

GEOMETRIC CONSTRUCTIONS ON CYCLES

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ABSTRACT. A point, plane or sphere in \mathbf{R}^n can be described as a point on the Lie quadric $\Omega \subset \mathbf{P}^{n+2}$, and a geometric construction on points, planes and spheres as a map which associates a point $y \in \Omega$ to a given k -tuple $(x_1, \dots, x_k) \in \Omega^k$. In this paper the Apollonius construction is described as a map $\mathcal{A} : \mathcal{D} \rightarrow \Omega$, where \mathcal{D} is a subset of Ω^{n+1} . A number of geometric constructions is obtained by composing the map \mathcal{A} with Lie reflections and some other projective transformations in \mathbf{P}^{n+2} .

1. Introduction. A geometric construction in the space \mathbf{R}^n can be viewed as a map on a set, containing geometric objects described in an appropriate way. In this paper constructions on points, and oriented hyperspheres and hyperplanes in \mathbf{R}^n are considered. A suitable way to describe such geometric constructions comes from Lie geometry. In Lie geometry, oriented planes and spheres of dimension $n - 1$ in \mathbf{R}^n , which are together called *geometric cycles*, are described as points on a quadric surface Ω in the projective space \mathbf{P}^{n+2} , while the angle of intersection is expressed in terms of the Lie product in \mathbf{R}^{n+3} . In this setting, a geometric construction on cycles can be thought of as a map from Ω^k to Ω , which associates to a given k -tuple of points, representing geometric objects in \mathbf{R}^n , a point in Ω , representing a solution of the construction. Lie geometry has been used to study geometric problems on circles for example in [3, 5, 6] and [7]. A thorough treatment of Lie geometry can be found in [1] or [2].

A basic example of a construction on cycles is the oriented Apollonius construction in \mathbf{R}^n , which asks for a sphere or plane, tangent to $(n + 1)$ given spheres and planes. In [7], a solution of an Apollonius construction is described as a point in the intersections of a projective line in \mathbf{P}^{n+2} with Ω , and a classification of Apollonius constructions,

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depending on the position of this projective line in the space \mathbf{P}^{n+2} , is given. In this paper we think of the Apollonius construction as a map \mathcal{A} , defined on a subset \mathcal{D} of Ω^{n+1} containing all $(n+1)$ -tuples of points in Ω which determine constructions with a solution (we call such constructions *consistent*). In general, a consistent Apollonius construction can have more than one solution. To define the single valued map \mathcal{A} , an additional condition is identified. This additional condition is first described in algebraic terms, and then its geometric meaning is analyzed.

In addition, we consider Lie reflections, and several other projective transformations in \mathbf{P}^{n+2} . We describe the geometric effect that such a transformation on a cycle $x \in \Omega$ has on the underlying geometric cycle and show that a wide class of classical geometric constructions on cycles can be described as compositions of the map \mathcal{A} with these transformations.

In this paper we use the same notation and terminology as in [7]. In Section 3 we introduce the single-valued map \mathcal{A} which associates to an $(n+1)$ -tuple of points from \mathcal{D} the unique solution satisfying an additional condition. In Section 4 we define some projective transformations, which we use, in Section 5, to generate a number of geometric constructions. In order to make the paper easier to read, we begin with a short description of the terminology and a summary of the basic facts about Lie geometry.

2. Basic concepts and definitions. An element x , denoted by a lower case letter, in the $(n+2)$ -dimensional real projective space \mathbf{P}^{n+2} is called a *cycle* and is given by a nonzero vector of homogeneous coordinates $X = (X^0, \dots, X^{n+2})$, denoted by the corresponding capital letter. The *Lie product* of vectors of homogeneous coordinates, given by

$$(1) \quad \begin{aligned} (X \mid Y) &= X^0 Y^{n+1} + X^1 Y^1 + \dots + X^n Y^n + X^{n+1} Y^0 - X^{n+2} Y^{n+2} \\ &= X^T \mathbf{A} Y, \end{aligned}$$

where

$$(2) \quad \mathbf{A} = \begin{bmatrix} 0 & \mathbf{0} & 1 & 0 \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & 0 \\ 1 & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

determines the quadric surface $\Omega = \{x \in \mathbf{P}^{n+2} \mid (X \mid X) = 0\}$ in \mathbf{P}^{n+2} , called the *Lie quadric*.

Every cycle $x \in \Omega$, except the cycle w with homogeneous coordinates $W = (1, 0, \dots, 0)$, represents an oriented *geometric cycle* \mathcal{C}_x , which is either an $(n-1)$ -plane or an $(n-1)$ -sphere in \mathbf{R}^n , as follows. Let $\{\varphi_i: \mathcal{U}_i \rightarrow \mathbf{R}^{n+2} \subset \mathbf{R}^{n+3}\}$ be the standard collection of local coordinate charts on \mathbf{P}^{n+2} , i.e., $\mathcal{U}_i = \{x \in \mathbf{P}^{n+2} \mid X^i \neq 0\}$, and

$$\begin{aligned} \varphi_i(x) &= \left(\frac{X^0}{X^i}, \dots, \frac{X^{i-1}}{X^i}, 1, \frac{X^{i+1}}{X^i}, \dots, \frac{X^{n+2}}{X^i} \right) \\ &\in (\{Y^0, \dots, Y^{i-1}, 1, Y^{i+1}, \dots, Y^{n+2}\}) \cong \mathbf{R}^{n+2}. \end{aligned}$$

- If $x \in \mathcal{U}_{n+1} \cap \Omega$, then the local coordinates

$$\varphi_{n+1}(x) = (v, \mathbf{p}, 1, \rho), \quad \mathbf{p} \in \mathbf{R}^n$$

represent the sphere with center \mathbf{p} and radius $|\rho|$. If $\rho > 0$ this sphere is positively oriented (outward normal), if $\rho < 0$ it is negatively oriented, and if $\rho = 0$ it is the point \mathbf{p} , a sphere with radius 0.

