ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 34, Number 3, Fall 2004

PDE AND EXTRAFUNCTIONS

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ABSTRACT. Many natural problems for partial differential equations (PDE) do not have solutions in the set of differentiable functions. In the development of the theory of PDE, this led to the introduction of weak solutions and then distributions. However, many PDE—some of them very simple—still do not have solutions in the set of distributions. In this paper we show that the theory of extrafunctions allows one to find generalized solutions to a much larger set of equations than those which are solvable in distributions. In a sense, the approach given here follows the traditional method of solving PDE by a series of functions. This succeeds because in spaces of extrafunctions all series of ordinary functions are convergent. There are other approaches, such as theories of generalized functions of Colombeau and Egorov, which also allow one to extend the scope of solvable PDE. However, the function spaces in which these generalized solutions are constructed do not even have a T_0 topology. In particular, in these spaces a limit of a sequence is not unique. In contrast to this, the spaces of extrafunctions have a Hausdorff topology. Moreover, these spaces are maximal with respect to this property. This makes extrafunctions universal for solving PDE under appropriate topological conditions.

1. Introduction. Many specialists consider the Schwartz distribution theory and Sobolev spaces as two of the most important devices in contemporary mathematical analysis. This is especially true for the theory of partial differential equations. However, in spite of the success of distribution theory, it is not able to solve many problems in mathematical physics that are related to divergence of certain integrals and series. Moreover, even "good" linear equations with infinitely differentiable coefficients, $P(x, D)u = f$, may have no solutions in the space of distributions even if $f \in C^\infty(\mathbf{R}^n)$ and the coefficients of P are analytic. The first such example was discovered by Lewy [**32**]. In recent years in the works by Fisher, [**25**–**27**], Rosinger [**38**–**40**], Colombeau [**13**–**19**], Delcroix and Scarpalezos [**20**], Oberguggenberger

Key words and phrases. Partial differential equation, extrafunction, hypernumber, sequential solution, distribution.
Received by the editors on June 14, 2002.

[**34**–**36**], Egorov [**22**–**24**], Burgin [**4**–**6**] and others, a new theory of generalized functions has been developed. However, this development of the theory has been aimed at the development of algebraic structures on the sets of distributions. Consequently, many problems of PDE theory have been left out of the scope of these new directions. The approach considered in this paper is a theory of a new family of extended functions lying between nonstandard analysis and the theory of distributions. It is based on the theory of hypernumbers and extrafunctions [**7**–**12**].

The theory of hypernumbers and extrafunctions emanated from physically directed thinking and was derived by a natural extension of the classical approach to the real number system. Namely, an important class of problems that appear in contemporary physics and involve infinite values inspired this theory. As is known, many mathematical models, which are used in modern theories of elementary particles (such as gauge theories), imply divergence of analytically calculated properties of physical systems. The simplest example is the case of a free electron when its interaction with photons changes the energy of the electron so that the energy becomes infinite (in a model). Mathematical investigation of various physical problems gives rise to divergent integrals and series that have, in some sense, infinite values. However, physical measurements give, as a result, only finite values. That is why many methods of divergence elimination (regularization), i.e., of elimination of infinity, have been elaborated. Nevertheless, the majority of these were not well grounded mathematically because they utilized operations with formal expressions that had neither mathematical nor physical meaning. Moreover, there are models in physics that contain infinities that cannot be eliminated by methods based on existing mathematical theories. In the theory of hyperintegration, based on the theory of hypernumbers, all divergent integrals and series that appear in the calculations with physical quantities become rigorously grounded as strict mathematical objects.

In this paper we consider two families of extended functions related to hypernumbers: compactwise extrafunctions and hyperdistributions. Compactwise extrafunctions have the topology of uniform convergence on compact sets. It is possible to define other families of extrafunctions, for instance, the family of all hypernumber-valued functions with the topology of pointwise convergence, but the compactwise extrafunctions appear to be most useful in applications to differential equations. In the universe of extrafunctions, it is possible to find very general conditions for the existence of a solution to the Cauchy problem for systems of linear differential equations of the form

$$\frac{\partial u}{\partial t} = \sum_{k=1}^{n} a_k(t,x)\frac{\partial u}{\partial x_k} + a_0(t,x)u + f(t,x) = 0.$$

If we compare the theory of extrafunctions with the theories of generalized functions of Egorov and Colombeau, we see that each of them has its advantages. Both the spaces G_E of Egorov generalized functions and G_C of Colombeau generalized functions may be projected onto some subspaces of the space of extrafunctions. When it is possible to make these projections correlated with differentiation and other operations utilized in some class of PDE, then from existence of solutions for these PDE in the subset of extrafunctions that is the image of G_E (or G_C), it follows that the same PDE have solutions in G_E (correspondingly, G_C). Uniqueness of solutions in G_E (or G_C) implies uniqueness of solutions in the corresponding subset of extrafunctions. However, as is demonstrated in [**24**], G_E is not a Hausdorff space. This means, for example, that if a sequence has a limit, then it has many different limits. Besides, there is no standard inclusion of the space C^∞ of smooth functions into G_E. The space G_C possesses a standard inclusion of the space C^∞, but this inclusion is not homomorphic, i.e., the inclusion of C^∞ into G_C does not preserve products of smooth functions. The space of extrafunctions has neither of these shortcomings. Moreover, if, for example, G_E is projected onto some Hausdorff space X to get a "good" topology, then the results of the third section of this paper imply that this projection can be factored through a projection of G_E into extrafunctions. Relations between G_E, G_C and extrafunctions are discussed in Section 6.

