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SPACES OF OPERATORS, c_0 AND l^1

ELIZABETH M. BATOR AND PAUL W. LEWIS

ABSTRACT. If Y is a Banach space so that l^1 embeds isomorphically as a complemented subspace of the separable space Y^* but c_0 does not embed as a subspace of Y, then it is shown that there is an infinite dimensional Banach space X so that l^1 embeds complementably in $X \otimes_\gamma Y^*$ but c_0 does not embed in $L(X, Y)$.

In a classic paper on the structure of Banach spaces [**2**], Bessaga and Pelczynski established the following result.

Theorem 1. *If c_0 embeds isomorphically in the dual X^* of the Banach space X, then l^∞ embeds in X^* and l^1 embeds complementably in X.*

The following complete generalization of Theorem 1 was established in [**7**]. In this theorem (e_n^*) denotes the canonical unit vector basis of l^1 and $X \otimes_\gamma Y^*$ denotes the greatest crossnorm tensor product completion of X and Y^*.

Theorem 2. *If X is an infinite dimension Banach space and c_0 embeds in $L(X, Y)$, then l^∞ embeds in $L(X, Y)$ and there is an isomorphism $J : l^1 \to X \otimes_\gamma Y^*$ so that $J(l^1)$ is complemented in $X \otimes Y^*$ and $J(e_n^*)$ is a finite rank tensor for each n.*

Of course, the converse of Theorem 1 is immediate, i.e., if l^1 embeds complementably in X, then certainly l^∞ (and thus c_0) embeds in X^*. The status of the converse of Theorem 2 is not clear at all. There is an example on page 215 of [**7**] which purports to show that the complementability of l^1 in the greatest crossnorm tensor product completion of X and Y^* does not imply that c_0 embeds in the space $L(X, Y)$ of all bounded linear transformations from X to Y. However,

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this example is based on an erroneous statement in [**3**]. Specifically, it is asserted on page 249 of [**3**] that $l^p \otimes_\gamma l^p$ contains a complemented copy of l^1 if $1 < p < \infty$. If this statement were true for $p > 2$, then l^1 would embed as a complemented subspace in the dual of $L(l^p, l^{p'})$, where $1/p + 1/p' = 1$. However, it is well documented that this space of operators is reflexive, e.g., see Kalton [**5**].

In this note we show that there is an isomorphism $J : l^1 \to X \otimes_\gamma Y^*$ so that $J(e_n^*)$ is finite rank for each n and $J(l^1)$ is complemented if and only if c_0 embeds in $L(X, Y^{**})$. Further a celebrated result of James [**4**] and a theorem of Bator [**1**] are used to construct a family of spaces $X \otimes_\gamma Y^*$ so that l^1 is complemented in each of these spaces but c_0 does not embed in $L(X, Y)$. This construction depends upon Theorem 2 above.

Theorem 3. *If X and Y are arbitrary Banach spaces, then c_0 embeds isomorphically in $L(X, Y^{**})$ if and only if there is an isomorphism $J : l^1 \to X \otimes_\gamma Y^*$ so that $J(e_n^*)$ is a finite rank tensor for every n and $J(l^1)$ is complemented in $X \otimes_\gamma Y^*$.*

However, if Y is a Banach space so that l^1 embeds isomorphically as a complemented subspace of the separable space Y^* but c_0 does not embed as a subspace of Y, then there is an infinite dimensional Banach space X and an isomorphism $J : l^1 \to X \otimes_\gamma Y^*$ so that $J(l^1)$ is complemented in $X \otimes_\gamma Y^*$, $J(e_n^*)$ is a finite rank tensor for each n and c_0 does not embed in $L(X, Y)$.