- If $x \in \Omega \setminus \mathcal{U}_{n+1}$ and $x \neq w$, the condition $(X \mid X) = 0$ implies $X^{n+2} \neq 0$, so $x \in \mathcal{U}_{n+2}$. The local coordinates

$$\varphi_{n+2}(x) = (\eta, \mathbf{n}, \omega, 1), \quad \omega = 0, \quad \mathbf{n} \in \mathbf{R}^n, \quad |\mathbf{n}|^2 = 1$$

represent the plane with normal \mathbf{n} , which is a unit vector since the condition $(X \mid X) = 0$ implies that $|\mathbf{n}|^2 - 1 = 0$, and $\eta = -\mathbf{n} \cdot \mathbf{q}$ where \mathbf{q} is a point on this plane (here ‘ \cdot ’ denotes the Euclidean product in \mathbf{R}^n).

Thus, every geometric cycle is represented by a point in $\Omega \setminus \{w\} \subset \mathcal{U}_{n+1} \cup \mathcal{U}_{n+2}$, the complement

$$(\Omega \setminus \{w\}) \setminus \mathcal{U}_{n+1} = (\Omega \setminus \{w\}) \cap \langle w \rangle^\perp$$

consists of cycles representing planes, and the complement

$$(\Omega \setminus \{w\}) \setminus \mathcal{U}_{n+2} = (\Omega \setminus \{w\}) \cap \langle r \rangle^\perp,$$

where r has homogeneous coordinates $R = (0, \mathbf{0}, 0, 1)$, consists of cycles representing points in \mathbf{R}^n .

The involution on \mathbf{P}^{n+2} , which associates to cycle $x \in \mathcal{U}_{n+2}$ with local coordinates $\varphi_{n+2}(x) = (\eta, \mathbf{n}, \omega, 1)$ the cycle x' with local coordinates $\varphi_{n+2}(x') = (-\eta, -\mathbf{n}, -\omega, 1)$, corresponds to a change of orientation. Cycles $x \in \Omega \setminus \mathcal{U}_{n+2}$, i.e., cycles representing points, are fixed by this involution.

The Lie product on homogeneous coordinates of cycles $x_1, x_2 \in \Omega$ has several geometric interpretations.

(L1) If $(X_1 \mid X_2) = 0$, then the corresponding geometric cycles \mathcal{C}_{x_1} and \mathcal{C}_{x_2} are coherently tangent. If both are planes or spheres, this means that they are tangent with compatible orientations, and if one is a point, then it lies on the other one (this is true also if one of the cycles is w , if we interpret w geometrically as the infinite point of \mathbf{R}^n).

(L2) For cycles $x_1, x_2 \in \mathcal{U}_{n+1} \cap \Omega$, i.e., nonplane cycles, the Lie product expressed in local coordinates in \mathcal{U}_{n+1} is

$$(X_1 \mid X_2) = v_1 + v_2 + \mathbf{p}_1 \cdot \mathbf{p}_2 - \rho_1 \rho_2 = -\frac{P(x_1, x_2)}{2},$$

where $P(x_1, x_2)$ is the *power* of x_1 to x_2 . If $P(x_1, x_2) \geq 0$, then it is the square of the tangential distance between the oriented spheres \mathcal{C}_{x_1} and \mathcal{C}_{x_2} . If x_1 is a point cycle, then $P(x_1, x_2) < 0$ implies that the geometric point \mathcal{C}_{x_1} is in the bounded component of $\mathbf{R}^n - \mathcal{C}_{x_2}$, and $P(x_1, x_2) > 0$ implies that \mathcal{C}_{x_1} is in the unbounded component of $\mathbf{R}^n - \mathcal{C}_{x_2}$.

(L3) For cycles $x_1, x_2 \in \mathcal{U}_{n+2} \cap \Omega$, i.e., nonpoint cycles, the Lie product expressed in local coordinates in \mathcal{U}_{n+2}

$$(X_1 \mid X_2) = \eta_1 \omega_2 + \eta_2 \omega_1 + \mathbf{n}_1 \cdot \mathbf{n}_2 - 1 = A(x_1, x_2)$$

is the *copower* of x_1 to x_2 , which is connected to the angle between geometric cycles in the following way. If $A(x_1, x_2)A(x'_1, x_2) \geq 0$, then \mathcal{C}_{x_1} and \mathcal{C}_{x_2} intersect, and

$$C(x_1, x_2) = A(x_1, x_2) + 1 = \cos \varphi,$$

where φ is the angle of intersection. If $A(x_1, x_2)A(x'_1, x_2) < 0$ then \mathcal{C}_{x_1} and \mathcal{C}_{x_2} do not intersect, and

$$C(x_1, x_2) = \frac{1}{\sin(\alpha/2)},$$

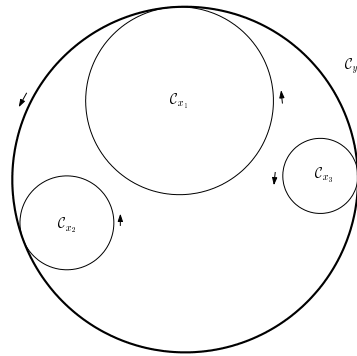


FIGURE 1. An Apollonius construction on three circles in the plane.

where α is the angle under which \mathcal{C}_{x_2} is seen from \mathcal{C}_{x_1} . More precisely, if \mathcal{C}_{x_1} is a sphere, α is the biggest angle between two geometric cycles that are tangent to \mathcal{C}_{x_2} and intersect \mathcal{C}_{x_1} in a main sphere, and if \mathcal{C}_{x_1} is a plane, α is the biggest angle between two lines, tangent to \mathcal{C}_{x_2} and intersecting on \mathcal{C}_{x_1} , i.e., the angle under which \mathcal{C}_{x_2} is seen from the closest point on \mathcal{C}_{x_1} .