Notation.

N is the set of natural numbers;

ω is the sequence of all natural numbers;

R is the set of all real numbers;

$\mathbf{R}_+$ is the set of all nonnegative real numbers;

$\mathbf{R}_{++}$ is the set of all positive real numbers;

$\mathbf{C}$ is the set of all complex numbers;

$\mathbf{C}^\omega$ is the set of all sequences of complex numbers;

if $a \in \mathbf{C}$, then $\|a\| = (|a_1|^2 + \cdots + |a_n|^2)^{1/2}$;

$\varnothing$ is the empty set;

if X and Y are topological spaces, then $F(X, Y)$ and $C(X, Y)$ are, respectively, the sets of all mappings of X into Y and the set of all continuous mappings of X into Y;

if $a = \{a_i\}_{i\in\omega}$ is a sequence of complex/real numbers, then $\alpha = Hn\{a_i\}_{i\in\omega}$ is the complex/real hypernumber determined by a;

$\mathbf{C}_\omega$ is the set of all complex hypernumbers;

$KF(D, \mathbf{C}_\omega)$ is the set of all complex compactwise extrafunctions on $D \subset \mathbf{R}^n$;

if $\{f_i\}_{i\in\omega}$ is a sequence of complex/real functions, then $f = Ep\{f_i\}_{i\in\omega}$ is the element of KF determined by $\{f_i\}_{i\in\omega}$ and,

if $\{f_i\}_{i\in\omega}$ is a sequence of complex/real functions, then $f = Ec\{f_i\}_{i\in\omega}$ is the complex/real compactwise extrafunction determined by $\{f_i\}_{i\in\omega}$.

2. Elements of the theory of hypernumbers.

Let $\mathbf{C}^\omega = \{\{a_i\}_{i\in\omega} : a_i \in \mathbf{C}\}$ be the set of all sequences of complex numbers.

Definition 2.1. For arbitrary $a = \{a_i\}_{i\in\omega}$ and $b = \{b_i\}_{i\in\omega}$ in $\mathbf{C}^\omega$ we say $a \sim b$ if $\lim_{i\to\infty} |a_i - b_i| = 0$.

Lemma 2.1. *The relation $\sim$ is an equivalence.*

Definition 2.2. The equivalence classes of the equivalence $\sim$ are called complex hypernumbers, and we denote the set of complex hypernumbers by $\mathbf{C}_\omega$.

The equivalence class of $a = \{a_i\}_{i\in\omega}$ will be denoted by $Hn\{a_i\}$. Real hypernumbers are defined by restricting the preceding definitions to sequences of real numbers.

The real and complex hypernumbers are equivalence classes of sequences just as rational numbers are sets of equivalent fractions and real numbers are sets of equivalent Cauchy sequences of rational numbers.

Remark 2.1. It is possible to replace the index set ω in the definition of hypernumbers by more general partially ordered sets. This produces new sets of hypernumbers. In particular, hypernumbers may be defined by ω^2-sequences, that is, sets of complex numbers that are indexed by elements from ω^2.

Relations on $\mathbf{C}$ induce corresponding relations on $\mathbf{C}_\omega$. For example, $\mathbf{C}$ is ordered by $c = a + bi < d = q + ri$ if and only if $a \le q$ and $b < r$ when $a = q$. This induces a partial order on $\mathbf{C}_\omega$ as follows:

Definition 2.3. If $a, b \in \mathbf{C}^\omega$, then $a \le b$ if and only if there is an n such that $a_i \le b_i$ for all $i \ge n$ and $a < b$ if and only if there is an n such that $a_i < b_i$ for all $i \ge n$.

From Definition 2.3 we proceed to

Definition 2.4. If $\alpha, \beta \in \mathbf{C}_\omega$, then $\alpha \le \beta$ if and only if $a \le b$ for some $a \in \alpha$ and $b \in \beta$, $\alpha < \beta$ if and only if $a < b$ for some $a \in \alpha$ and $b \in \beta$ and $\alpha \ne \beta$.

Lemma 2.2. *The relations $\le$ and $<$ on $\mathbf{C}_\omega$ are a partial order and a strict partial order, respectively.*

Proof. We begin with the relation $<$. By the definition, a strict partial order is a transitive antisymmetric relation. Thus, we have to test these properties for $<$ on $\mathbf{C}_\omega$.

