summing operator. For each $1 \leq i \leq m-1$ there are operators

$$j_i : \otimes^i_{\pi,s} C([0,1]) \longrightarrow \otimes^{i+1}_{\pi,s} C([0,1])$$

and

$$\pi_i : \otimes^{i+1}_{\pi,s} C([0,1]) \longrightarrow \otimes^i_{\pi,s} C([0,1])$$

such that $\pi_i \circ j_i$ is the identity map on $\otimes^i_{\pi,s} C([0,1])$ (see [**2**, p. 168]).

Consider the polynomial

$$P := T \circ \pi_1 \circ \cdots \circ \pi_{m-1} \circ \delta_m : C([0,1]) \longrightarrow F$$

where $\delta_m : C([0,1]) \to \otimes^m_{\pi,s} C([0,1])$ is the polynomial given by $\delta_m(f) := \otimes^m f$. Then P is 1-dominated (see details in [**3**], p. 910). Hence, by (f), P is compact. Since

$$T \circ \pi_1 \circ \cdots \circ \pi_{m-1} : \otimes^m_{\pi,s} C([0,1]) \longrightarrow F$$

is the linearization of P, it is compact as well [**16**, Lemma 4.1]. Therefore, the operator

$$T = T \circ \pi_1 \circ \cdots \circ \pi_{m-1} \circ j_{m-1} \circ \cdots \circ j_1$$

is compact and, by Theorem 1, we conclude that F has the CRP. ◻

REFERENCES

1. R. Alencar, *On reflexivity and basis for* $\mathcal{P}(^mE)$, Proc. Roy. Irish Acad. **85A** (1985), 131–138.

2. F. Blasco, *Complementation in spaces of symmetric tensor products and polynomials*, Studia Math. **123** (1997), 165–173.

3. R. Cilia, M. D'Anna and J.M. Gutiérrez, *Polynomial characterization of* $\mathcal{L}_\infty$*-spaces*, J. Math. Anal. Appl. **275** (2002), 900–912.

4. R. Cilia and J.M. Gutiérrez, *Polynomial characterization of Asplund spaces*, Bull. Belgian Math. Soc. Simon Stevin, to appear.

5. J. Diestel, *A survey of results related to the Dunford-Pettis property*, in *Proc. of Conf. on Integration, Topology and Geometry in Linear Spaces* (Chapel Hill 1979) (W.H. Graves, ed.), Contemp. Math., vol. 2, Amer. Math. Soc., Providence, 1980, pp. 15–60.

6. J. Diestel, H. Jarchow and A. Tonge, *Absolutely summing operators,* Cambridge Stud. Adv. Math., vol. 43, Cambridge Univ. Press, Cambridge, 1995.

7. J. Diestel and J.J. Uhl, Jr., *Vector measures*, Math. Surveys Monographs, vol. 15, Amer. Math. Soc., Providence, 1977.

8. S. Dineen, *Complex analysis on infinite dimensional spaces*, Springer Monographs in Math., Springer, Berlin, 1999.

9. G. Emmanuele, *Dominated operators on* $C[0,1]$ *and the* (CRP), Collect. Math. **41** (1990), 21–25.

10. ———, *Banach spaces with the* (CRP) *and dominated operators on* $C(K)$, Ann. Acad. Sci. Fenn. Ser. A I Math. **16** (1991), 243–248.

11. K. Floret, *Natural norms on symmetric tensor products of normed spaces*, Note Mat. **17** (1997), 153–188.

12. M.C. Matos, *Absolutely summing holomorphic mappings*, An. Acad. Brasil. Ciênc. **68** (1996), 1–13.

13. Y. Meléndez and A. Tonge, *Polynomials and the Pietsch domination theorem*, Math. Proc. Roy. Irish Acad. **99A** (1999), 195–212.

14. J. Mujica, *Complex analysis in Banach spaces*, North-Holland Math. Studies, vol. 120, North-Holland, Amsterdam, 1986.

15. K. Musial, *Martingales of Pettis integrable functions*, in *Measure theory* (Oberwolfach 1979) (D. Kölzow, ed.), Lecture Notes in Math., vol. 794, Springer, Berlin, 1980, 324–339.

16. R.A. Ryan, *Applications of topological tensor products to infinite dimensional holomorphy*, Ph.D. Thesis, Trinity College, Dublin, 1980.

17. M. Talagrand, *Pettis integral and measure theory*, Mem. Amer. Math. Soc. **307**, Amer. Math. Soc., Providence, 1984.

18. I. Villanueva, *Integral mappings between Banach spaces*, J. Math. Anal. Appl. **279** (2003), 56–70.

Dipartimento di Matematica, Facoltà di Scienze, Università di Catania, Viale Andrea Doria 6, 95100 Catania, Italy
E-mail address: cilia@dmi.unict.it

Departamento de Matemática Aplicada, ETS de Ingenieros Industriales, Universidad Politécnica de Madrid, C. José Gutiérrez Abascal 2, 28006 Madrid, Spain
E-mail address: jgutierrez@etsii.upm.es