The sign of $C(x_1, x_2)$ is connected to the orientations of \mathcal{C}_{x_1} and \mathcal{C}_{x_2} . If $C(x_1, x_2) > 0$ we say that x_1 and x_2 are *coherent*, where two intersecting cycles x_1 and x_2 are coherent if the angle of intersection of \mathcal{C}_{x_1} and \mathcal{C}_{x_2} is acute, and two non-intersecting finite cycles x_1 and x_2 are coherent if they induce the same orientation on \mathbf{R}^n . In general, two non-intersecting cycles x_1 and x_2 are coherent if there exists a continuous rigid motion of \mathbf{R}^n which moves \mathcal{C}_{x_2} within $\mathbf{R}^n \setminus \mathcal{C}_{x_1}$ to a cycle that is coherently tangent to \mathcal{C}_{x_1} .

3. The Apollonius problem. Given $n + 1$ oriented spheres and planes of dimension $n - 1$ in \mathbf{R}^n , the Apollonius problem is to find a common coherently tangent sphere or plane. In Lie geometry, an Apollonius problem is given by an $(n + 1)$ -frame of points $\{x_1, \dots, x_{n+1}\}$ on the Lie quadric $\Omega \subset \mathbf{P}^{n+2}$.

By (L1), solutions are the intersections of the Lie-orthogonal complement to this frame and the quadric. Thus, a solution y can be expressed as a solution of a system of $n + 1$ homogeneous linear equations and a

homogeneous quadratic equation

$$(3) \quad (Y \mid X_i) = 0, \quad i = 1, \dots, n+1, \quad (Y \mid Y) = 0.$$

In order to avoid constructions with infinitely many solutions we require that the homogeneous coordinates X_1, \dots, X_{n+1} are linearly independent vectors.

An $(n+1)$ -frame of points $\mathcal{X} = \{x_1, \dots, x_{n+1}\} \subset \Omega$ such that the homogeneous coordinates X_1, \dots, X_{n+1} are linearly independent is called a *configuration*. The Lie-orthogonal complement to a configuration is a projective line, which intersects the quadric in two points, in one point, or it misses the quadric, so the corresponding Apollonius problem has either two, one or no solutions. We say that a configuration is *consistent* if the corresponding Apollonius problem has at least one solution.

The configuration \mathcal{X} is a *Steiner configuration* if the vectors X_1, \dots, X_{n+1}, R are linearly independent, and a *cone configuration* if the vectors X_1, \dots, X_{n+1}, W are linearly independent. An Apollonius construction given by a Steiner configuration cannot have two point solutions (it cannot consist of cycles which intersect in two common points), while an Apollonius construction given by a cone configuration cannot have two solutions that are planes. A classification of Steiner configurations with respect to the existence and properties of the solutions is given in [7]. A configuration that is neither Steiner nor cone determines an Apollonius problem with no solutions, since the Lie-orthogonal complement is the projective line $\{\lambda W + \mu R \mid \lambda, \mu \in \mathbf{R}\}$ spanned by w and r which intersects the quadric only in the point w .

In this section, we will describe a consistent way to choose one preferred solution of every Apollonius problem with two solutions, depending on the order of the cycles in the configuration. In this way, we will be able to look at the Apollonius construction as a single-valued map \mathcal{A} (or as a well-defined algorithm), defined on the set of ordered consistent configurations $\mathcal{X} = (x_1, \dots, x_{n+1})$, with values in Ω .

3.1 Steiner configurations. Let \mathcal{X} be an ordered consistent Steiner configuration. We choose *standard homogeneous coordinates* \overline{Z} for every cycle $z \in \Omega$ in the following way:

(i) if $z \in \mathcal{U}_{n+2}$, standard coordinates are the local coordinates in \mathcal{U}_{n+2} : $\overline{Z} = (\eta, \mathbf{n}, \omega, 1)$, $\omega = 1/\rho$,

(ii) if $\rho = 0$, then $z \in \mathcal{U}_{n+1}$ and standard coordinates are the local coordinates in \mathcal{U}_{n+1} : $\overline{Z} = (v, \mathbf{p}, 1, 0)$.

There exists precisely one cycle u , such that U is Lie-orthogonal to the vectors X_1, \dots, X_{n+1} and R . Homogeneous coordinates of u can be computed from the $(n+2)$ -fold cross product

$$(4) \quad U = \times(\overline{X}_1, \dots, \overline{X}_{n+1}, R) := (-1)^{n+1} \mathbf{A}(\overline{X}_1 \times \dots \times \overline{X}_{n+1} \times R)$$

where \mathbf{A} is the matrix of the Lie bilinear form (2). The vector U depends on the order of the cycles in the configuration. For any vector $Z \in \mathbf{R}^{n+3}$,

$$\begin{aligned} (Z | U) &= Z^T \mathbf{A} U = (-1)^{n+1} Z^T (\overline{X}_1 \times \dots \times \overline{X}_{n+1} \times R) \\ &= D(Z, \overline{X}_1, \dots, \overline{X}_{n+1}, R) \end{aligned}$$

where $D(Z_0, \dots, Z_{n+2})$ denotes the determinant with columns Z_0, \dots, Z_{n+2} .

Lemma 3.1. *Let $\mathcal{X} = (x_1, \dots, x_{n+1})$ be an ordered Steiner configuration which generates an Apollonius problem with two different solutions y_1 and y_2 , and let the $D_i = (\overline{Y}_i | U)$, $i = 1, 2$. Then $D_1 \neq D_2$. If both D_1 and D_2 are nonzero, then $D_1 = -D_2$.*

Proof. First let us assume that $(U | U) = 0$. Since U is Lie-orthogonal to all X_i , $u \in \Omega$ represents a solution of the Apollonius problem. Since U is Lie-orthogonal to R , this solution is a common point of all cycles of the configuration. The second solution y_2 is not a point, since we have a Steiner configuration, and it does not contain the point $u = y_1$, so $(\overline{Y}_2 | U) \neq 0$ is either positive or negative.