1. The relation $<$ is transitive on $\mathbf{C}_\omega$.

Let $\alpha < \beta$ and $\beta < \delta$ for some $\alpha, \beta, \delta \in \mathbf{C}_\omega$. Then, by the definition of $<$, there are sequences $\{a_i\} \in \alpha$, $\{b_i\} \in \beta$, $\{c_i\} \in \beta$ and $\{d_i\} \in \delta$ for which the following conditions are valid: for some natural number n if $i \ge n$, then $a_i < b_i$ and, for some natural number m, if $i \ge m$, then $c_i < d_i$. Then let $M = \max\{n, m\}$ and define $d'_i = d_i + |b_i - c_i|$. Then $\{d'_i\} \in \delta$ and for $i \ge M$, $a_i < d'_i$. Thus, by definition, $\alpha < \delta$.

2. The relation $<$ is antisymmetric.

Suppose this is false. Then we have $\alpha < \beta$ and $\beta < \alpha$. By the definition of the relation $<$, it follows that we have $\{a_i\}$, $\{a'_i\} \in \alpha$ and $\{b_i\}$, $\{b'_i\} \in \beta$ such that $a_i < b_i$ for $i \geq n$ and $a'_i > b'_i$ for $i \geq m$. However, this implies that $\alpha = \beta$, since we have for $i \geq \max\{n, m\}$,

$$a_i < b_i < a'_i + |b_i - b'_i|.$$

By definition, neither of the relations $\alpha < \beta$ or $\alpha > \beta$ can hold if $\alpha = \beta$, so we have reached a contradiction.

If one simply changes $<$ to $\leq$ in all places, the preceding proof shows that $\leq$ is also transitive and that $\alpha \leq \beta$ and $\beta \leq \alpha$ together imply $\alpha = \beta$.

Definition 2.5. Given a sequence $\{\alpha_n\}$ in $\mathbf{C}^\omega$, we define $\lim_{n\to\infty} \alpha_n = \alpha$ by the following. Given $\varepsilon > 0$ and sequences $\{a_i^n\} = \alpha_n$ and $\{a_i\} = \alpha$, there are integers $N(\varepsilon)$ and $I(\varepsilon, n)$, defined for $n \geq N(\varepsilon)$ such that, for $n \geq N(\varepsilon)$ and $i \geq I(\varepsilon, n)$, one has $|a_i^n - a_i| < \varepsilon$.

Proposition 2.1. *The convergence defined in Definition* 2.5 *does not depend on the choices of elements from the equivalence classes.*

Proof. Let $\alpha_n = \{a_i^n\}$ converge to $\alpha = \{a_i\}$, and assume that $\{a_i^n\} \sim \{b_i^n\}$. To prove Proposition 2.1 we have to show that $\beta_n = \{b_i^n\}$ also converges to α. However, by definition, given $\varepsilon > 0$, for each n there is a $J(n, \varepsilon)$ such that $|b_i^n - a_i^n| < \varepsilon/2$ when $i \geq J(n, \varepsilon)$. In addition, Definition 2.5 gives us $M(\varepsilon/2)$ and $L(n, \varepsilon/2)$ such that $|a_i - a_i^n| < \varepsilon/2$ when $n \geq M(\varepsilon/2)$ and $i \geq L(n, \varepsilon/2)$. Thus, taking the $N(\varepsilon)$ and $I(n, \varepsilon)$ in Definition 2.5 to be $M(\varepsilon/2)$ and $\max\{J(n, \varepsilon/2), L(n, \varepsilon/2)\}$, respectively, we see that $\lim_{n\to\infty} \beta_n = \alpha$.

On $\mathbf{C}^\omega$ one can introduce the function

$$d(\{a_i\}, \{b_i\}) = \frac{\limsup |a_i - b_i|}{1 + \limsup |a_i - b_i|}$$

with $d(\{a_i\}, \{b_i\}) = 1$ when $\limsup |a_i - b_i| = \infty$. Since $\limsup |a_i = b_i| = 0 \Leftrightarrow \{a_i\} \sim \{b_i\}$ and d satisfies the triangle inequality, one sees that d is a metric on $\mathbf{C}_\omega$.

Lemma 2.6. *The convergence of Definition* 2.5 *is equivalent to metric convergence in* $(\mathbf{C}_\omega, d)$.

Theorem 2.1. $(\mathbf{C}_\omega, d)$ *is a complete metric space.*

Proof. Given a Cauchy sequence in the metric d, $\{\alpha_n\}$, we must show that it converges. We assume that $\{\alpha_i^n\} \in \alpha_n$. Then, for each $l \in \mathbf{N}$, there is an $N(l)$ such that

$$\limsup_{i\to\infty} |a_i^n - a_i^m| < 2^{-l} \quad \text{for } n, m \geq N(l).$$

Increasing the values of the $N(l)$s, starting with $N(1)$, we can assume that $\{N(l)\}$ is a strictly increasing sequence. From the definition of "limsup," it follows that for each l there is an $I(l)$ such that

$$|a_i^{N(l)} - a_i^{N(l+1)}| < 2^{-l+1} \quad \text{for } i \geq I(l).$$

As before, increasing the values of the $I(l)$s we can assume that $\{I(l)\}$ is also a strictly increasing sequence.