Proof. Since $(X \otimes_\gamma Y^*)^*$ is isometrically isomorphic to $L(X, Y^{**})$, it is clear from Theorem 2 (or the classical Bessaga-Pelczynski theorem) that c_0 embeds in $L(X, Y^{**})$ if and only if l^1 embeds as a complemented subspace in $X \otimes_\gamma Y^*$. Thus, to finish the proof of the first assertion in the theorem, it suffices to show that if $c_0 \hookrightarrow L(X, Y^{**})$, then there is an isomorphism $J : l^1 \to X \otimes_\gamma Y^*$ so that $J(l^1)$ is complemented and $J(e_n^*)$ is finite rank for each n.

Suppose then that $T : c_0 \to L(X, Y^{**})$ is an isomorphism, (x_n) is a bounded sequence in X and (y_n^*) is a bounded sequence in Y^* so that $\langle T(e_n)x_n, y_n^* \rangle = 1$ for each n. The proof of Theorem 1 in [**6**] and Theorem 1 in [**7**] shows that there is a sequence (u_n) of *differences* of the rank one tensors $(x_n \otimes y_n^*)_{n=1}^\infty$ so that (u_n) is equivalent to (e_n^*)

and $[u_n]$ is complemented in $X \otimes_\gamma Y^*$.

Now suppose that Y satisfies the hypotheses of the second portion of the theorem, e.g., see [**4**]. Use Theorem 4 of [**1**] and let X be an infinite dimensional Banach space so that every member of $L(X,Y)$ is compact, i.e., $L(X,Y) = K(X,Y)$.

First we show that c_0 does not embed isomorphically in $L(X,Y)$. Suppose (to the contrary) that $c_0 \hookrightarrow L(X,Y)$. By Theorem 2 above, $l^\infty \hookrightarrow L(X,Y)$. Since $L(X,Y) = K(X,Y)$, a result of Kalton [**5**, p. 271], shows that $l^\infty \hookrightarrow X^*$ or $l^\infty \hookrightarrow Y$. The hypothesis that c_0 does not embed in Y precludes the second possibility. Therefore we assume that $l^\infty \hookrightarrow X^*$. An application of Theorem 1 or Theorem 2 above ensures that l^1 embeds complementably in X. Theorem 5 of [**1**] produces the desired contradiction. That is, if Z is any separable infinite dimensional subspace of Y, then there is a bounded linear operator S from l^1 onto Z. Projecting X onto l^1 and following this projection with S produces a noncompact member of $L(X,Y)$.

To finish the argument, it suffices to show that l^1 embeds appropriately as a complemented subspace of $X \otimes_\gamma Y^*$. Suppose that W is a subspace of Y^* so that W is isomorphic to l^1 and $P : Y^* \to W$ is a projection. Let x be any norm-1 element of X, and let $Q : X \to [x]$ be a projection. Then $[x] \otimes_\gamma W$ is isomorphic to l^1, e_n^* is identified with a rank one tensor with respect to this isomorphism, and $Q \otimes P$ is a projection onto $[x] \otimes_\gamma W$. $\square$

REFERENCES

1. Elizabeth M. Bator, *Unconditionally converging and compact operators on* c_0, Rocky Mountain J. Math. **22** (1992), 417–422.

2. C. Bessaga and A. Pelczynski, *On bases and unconditional convergence in Banach spaces*, Studia Math. **17** (1958), 151–164.

3. J. Diestel and J.J. Uhl, Jr., *Vector measures*, Math. Surveys **15**, Amer. Math. Soc., Providence, 1977.

4. R.C. James, *Separable conjugate spaces*, Pacific J. Math. **10** (1960), 563–571.

5. N. Kalton, *Spaces of compact operators*, Math. Ann. **208** (1974), 267–278.

6. P. Lewis, *Mapping properties of* c_0, Colloq. Math. **80** (1999), 235–244.

7. ———, *Spaces of operators and* c_0, Studia Math. **145** (2001), 213–218.

Department of Mathematics, University of North Texas, Denton, Texas 76203
E-mail address: bator@unt.edu

Department of Mathematics, University of North Texas, Denton, Texas 76203
E-mail address: lewis@unt.edu