Now let $(U | U) \neq 0$. In this case the problem has two non-point solutions $y_1, y_2 \in \mathcal{U}_{n+2}$. The vector $\overline{Y}_1 - \overline{Y}_2$ is Lie-orthogonal to all cycles of the configuration and its last coordinate is zero, so it is a vector of homogeneous coordinates of u . Let

$$(5) \quad \overline{Y}_1 - \overline{Y}_2 = \lambda U, \quad \lambda \neq 0.$$

Since

$$\lambda(\overline{Y}_1 \mid U) = (\overline{Y}_1 \mid \overline{Y}_1 - \overline{Y}_2) = -(\overline{Y}_1 \mid \overline{Y}_2)$$

and

$$\lambda(\overline{Y}_2 \mid U) = (\overline{Y}_2 \mid \overline{Y}_1 - \overline{Y}_2) = (\overline{Y}_2 \mid \overline{Y}_1),$$

it follows that $(\overline{Y}_1 \mid U) = -(\overline{Y}_2 \mid U)$. \square

Definition 3.1. If the Apollonius problem is given by a consistent ordered Steiner configuration \mathcal{X} , we define $\mathcal{A}(\mathcal{X}) = y_1$, where y_1 is chosen so that $D_1 \geq D_2$.

Thus, if the problem has two solutions, neither of which is a point, y_1 is the one with $D_1 > 0$. If one solution is a point, then the choice of y_1 depends on the sign of the determinant of the non-point solution.

In order to give the choice y_1 a geometric meaning, let us first look at the special case where all cycles x_i of the configuration are points. The two solutions $y_{1,2}$ represent the unique nonoriented sphere or plane through these points with both orientations, so $Y_{1,2} = U \pm aR$ where $a = \sqrt{(U \mid U)} > 0$. Let us show that $Y_1 = U + aR$. Since $\overline{Y}_1 = U/a + R$ and

$$\begin{aligned} D_1 &= D(\overline{Y}_1, \overline{X}_1, \dots, \overline{X}_{n+1}, R) = \frac{1}{a} D(U, \overline{X}_1, \dots, \overline{X}_{n+1}, R) \\ &= \frac{(U \mid U)}{a} = a > 0. \end{aligned}$$

The component

$$(6) \quad U^{n+1} = (\mathbf{A}U)^1 = \begin{vmatrix} \mathbf{p}_1 & \cdots & \mathbf{p}_{n+1} \\ 1 & \cdots & 1 \end{vmatrix}$$

is the volume of the simplex spanned by the points \mathbf{p}_i . Since

$$\varphi_{n+1}(y_1) = (v_1, \mathbf{p}_1, 1, \rho_1) = \frac{1}{U^{n+1}}(U + aR),$$

it follows that $\rho_1 = a/U^{n+1}$ is of the same sign as U^{n+1} . Therefore, \mathcal{C}_{y_1} is oriented consistently with the simplex spanned by the points of the configuration.

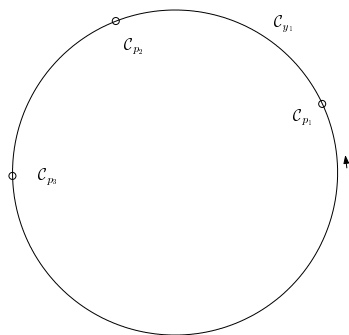


FIGURE 2. An ordered configuration on points.

On the other extreme, let \mathcal{X} be an ordered configuration with no point cycles. For every $i = 1, \dots, n+1$, and for every j , such that y_j is not a point cycle, the vector $P_{i,j} = \overline{X}_i - \overline{Y}_j$ is Lie-orthogonal to X_i , to Y_j , and to R , and $(P_{i,j} \mid P_{i,j}) = 0$, so it is a vector of homogeneous coordinates of the point of tangency of \mathcal{C}_{x_i} and \mathcal{C}_{y_j} . If

$$V'_j = \times(\overline{X}_1 - \overline{Y}_j, \dots, \overline{X}_{n+1} - \overline{Y}_j, R),$$

then

$$\begin{aligned} (\overline{Y}_j \mid V'_j) &= D(\overline{Y}_j, (\overline{X}_1 - \overline{Y}_j), \dots, (\overline{X}_{n+1} - \overline{Y}_j), R) \\ &= D(\overline{Y}_j, \overline{X}_1, \dots, \overline{X}_{n+1}, R) = (\overline{Y}_j \mid U). \end{aligned}$$

The standard homogeneous coordinates of $P_{i,j}$ are

$$\overline{P}_{i,j} = \frac{1}{s_{i,j}} P_{i,j}, \quad s_{i,j} = (P_{i,j})^{n+1} = \frac{1}{\rho_{x_i}} - \frac{1}{\rho_{y_j}}.$$

The orthogonal vector corresponding to the point-configuration $(p_{1,j}, \dots, p_{n+1,j})$ is thus

$$V_j = \times(\overline{P}_{1,j}, \dots, \overline{P}_{n+1,j}, R) = \frac{1}{s_{1,j} \cdots s_{n+1,j}} V'_j,$$

and

$$(7) \quad (\overline{Y}_j \mid U) = (\overline{Y}_j \mid V'_j) = s_{1,j} \cdots s_{n+1,j} (\overline{Y}_j \mid V_j).$$

The sign of $D_j = (\overline{Y}_j \mid U)$ thus depends on the sign of $(\overline{Y}_j \mid V_j)$ and on the signs the coefficients $s_{i,j}$.