Define the sequence $\{a_i\}$ by $a_i = 0$ for $i < I(1)$ and $a_i = a_i^{N(l)}$ for $I(l) \leq i < I(l+1)$. Letting α denote the equivalence class of $\{a_i\}$, we claim that α is the limit of $\{\alpha_n\}$. Since the limit of any subsequence of a Cauchy sequence is the limit of the whole sequence, it will suffice to show that α is the limit of $\{\alpha_{N(l)}\}$. Assume that $k \leq l$ and $I(l) \leq i < I(l+1)$. Then we have

$$\begin{aligned} |a_i^{N(k)} - a_i| &= |a_i^{N(k)} - a_i^{N(l)}| \\ &\leq |a_i^{N(k)} - a_i^{N(k+1)}| + \cdots + |a_i^{N(l-1)} - a_i^{N(l)}| \\ &\leq 2^{-k+1} + \cdots + 2^{-l+2} < 2^{-k+2}. \end{aligned}$$

Since this inequality holds for all $i \geq I(k)$, $\limsup |a_i^{N(k)} - a_i| \leq 2^{-k+2}$, and

$$\lim_{k\to\infty} d(\alpha_{N(l)}, \alpha) = 0.$$

Remark 2.2. For the space of real hypernumbers $\mathbf{R}_\omega$, the result of Theorem 2.1 was obtained in [**7**] for a more general construction of hypernumbers.

Theorem 2.2. $\mathbf{C}_\omega$ *is a vector space over* $\mathbf{C}$.

Proof. We have for $a, b \in \mathbf{C}$ and $\alpha, \beta \in \mathbf{C}_\omega$ we define $a\alpha + b\beta$ to be the equivalence class of $\{aa_i + bb_i\}$ where $\alpha \sim \{a_i\}$ and $\beta \sim \{b_i\}$. One checks easily that this is independent of the representatives chosen and satisfies the vector space axioms.

On $\mathbf{C}^\omega$ one can introduce the topology in which a set is open if each of its points has a neighborhood

$$N_r(\{p_i\}) = \{\{a_i\}; d(\{a_i\}, \{p_i\}) < r\}$$

for some $r > 0$, where

$$d(\{a_i\}, \{p_i\}) = \frac{\limsup |a_i - p_i|}{1 + \limsup |a_i - p_i|}$$

as before. However, this topology does not satisfy any separation axiom: if $\{a_i\} \sim \{b_i\}$, then any open set containing $\{a_i\}$ will contain $\{b_i\}$. When one considers $\mathbf{C}^\omega$ with this topology, the space $\mathbf{C}_\omega$ has the following maximality property.

Theorem 2.3. $\mathbf{C}_\omega$ *is the largest Hausdorff quotient space of* $\mathbf{C}^\omega$.

Proof. Since $\mathbf{C}_\omega$ is a Hausdorff space, to prove the theorem, it is necessary to demonstrate that if a Hausdorff space X is a quotient space of $\mathbf{C}^\omega$ with the projection $q : \mathbf{C}^\omega \to X$, then there is a continuous projection $v : \mathbf{C}_\omega \to X$ for which $q = pv$. Let us consider such a Hausdorff space X with the continuous projection $q : \mathbf{C}^\omega \to X$. Then, for any points $x, y \in X$, $x \neq y$, we have disjoint neighborhoods $\mathcal{O}_x$ of x and 0_y of y. Since q is continuous $q^{-1}(\mathcal{O}_x)$ and $q^{-1}(\mathcal{O}_y)$ will be disjoint open sets in $\mathbf{C}^\omega$. Since these sets are open, each must contain the entire equivalence class in $\mathbf{C}_\omega$ of each of its elements. In particular, $p^{-1}(p(q^{-1}(\{x\})) \subset q^{-1}(\mathcal{O}_x)$ and $p^{-1}(p(q^{-1}(\{y\})) \subset q^{-1}(\mathcal{O}_y)$. Thus we can define r by $r(x) = p(q^{-1}(\{x\})$ for $x \in X$. This makes r well-defined and continuous and completes the proof. $\square$

This theorem shows that the set of complex hypernumbers is a topological extension of the set of all complex numbers, while the set of hypercomplex numbers which is introduced in nonstandard analysis [**37**], is a set-theoretical extension of the set of all complex numbers [**11**].

3. Extrafunctions and hyperdistributions. To apply the idea behind the construction of hypernumbers to differential equations, one needs to have spaces of extended functions based on hypernumbers. We will consider two such spaces, "compactwise extrafunctions," denoted by $KF(\mathbf{R}^n, \mathbf{C}^m_\omega)$ and "hyperdistributions," denoted by $HD(\mathbf{R}^n)$. In what follows we will always denote extended functions by capital letters, U, F, Φ, etc., to help distinguish them from lowercase ordinary functions, u, f, ϕ, etc.