By (6), the sign of $(V_j \mid \overline{Y}_j)$ is positive if the orientation of the solution \mathcal{C}_{y_j} coincides with the orientation of the simplex spanned by the points of tangency $p_{i,j}$, $i = 1, \dots, n+1$. The sign of

$$s_{i,j} = \frac{1}{\rho_{x_i}} - \frac{1}{\rho_{y_j}}$$

depends on the position of \mathcal{C}_{y_j} with respect to \mathcal{C}_{x_i} and is positive when the solution \mathcal{C}_{y_j} lies in the negative side of the oriented geometric cycle \mathcal{C}_{x_i} .

In general, if the configuration \mathcal{X} contains point cycles x_i , then the corresponding coefficients $s_{i,j}$ in (7) are substituted by 1.

Thus we have proved:

Proposition 3.1. *If a Steiner configuration determines an Apollonius problem with two nonpoint solutions, then the chosen solution $y_1 = \mathcal{A}(\mathcal{X})$ is either oriented consistently with the simplex spanned by the points of tangency and an even number of cycles in the configuration are on its positive side, or it is oriented nonconsistently, and an odd number of cycles in the configuration are on its positive side. This is true also if y_2 is a point. On the other hand, if one solution is a point, and the only nonpoint solution does not have this property, then y_1 is the point solution.*

The sign of $(U \mid U)$ is connected to the angle between the two solutions in the following way.

Proposition 3.2. *Assume that the Apollonius problem has two nonpoint solutions, i.e., that $(U \mid U) \neq 0$. Then the copower $A(y_1, y_2) = (\overline{Y}_1 \mid \overline{Y}_2)$ is of opposite sign as $(U \mid U)$. Thus,*

- *If $(U \mid U) > 0$, then $C(y_1, y_2) = A(y_1, y_2) + 1 < 1$. This implies that either \mathcal{C}_{y_1} and \mathcal{C}_{y_2} intersect, when $|C(y_1, y_2)| < 1$, or they have opposite orientations when $C(y_1, y_2) \leq -1$.*
- *If $(U \mid U) < 0$, then $A(y_1, y_2) > 0$ and $C(y_1, y_2) > 1$ and the geometric solutions do not intersect and are coherent.*

Proof. By (5), $(U \mid U) = 1/\lambda^2(\overline{Y}_1 - \overline{Y}_2 \mid \overline{Y}_1 - \overline{Y}_2) = -2/\lambda^2(\overline{Y}_1 \mid \overline{Y}_2) = -2/\lambda^2 A(y_1, y_2)$. \square

Proposition 3.3. *The collinearity factor λ in (5) is of the same sign as $(U \mid U)$.*

Proof. $D_1 = (\overline{Y}_1 \mid U) = (\overline{Y}_1 - \overline{Y}_2 \mid U) + (\overline{Y}_2 \mid U) = \lambda(U \mid U) + D_2 > D_2$, so $\lambda(U \mid U) > 0$. \square

3.2 Cone configurations. A consistent ordered non-Steiner configuration \mathcal{X} determines an Apollonius problem that has only point solutions. In order to choose the preferred point y_1 , we treat such a configuration as a cone configuration. We choose standard homogeneous coordinates \overline{Z} for every cycle $z \in \Omega$ in the following way:

(i) if $z \in \mathcal{U}_{n+1}$, then we use local coordinates in \mathcal{U}_{n+1} : $\overline{Z} = (v, \mathbf{p}, 1, \rho)$,

(ii) if $\omega = 0$, then $z \in \mathcal{U}_{n+2}$ and we use local coordinates in \mathcal{U}_{n+2} : $\overline{Z} = (\eta, \mathbf{n}, 0, 1)$.

In this case there exists precisely one cycle u , such that U is Lie-orthogonal to the vectors X_1, \dots, X_{n+1} and W . Homogeneous coordinates of u can be computed from the $(n+2)$ -fold cross product

$$(8) \quad U = \times(\overline{X}_1, \dots, \overline{X}_{n+1}, W) := (-1)^{n+2} \mathbf{A}(\overline{X}_1 \times \dots \times \overline{X}_{n+1} \times W).$$

For any vector $Z \in \mathbf{R}^{n+3}$,

$$\begin{aligned} (Z \mid U) &= Z^T \mathbf{A}U = (-1)^{n+2} Z^T (\overline{X}_1 \times \dots \times \overline{X}_{n+1} \times W) \\ &= D(Z, \overline{X}_1, \dots, \overline{X}_{n+1}, W). \end{aligned}$$

Lemma 3.2. *Let $\mathcal{X} = (x_1, \dots, x_{n+1})$ be an ordered cone configuration which generates an Apollonius problem with two different solutions y_1 and y_2 , and let the $D_i = (\overline{Y}_i \mid U)$, $i = 1, 2$. Then $D_1 \neq D_2$. If both D_1 and D_2 are nonzero, then they are of opposite sign.*

Proof. We first assume that $(U \mid U) = 0$. In this case $u \in \Omega$ is a plane-solution of the Apollonius problem. The second solution y_2 is

not a plane, since we have a cone configuration, and it is not tangent to the plane $u = y_1$ (if it were, the problem would have infinitely many solutions, since every cycle in the projective line spanned by u and y_2 would be a solution), so $(\bar{Y}_2 | U) \neq 0$ is either positive or negative.

Now let $(U | U) \neq 0$. In this case the problem has two non-plane solutions $y_1, y_2 \in \mathcal{U}_{n+1}$. The vector $\bar{Y}_1 - \bar{Y}_2$ is a vector of homogeneous coordinates of u , so $\bar{Y}_1 - \bar{Y}_2 = \lambda U$, $\lambda \neq 0$. Since

$$\lambda(\bar{Y}_1 | U) = (\bar{Y}_1 | \bar{Y}_1 - \bar{Y}_2) = -(\bar{Y}_1 | \bar{Y}_2)$$

and

$$\lambda(\bar{Y}_2 | U) = (\bar{Y}_2 | \bar{Y}_1 - \bar{Y}_2) = (\bar{Y}_2 | \bar{Y}_1),$$

it follows that $(\bar{Y}_1 | U) = -(\bar{Y}_2 | U)$. \square

Definition 3.2. If the Apollonius problem is given by a consistent ordered non-Steiner configuration \mathcal{X} , we define $\mathcal{A}(\mathcal{X}) = y_1$, where y_1 is chosen so that $D_1 \geq D_2$.

Remark 1. A generic configuration is both Steiner and cone. Here we have chosen to treat such a configuration as a Steiner configuration. We can just as well treat such a configuration as a cone configuration. In this case, standard coordinates of a cycle in $\mathcal{U}_{n+1} \cap \mathcal{U}_{n+2}$ are the local coordinates in \mathcal{U}_{n+1} and not in \mathcal{U}_{n+2} , and the algorithm for the choice y_1 is similar. This approach is better suited for example for studying continuous families of Apollonius problems consisting only of cone configurations but containing also Steiner configuration.

4. Transformations of cycles. The goal of this paper is to provide the tools for symbolic solutions of as many geometric constructions on cycles as possible. This can be achieved by combining the map \mathcal{A} with some other projective transformations on \mathbf{P}^{n+2} . These are divided into two classes. Projective transformations, arising from linear transformations on homogeneous coordinates preserving the Lie product, preserve the Lie quadric. All such transformations are generated by Lie reflections, which we describe first. Among these, certain transformations preserve either angles or powers. On the other hand, projective transformations arising from linear transformations that do not preserve the

Lie product generally move cycles from the quadric off the quadric, and deform angles and powers.

4.1 Lie reflections. For any cycle $z \in \mathbf{P}^{n+2} \setminus \Omega$, the projective map

$$\mathbf{L}[z]: \mathbf{P}^{n+2} \rightarrow \mathbf{P}^{n+2}$$

given by

$$(9) \quad \mathbf{L}[z](X) = X - 2 \frac{(X | Z)}{(Z | Z)} Z$$

is a *Lie reflection*. The name ‘reflection’ is justified by the fact that the corresponding matrix

$$\mathbf{L}[z] = \mathbf{I} - \frac{2}{(Z | Z)} (\mathbf{A}Z) Z^T$$

has $\det(\mathbf{L}[z]) = -1$ and $\mathbf{L}[z]^2 = \mathbf{I}$. Since

$$\begin{aligned} (\mathbf{L}[z](X_1) | \mathbf{L}[z](X_2)) &= (X_1 - 2 \frac{(X_1 | Z)}{(Z | Z)} Z | X_2 - 2 \frac{(X_2 | Z)}{(Z | Z)} Z) \\ &= (X_1 | X_2), \end{aligned}$$

a Lie reflection $\mathbf{L}[z]$ preserves the Lie quadric,

$$\mathbf{L}[z]: \Omega \rightarrow \Omega.$$

Actually, Lie reflections generate the group $O_A(n+3)$ of Lie-orthogonal transformations in \mathbf{R}^{n+3} , [2, Theorem 2.3]. A simple computation shows that the Lie reflection $\mathbf{L}[z]$ fixes the cycle z as well as any cycle x , Lie-orthogonal to z .

Here are some interesting special cases.

- The Lie reflection $\mathbf{L}[r]$ is fixed on point cycles x . If x is a nonpoint cycle, then $\mathbf{L}[r]$ changes the orientation, i.e.,

$$\mathbf{L}[r](X) = X - 2 \frac{(X | R)}{(R | R)} R = X + 2(X | R)R, \quad \mathbf{L}[r](x) = x'.$$

• If $(Z \mid R) = 0$, then the Lie reflection $\mathbf{L}[z]$ preserves copowers. Lie reflections $\mathbf{L}[z]$ with this property generate a subgroup of the group $O_A(n+3)$ which is called the Möbius group. If $(Z \mid Z) > 0$, then the two cycles \tilde{z}, \tilde{z}' with homogeneous coordinates $\tilde{Z}, \tilde{Z}' = Z \pm \sqrt{(Z \mid Z)}R$ are in Ω . The Lie reflection $\mathbf{L}[z]$ corresponds to geometric inversion in \mathbf{R}^n with respect to the underlying nonoriented geometric cycle. If $(Z \mid Z) < 0$, then the cycles $\tilde{z}, \tilde{z}' \in \Omega$ with homogeneous coordinates $\tilde{Z}, \tilde{Z}' = Z \pm \sqrt{-(Z \mid Z)}R$ represent a sphere with both orientations, since $(Z \mid Z) < 0$ is possible only if $Z_{n+1} \neq 0$. The Lie reflection $\mathbf{L}[z]$ in this case corresponds to the composition of geometric inversion over this sphere and over its center.

• If $(Z \mid W) = 0$ then $\mathbf{L}[z]$ preserves powers. All such reflections generate a subgroup of $O_A(n+3)$ called the Laguerre group.

Let \mathcal{X} be an ordered Steiner configuration with two nonpoint tangent cycles, and let u be the orthogonal cycle with homogeneous coordinates given by (4). Then $(U \mid U) \neq 0$ so the orthogonal cycle u is not on the quadric and the Lie reflection $\mathbf{L}[u]$ is defined. Since $(U \mid X_i) = 0$, $i = 1, \dots, n+1$, the cycles of the configuration are fixed by $\mathbf{L}[u]$.