The space $KF(\mathbf{R}^n, \mathbf{C}^m_\omega)$ is the space of sequences $\{f_n\}$ of continuous functions on $\mathbf{R}^n$ with values in $\mathbf{C}^m$ with the equivalence relation

$$\{f_n\} \sim \{g_n\} \Longleftrightarrow \lim_{n\to\infty} \max_{x\in K} |f_n(x) - g_n(x)| = 0$$

for all compact subset sets K of $\mathbf{R}^n$. On $KF(\mathbf{R}^n, \mathbf{C}^m_\omega)$ we have the family of pseudo-metrics parametrized by the compact subsets of $\mathbf{R}^n$

$$d_K(\{f_n\}, \{g_n\}) = \frac{\limsup\{\max_{x\in K} |f_n(x) - g_n(x)|\}}{1 + \limsup\{\max_{x\in K} |f_n(x) - g_n(x)|\}}.$$

As before, $KF(\mathbf{R}^n, \mathbf{C}^m_\omega)$ is a Hausdorff space in the topology associated with this family of pseudo-metrics and, by standard constructions (cf. [**30**, Theorem 12, p. 231]) one can introduce a uniformity so that it is complete as well.

For applications to differential equations one needs to have derivatives. For this one can consider the subsets of $KF(\mathbf{R}^n, \mathbf{C}^m_\omega)$ corresponding to the sequences $\{f_n\}$ where each f_n is infinitely differentiable, and define the partial derivative $\partial^\alpha F$, the "sequential derivative of F" to be the equivalence class of $\{\partial^\alpha f\}$. This definition has the drawback that $\partial^\alpha F$ is not uniquely determined by F. This led us to introduce the space of "hyperdistributions," $HD(\mathbf{R}^n)$. This space is defined in strict analogy to the definition of hypernumbers as follows. The elements of $HD(\mathbf{R}^n)$ are sequences of locally integrable functions with the equivalence relation

$$\{f_n\} \sim \{g_n\} \Longleftrightarrow \lim_{n\to\infty} \int_{\mathbf{R}^n} (f_n - g_n)\phi \, dx = 0$$

for all $\phi \in C_c^\infty(\mathbf{R}^n)$. Note that every equivalence class here contains a sequence of smooth functions: if we choose a sequence of compact sets $\{K_n\}$ expanding to $\mathbf{R}^n$ and consider the mollifications $(f_n)_\varepsilon$ (cf. [**28**, Theorem 8.14]), we can choose a sequence $\{\varepsilon_n\}$ so that

$$\|f_n - (f_n)_{\varepsilon_n}\|_{L^1(K_n)} < \frac{1}{n}.$$

Clearly, $\{(f_n)_{\varepsilon_n}\}$ is equivalent to $\{f_n\}$. Now, given $F \in HD(\mathbf{R}^n)$, we define $\partial^\alpha F$ to be the equivalence class of $\{\partial^\alpha f_n\}$ where $\{f_n\}$ is an element of the equivalence class of F consisting of smooth functions. This definition only depends on the equivalence class of F: if $\{g_n\}$ is another sequence of smooth functions from the equivalence class of F, we have

$$\lim_{n\to\infty} \int_{\mathbf{R}^n} (\partial^\alpha f_n - \partial^\alpha g_n)\phi\, dx = \lim_{n\to\infty} (-1)^{|\alpha|} \int_{\mathbf{R}^n} (f_n - g_n)\partial^\alpha \phi\, dx = 0.$$

If we replace $\mathbf{R}^n$ by an open set $D \subset \mathbf{R}^n$, then we have $KF(D, \mathbf{C}_\omega^m)$ and $HD(D)$ defined in the obvious ways.

4. The Cauchy problem for extrafunctions. We will consider the linear Cauchy problems

$$(1) \qquad \frac{\partial U}{\partial t} = \sum_{k=1}^n a_k(t,x)\frac{\partial U}{\partial x_k} + a_0(t,x)U + F(t,x), \quad U(0,x) = \Phi(x),$$

where U and F belong to $KF(D, \mathbf{C}^m)$ and a_k, $k = 0, 1, \dots, n$ are $m\times m$ matrices of continuous functions on $[-1,1] \times \mathbf{R}^n$. The domains D are open subsets of $\mathbf{R}_t \times \mathbf{R}_x^n$. We require the differential equations to hold in D and the initial conditions to hold on $D \cap \{t = 0\}$. The initial value, Φ, is in $KF(\mathbf{R}^n, \mathbf{C}^m)$. However, we will assume that Φ and F have defining sequences consisting of continuous functions. As is well known (see, for instance, [**29**]) linear, noncharacteristic Cauchy problems of arbitrary order can be reduced to the form (1). Note that (1) is well defined since each of our three spaces is a module over the algebra of smooth functions. However, since derivatives are not uniquely defined in the KF spaces, we only require that (1) hold for *some* choices of the derivatives in those spaces.

We do not consider the Cauchy problem for hyperdistributions. Note that for any smooth function $a(x)$ the hyperdistribution U corresponding to the sequence $\{a(x)(1-\cos jt)\}_{j=1}^{\infty}$ satisfies $\partial U/\partial t = 0$ in the sense of hyperdistributions and $U|_{t=0} = 0$. Since the constant sequence $\{a(x)\}$ belongs to the equivalence class of U, it is clear that even in this simplest case there is no uniqueness in the solution of the Cauchy problem. On the other hand, one can check easily that the only solution to $\partial U/\partial t = 0$ with $U|_{t=0} = 0$ in the sense of compactwise extrafunctions is $U = 0$.

The main result of this paper is the following.