Proposition 4.1. *If y and y' are two different nonpoint solutions of the Apollonius problem determined by a Steiner configuration \mathcal{X} , then $\mathbf{L}[u](y) = y'$.*

Proof. Since $(U \mid R) = 0$, the Lie reflection $\mathbf{L}[u]$ preserves angles, so it maps a solution of the Apollonius problem to a solution. Thus the two solutions y and y' are either both fixed by $\mathbf{L}[u]$, or $\mathbf{L}[u](y) = y'$. Assume that y is a fixed point of $\mathbf{L}[u]$. Then $y \in \langle u \rangle^\perp$ and it is easy to see that the system (3) has only one solution [7, Case 2.a], so $y = y'$ which contradicts our assumptions. Therefore $\mathbf{L}[u](y) = y'$. \square

Let y be a solution of an Apollonius problem given by a Steiner configuration and let

$$V = \times(\overline{P}_1, \dots, \overline{P}_{n+1}, R)$$

be the orthogonal cycle of the configuration consisting of the points of tangency. Then $(V \mid V) > 0$ and V_{n+1} is the volume of the simplex

spanned by the points \mathcal{C}_{p_i} in \mathbf{R}^n . Since $\mathbf{L}[u]$ preserves tangency,

$$\mathbf{L}[u](\overline{P}_i) = P'_i, \quad i = 1, \dots, n+1$$

where p'_i represent the points of tangency of the second solution y' . Let $\overline{U} = \varphi_{n+1}(U)$. Then it follows from Definition 9 that

$$(P')^{n+1} = 1 - \frac{(\overline{U} | \overline{P}_i)}{(\overline{U} | \overline{U})} = \frac{\lambda_i^2}{(\overline{U} | \overline{U})},$$

where

$$\lambda_i^2 = \sum_1^n (\overline{U}^k - (\overline{P}_i)^k)^2.$$

The standard homogeneous coordinates of P'_i are

$$\overline{P}'_i = \frac{P'_i}{(P'_i)^{n+1}} = \frac{P'_i(\overline{U} | \overline{U})}{\lambda_i^2}.$$

If $V' = \times(\overline{P}'_1, \dots, \overline{P}'_{n+1}, R)$ is the orthogonal cycle of the configuration of tangent points of the second solution, then

$$\begin{aligned} (\mathbf{L}[u](V) | V') &= \frac{(U | U)^{n+1} D(\mathbf{L}[u](V), \mathbf{L}[u](\overline{P}_1), \dots, \mathbf{L}[u](\overline{P}_{n+1}), R)}{\lambda_1^2 \cdots \lambda_{n+1}^2} \\ &= \frac{(U | U)^{n+1} \det(\mathbf{L}[u]) D(V, \overline{P}_1, \dots, \overline{P}_{n+1}, R)}{\lambda_1^2 \cdots \lambda_{n+1}^2} \\ &= -\frac{(U | U)^{n+1} (V | V)}{\lambda_1^2 \cdots \lambda_{n+1}^2} \end{aligned}$$

Since $(V | V) > 0$,

$$\text{sign}(\mathbf{L}[u](V) | V') = -\text{sign}(U | U)^{n+1}.$$

What does this tell us? If $(U | U) > 0$, and if y the solution to the Apollonius problem which is oriented consistently with the simplex on the points of tangency $\mathcal{C}_{p_1}, \dots, \mathcal{C}_{p_{n+1}}$, then the second solution $y' = \mathbf{L}[u](y)$ will not be oriented consistently with the simplex on the points of tangency $\mathcal{C}_{p'_1}, \dots, \mathcal{C}_{p'_{n+1}}$. If $(U | U) < 0$, then the situation

depends on the dimension n of the problem. If n is even, for example in the plane, Lie reflection across U preserves the consistency of orientations, which implies that the parity of the number of coefficients $s_{i,j}$ that are positive has to change). If n is odd, for example in the space \mathbf{R}^3 , Lie reflection across U does not preserve consistency of orientations of the solution with the simplex on the point of tangency.

4.2 Angle and power deforming transformations. The map \mathcal{A} on cycles from the Lie quadric gives solutions of the classical Apollonius construction on these cycles. In order to solve generalized Apollonius constructions which ask for a geometric object that intersects given $(n + 1)$ geometric objects under prescribed angles or powers, we introduce several projective transformations which, in general, do not preserve the Lie product, and thus deform angles and powers.

These angle deforming transformations are special cases of the following general projective transformation. For a triple of cycles z , z_1 and z_2 , such that $(Z | Z_1) \neq 0$, $(Z | Z_2) \neq 0$, let

$$\mathbf{H}[z, z_1, z_2](X) = X - \frac{(Z_1 | Z_2)(Z | X)}{(Z_1 | Z)(Z_2 | Z)}Z.$$

In order to simplify notation, we will use the same symbol for the linear transformation on homogeneous coordinates as for the induced projective transformation on cycles.

The following special cases of this general transformation are particularly interesting from the geometric point of view, and can be used for solving generalized Apollonius constructions and other constructions which involve non-zero angles and powers between geometric objects.

- For any cycle $x \neq r$,

$$\mathbf{H}[r, r, r](X) = (X_0, \dots, X_{n+1}, 0),$$

so $\mathbf{H}[r, r, r]$ is the projection in $\mathbf{P}^{n+2} - \{r\}$ onto the projective subspace $\langle r \rangle^\perp$.

- Let $Z = R$, and let $z_1, z_2 \in \mathcal{U}_{n+2}$ be nonpoint cycles with Z_1, Z_2 their local coordinates in \mathcal{U}_{n+2} . Define

$$\begin{aligned} \tilde{\mathbf{H}}[z_1, z_2](X) &= \mathbf{H}[r, z_1, z_2](X) = X - (Z_1 | Z_2)(X | R)R \\ &= X - A(z_1, z_2)(X | R)R. \end{aligned}$$

Every point cycle $x \notin \mathcal{U}_{n+2}$ is fixed by $\tilde{\mathbf{H}}[z_1, z_2]$. If $x \in \mathcal{U}_{n+2}$ is a nonpoint cycle, and $c \in \mathcal{U}_{n+2}$ is Lie-orthogonal to $\tilde{\mathbf{H}}[z_1, z_2](X)$, then

$$0 = (\tilde{\mathbf{H}}[z_1, z_2](X) \mid C) = (X \mid C) - A(z_1, z_2)(X \mid R)(R \mid C),$$

so $A(x, c) = A(z_1, z_2)$. A nonpoint cycle c is thus Lie orthogonal to $\tilde{\mathbf{H}}[z_1, z_2](X)$ precisely when the copower $A(x, c)$ equals the copower $A(z_1, z_2)$.