Theorem 4.1. *The Cauchy problem* (1) *has a solution* U *in* KF *on a neighborhood of* $\{t=0\}$ *in* $\mathbf{R}_t \times \mathbf{R}_x^n$. *Moreover, the* U *has a defining sequence of real-analytic functions.*

Proof. By the Weierstrass approximation theorem, we can assume that we have sequences of matrices with polynomial entries $\{a_k^j\}$, $k = 0, 1, \dots, n$ such that a_k^j converges uniformly to a_k on $D_l = [-1,1] \times \{|x| \le l\}$ for all l. Likewise, since F and Φ are assumed to have defining sequences $\{f_j\}$ and $\{\phi_j\}$ of continuous functions, the Weierstrass approximation theorem implies that we can choose vector-valued functions with polynomial entries $\hat{f}_j$ and $\hat{\phi}_j$ such that $f_j - \hat{f}_j$ converges uniformly to zero on compact subsets of $[-1,1] \times \mathbf{R}^n$ and $\phi_j - \hat{\phi}_j$ converges uniformly to zero on compact subsets of $\mathbf{R}^n$. Note that this means that $\{\hat{f}_j\}$ is a defining sequence for F and $\{\hat{\phi}_j\}$ is a defining sequence for Φ in each of our spaces of generalized functions. We will apply the Cauchy-Kowalewski theorem to the problems

$$(2_j)\quad \frac{\partial u_j}{\partial t} = \sum_{k=1}^{n} a_k^j(x,t) \frac{\partial u_j}{\partial x_k} + a_0^j(x,t) u_j + \hat{f}_j(x,t), \quad u_j(0,x) = \phi_j(x).$$

If, for each j, u_j satisfies (2_j), then $U \sim \{u_j\}$ is a solution of (1). The Cauchy-Kowalewski theorem (see [**21, 42**]) states that (2_j) has a unique real-analytic solution on $D_R = \{(t,x) : |x| \le R, |t| \le T(R)\}$ for some $T(R) > 0$ for each R. Moreover, one can choose $T(R)$, $0 < T(R) \le 1$, so that it depends only on n, m and the maximum norms of the entries in a_k^j, $k = 0, 1, \dots, n$, on $[-1,1] \times \{x : |x| \le R\}$. By construction these norms are bounded independently of j for each $R \in \mathbf{R}_{++}$. Thus we can assume that all the functions in the sequence of solutions $\{u_j^R\}$ is defined on a fixed domain $D_R = [-T(R), T(R)] \times \{|x| \leq R\}$. Moreover, when we change R, the uniqueness of real analytic solutions implies that $u_j^R = u_j^{R'}$ on $D_R \cap D_{R'}$. Thus, taking u_j to be the unique extension of the $\{u_j^R : R \geq 1\}$ to $\cup_{R\geq 1} D_R$ and, defining $U \sim \{u^j\}$, the theorem follows.

If one assumes that the coefficients a_k are bounded, then one has a stronger existence theorem.

Theorem 4.2. *If the coefficients* $a_k(x,t)$, $k = 0, 1, \ldots, n$, *are bounded on* $[-1,1] \times \mathbf{R}^n$, *then* (1) *has a solution in* KF *on* $D_T = (-T, T) \times \mathbf{R}^n$ *for some* $T > 0$.

Proof. In this case the polynomial approximations $\{a_k^j\}$, $k = 0, 1, \ldots, n$, are converging uniformly on compact subsets to functions which are globally bounded. Thus, assuming that all entries of the matrices $a_k(x,t)$, $j = 0, 1, \ldots, n$, are bounded by B on $[-1,1] \times \mathbf{R}^n$, for each R there is a $j(R)$ such that each entry of $\{a_k^j\}$, $k = 0, 1, \ldots, n$, is bounded by $B+1$ on $[-1,1] \times \{|x| \leq R\}$ for $j \geq j(R)$. Hence, by the Cauchy-Kowalewski theorem in the form quoted above, there is a $T_0 > 0$ such that, for all $R \geq 1$ and $j \geq j(R)$, the solution of (2_j) is defined on $[-T_0, T_0] \times \{|x| \leq R\}$. In other words, in this case, instead of having the solutions u_j of (2_j) defined on a fixed neighborhood of $\{t = 0\}$ as in Theorem 4.1, here they are defined on a sequence of neighborhoods D_j which increase to $[-T_0, T_0] \times \mathbf{R}^n$ as $j \to \infty$. Hence, if we extend the u_js continuously to all of $[-T_0, T_0] \times \mathbf{R}^n$ not requiring that the extensions solve (2_j), the resulting generalized function $U \sim \{u_j\}$ will nonetheless be a solution of (1) on $[-T_0, T_0] \times \mathbf{R}^n$. This completes the proof. $\square$

5. Relation of extrafunctions to distributions. There are different equivalent ways to define distributions. The definition of a distribution as a functional was historically the first [**41**]. Another approach is called sequential because in it distributions are defined as classes of equivalent sequences of ordinary functions [**1**]. The latter definition is useful for comparing distributions and extrafunctions, and we will discuss it here.

Definition 5.1. A sequence $\{f_j\}$ of functions in $C^\infty(D)$, $D \subset \mathbf{R}^n$, is called fundamental on D if, for any compact set $K \subset D$, there are a sequence $\{F_j\}$ and a multi-index $\alpha \in \mathbf{N}^n$ such that $\partial^\alpha F_j = f_j$ on K and F_j converges uniformly on K.

Given a fundamental sequence $\{f_j\}$ for each compact $K \subset D$, we have the continuous function F_K on K, $F_K = \lim F_j$ on K, and we associate the functional distribution

$$l(\phi) = (-1)^{|\alpha_K|} \int F \partial^{\alpha_K} \phi \, dx, \quad \phi \in C_0^\infty(D) \quad \text{with support in } K$$

with $\{f_j\}$. We will say two fundamental sequences are equivalent if they give the same distributions.

If $\{f_j\}$ is a fundamental sequence, then $D\{f_j\}$ denotes the class of all sequences equivalent to $\{f_j\}$ as well as the corresponding distribution. Let $f = D\{f_j\}$ be a distribution. Then for any β, $\partial^\beta f$ is defined by $\partial^\beta f = D\{\partial^\beta f_j\}$.

Any sequence of complex functions defines both a compactwise and restricted pointwise extrafunctions. However, only some of the sequences of complex functions define distributions, [**1**]. In what follows, we consider only compactwise extrafunctions on $\mathbf{R}^n$. Let $KF(\mathbf{R}^n)$ be the class of complex compactwise extrafunctions $\mathbf{R}^n$. As was observed in Section 3, this is a linear space over the field $\mathbf{C}$.

Theorem 5.1. *There is the linear subspace $DKF(\mathbf{R}^n) \subset KF(\mathbf{R}^n)$ for which a linear projection $p : DKF \to \mathcal{D}'$ exists.*

The proof of this theorem is essentially based on the following lemma.

Lemma 5.1. *If two fundamental sequences $\{f_j\}$ and $\{g_j\}$ of smooth complex functions define the same compactwise extrafunction, then they define the same distribution.*

Proof. Let $\{f_j\}$ and $\{g_j\}$ be fundamental sequences defining the same compactwise extrafunction f. Then $\lim_{j\to\infty} \sup_K |f_j(x) - g_j(x)| = 0$ for all compact sets $K \subset \mathbf{R}^n$. For any $\phi \in C_c^\infty(\mathbf{R}^n)$, letting $\partial^\alpha F_j = f_j$

and $\partial^\beta G_j = g_j$ in Definition 5.1, we have

$$(-1)^{|\alpha|}\int F\partial^\alpha\phi\,dx - (-1)^{|\beta|}\int G\partial^\beta\phi\,dx = \lim_{j\to\infty}\int (f_j - g_j)\phi\,dx = 0.$$

Proof of Theorem 5.1. Let us take FEE to be the space of all fundamental sequences of smooth complex functions. On one hand, there is a morphism $q : FEE \to KF$ because any sequence of complex functions defines some compactwise extrafunction. Let DKF be the image of FEE, i.e., $DKF = q(FEE)$. On the other hand, there is a projection $r : FEE \to \mathcal{D}'$. By Lemma 5.1, it is possible to factor q through r, i.e., there is a morphism $p : DKF \to \mathcal{D}'$ such that $r = qp$. As r and q are linear projections, p is also a linear projection.

The projection p defines an equivalence relation on $DKF : f \sim g$ if $p(f) = p(g)$, i.e., $f \sim g$ if and only if f and g define the same distribution. The proof of Theorem 5.1 shows that the mapping p commutes with the operation of differentiation. Thus, we have the following result.

Theorem 5.2. *For any complex compactwise extrafunction $f \in DKF$, the sequential derivative $\partial^\alpha f$ defines a distribution g that is equal to the derivative $\partial^\alpha p(f)$ of the distribution $p(f)$.*

Thus, when a compactwise extrafunction defines a distribution, its sequential extraderivatives coincide with the derivatives of the distribution. These results show that it is possible to reduce the problem of integrating differential equations in the spaces of distributions to the problem of integrating differential equations in the spaces of extrafunctions.

6. Relation of extrafunctions to the generalized functions of Colombeau and Egorov. To extend the scope of solvable differential equations, Egorov [**23**] constructs an algebra of generalized functions G_E. He begins by defining a space of generalized complex numbers, $\tilde{\mathbf{C}}$. The elements of $\tilde{\mathbf{C}}$ are equivalence classes of sequences $\{c_i\}$ from the one-point compactification, $\mathbf{C}^0$, of the complex plane. Two sequences $\{c_i\}$ and $\{d_i\}$ are equivalent if they coincide beyond some point ($c_i = d_i$

for $i \geq I$). The set $\tilde{\mathbf{C}}$ is an algebra but not a field. It is also a topological space. This topological space is complete in a generalized sense, but it is not a Hausdorff space. In particular, limits of sequences are not unique in $\tilde{\mathbf{C}}$.

If we compare $\tilde{\mathbf{C}}$ with the set of hypercomplex numbers from nonstandard analysis, we see that both sets are constructed as quotients of the set of sequences of complex numbers. The difference is that $\tilde{\mathbf{C}}$ is obtained by factorization by the filter of all cofinite subsets of $\mathbf{N}$, while the hypercomplex numbers are defined through factorization by an ultrafilter that contains the filter of all cofinite subsets of $\mathbf{N}$. As a result, $\tilde{\mathbf{C}}$ is only an algebra, but the hypercomplex numbers form a field.

Let D be a domain in $\mathbf{R}^n$. The algebra of generalized functions G_E on D is defined as follows. The elements of G_E are equivalence classes of sequences of functions in $C^\infty(D)$. Two sequences $\{f_i\}$ and $\{g_i\}$ are equivalent if, for each compact set $K \subset D$, there is an integer I_K such that $g_i = f_i$ on K for $i > I_K$. The set G_E is a topological space which is complete in a generalized sense, but it is not a Hausdorff space.

Some generalized functions determine distributions: if the limit

$$\lim_{i\to\infty} \int f_n \phi \, dx = l(\phi)$$

exists for all $\phi \in C_0^\infty(D)$, then one identifies the generalized function defined by $\{f_n\}$ with l.

Partial derivatives of generalized functions in G_E are determined by the corresponding sequences of derivatives, i.e., if $f \sim \{f_i\}$, then $\partial^\alpha f \sim \partial^\alpha f_i$. When f is identified with a distribution l, $\partial^\alpha f$ is identified with $\partial^\alpha l$ (Egorov [**23**]). Hence the projection of G_E onto distributions commutes with differentiation.

Theorem 6.1. *There is a homomorphism p of G_E as a module over $C^\infty(D)$ into $KF(D)$ as a module over $C^\infty(D)$, and we have $p(\partial^\alpha f) = \partial^\alpha p(f)$.*

Proof. If $\{f_i\}$ is a sequence defining $f \in G_E$, we define $p(f)$ to be the extrafunction defined by $\{f_i\}$. Since the equivalence relation on G_E is stronger than the equivalence relation on KF, p is well defined.

Operations on G_E (addition, subtraction and multiplication by smooth functions) are induced by the corresponding operations on sequences of functions (Egorov [**23**]). The same is true for KF. Consequently, p is a homomorphism of the module G_E over the algebra $C^\infty(D)$ into $KF(D)$. The final statement, $p(\partial^\alpha f) = \partial^\alpha p(f)$, follows since derivatives are defined sequentially in G_E and KF.

Theorem 6.1 has the following corollary.

Corollary 6.1. *If a system of linear differential equations $Hu = f$ has a solution in G_E, then one has $Hp(u) = p(f)$ in KF. Thus, any linear system of differential equations with smooth coefficients with a solution in G_E has a solution in KF.*

As Egorov's theory of generalized functions includes the theory of Colombeau (Egorov [**23**]), it follows that linear systems of equations which are solvable in Colombeau's generalized functions are solvable in KF. However, one can explain the relations between extrafunctions and Colombeau's generalized functions directly. The basic idea underlying Colombeau theory (in its simplest form) is that of embedding the space of distributions into a factor algebra of $C^\infty(D)^I$ with $I = (0,1]$ and $D \subset \mathbf{R}^n$, namely, the Colombeau algebra $G(D)$ or $G_C(D)$ is the quotient algebra $E_M(D)/N(D)$ where

$$\begin{aligned} E_M(D) = \{\{u_t\}_{t\in I} : {} & u_t \in C^\infty(D), \text{ and for all compact } K \subset D \\ & \text{and all multi-indices } \alpha \text{ there is an integer } p \text{ such that} \\ & \sup_K |\partial^\alpha u_t| = O(t^p)\}, \\ N(D) = \{\{u_t\}_{t\in I} : {} & u_t \in C^\infty(D), \text{ and for all compact } K \subset D \\ & \text{and all multi-indices } \alpha, \\ & \sup_K |\partial^\alpha u_t| = O(t^q) \text{ for all } q\}. \end{aligned}$$

If we consider the compactwise extra-functions, $KF(D, I)$, based on the index set I instead of $\mathbf{N}$, then there is a projection of the algebra $G(D)$ into $KF(D, I)$. This makes it possible to build a differential projection leading to the same results as in Theorem 6.1 and its corollary for Colombeau's generalized functions.

In a more general setting (see Kunzinger and Oberguggenberger [**31**]) the Colombeau algebra $G(D)$ is a factor algebra of $C^\infty(D)^F$, where F is a set of real-valued functions. However, as compact-wise extrafunctions may be defined for arbitrary partially ordered sets of indices (Burgin [**8**]), we have the same results on the solvability of linear PDE in this setting as well.

7. Conclusion. Thus we have demonstrated that introducing extrafunctions makes it possible to "solve" a much larger class of PDE than can be solved in the space of distributions. Theories of generalized functions, which appeared as extensions of distributions, also do not add anything essential to the collection of soluble equations in comparison with those that are soluble by means of extrafunctions. With these properties the theory of extrafunctions changes the problem of integrating differential equations. Instead of seeking to get a solution in terms of ordinary functions, one would ask what is the class of extrafunctions defined by a given equation. For example, the class of extrafunctions that corresponds to the equation of Lewy does not contain either ordinary functions or distributions.

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