- Let $Z = W$ and $z_1, z_2 \in \mathcal{U}_{n+1}$ be nonplane cycles with local coordinates Z_1, Z_2 in \mathcal{U}_{n+1} . Define

$$\begin{aligned} \hat{\mathbf{H}}[z_1, z_2](X) &= \mathbf{H}[w, z_1, z_2](X) \\ &= X - (Z_1 \mid Z_2)(X \mid W)W \\ &= X + \frac{1}{2}P(z_1, z_2)(X \mid W)W \end{aligned}$$

If $x \in \mathcal{U}_{n+1}$ is a nonplane cycle, and $c \in \mathcal{U}_{n+1}$ is Lie-orthogonal to X , then

$$0 = (\hat{\mathbf{H}}[z_1, z_2](X) \mid C) = (X \mid C) + \frac{1}{2}P(z_1, z_2)(X \mid C),$$

so

$$P(z_1, z_2) = -\frac{2(X \mid C)}{(X \mid W)(C \mid W)} = P(x, c).$$

- For a given $a \in \mathbf{R}$, let

$$\tilde{\mathbf{H}}_a(X) = X + (1 - a)(X \mid R)R.$$

The transformation $\tilde{\mathbf{H}}_a$ fixes all cycles $x \in \langle r \rangle^\perp$. The map $\mathbf{R} \rightarrow PGL(n + 3)$ which maps a to $\tilde{\mathbf{H}}_a$ is equivariant with respect to multiplications in \mathbf{R} , since

$$\tilde{\mathbf{H}}_{a_1}\tilde{\mathbf{H}}_{a_2} = \tilde{\mathbf{H}}_{a_1a_2}.$$

If $x, y \in \mathcal{U}_{n+2}$, and $(Y \mid \tilde{\mathbf{H}}_a(X)) = 0$, then

$$(Y \mid \tilde{\mathbf{H}}_a(X)) = (Y \mid X) - (a - 1)(X \mid R)(Y \mid R) = 0,$$

so $A(x, y) = (a - 1)$ and $C(x, y) = a$. A cycle y is thus Lie-orthogonal to $\tilde{\mathbf{H}}_a(X)$ precisely when $C(x, y) = a$.

- For a given $p \in \mathbf{R}$, let

$$\hat{\mathbf{H}}_p(X) = X + \frac{p}{2}(X | W)W.$$

The transformation $\hat{\mathbf{H}}_p$ fixes all cycles $x \in \langle w \rangle^\perp$. Since

$$\hat{\mathbf{H}}_{p_1} \hat{\mathbf{H}}_{p_2} = \hat{\mathbf{H}}_{p_1+p_2},$$

the map $\mathbf{R} \rightarrow PGL(n+3)$ which maps p to $\hat{\mathbf{H}}_p$ is equivariant with respect to addition in \mathbf{R} , so $\{\hat{\mathbf{H}}_p, p \in \mathbf{R}\}$ is a one-parametric subgroup of the group of projective transformations $PGL(n+3)$. If $x, y \in \mathcal{U}_{n+1}$, and $(Y | \hat{\mathbf{H}}_p(X)) = 0$, then

$$(Y | \hat{\mathbf{H}}_p(X)) = (Y | X) + \frac{p}{2}(X | W)(Y | W) = 0,$$

so $P(x, y) = p$, and \mathcal{C}_y has power p with respect to \mathcal{C}_x .

5. Geometric constructions. Let $\mathcal{D} \subset \Omega \times \cdots \times \Omega$ be the set of ordered consistent configurations. Let

$$\mathcal{A}: \mathcal{D} \rightarrow \Omega$$

be the map which associates to a given configuration the solution y_1 . The map \mathcal{A} is continuous on the set of Steiner configurations with no point cycles. A continuous deformation of a small positively oriented sphere in the configuration to a small negatively oriented sphere may cause a switch in the choice of the solution y_1 , and therefore a jump in the value of $\mathcal{A}(\mathcal{X})$. The map \mathcal{A} , restricted to the closed subset of \mathcal{D} consisting of non-Steiner, i.e., cone, configurations, is continuous on the subset of configurations with no plane cycles, while configurations containing plane cycles are points of discontinuity.

In this section we give examples of classical geometric constructions on spheres and planes, which are generated as compositions of Lie reflections, the transformations $\hat{\mathbf{H}}$ and $\tilde{\mathbf{H}}$, and the map \mathcal{A} .

Apollonius constructions. 1. An $(n+1)$ -tuple of points $\{\mathbf{p}_1, \dots, \mathbf{p}_{n+1}\}$ in \mathbf{R}^n determines a nonoriented $(n-1)$ -sphere precisely when the corresponding point cycles $x_i = (v_i, \mathbf{p}_i, 1, 0) \in \mathbf{P}^{n+2}$, $i = 1, \dots, n+1$ form

a Steiner configuration. The oriented $(n-1)$ -sphere with orientation consistent with the simplex $\langle \mathbf{p}_1, \dots, \mathbf{p}_{n+1} \rangle$, is given by

$$s = \mathcal{A}(x_1, \dots, x_{n+1}).$$

2. An n -tuple of points $\mathbf{p}_1, \dots, \mathbf{p}_n$ in \mathbf{R}^n spans an $n-1$ -plane in \mathbf{R}^n precisely when the configuration $\mathcal{X} = (x_1, \dots, x_n, w)$, $x_i = (v_i, \mathbf{p}_i, 1, 0)$, $i = 1, \dots, n$ is Steiner. The plane is given by

$$\Pi = \mathcal{A}(\mathcal{X}).$$

3. If $x_1, x_2 \in \mathbf{P}^4$ represent two nonpoint geometric cycles in the plane, then these cycles intersect precisely when the configuration $\mathcal{X} = (x_1, x_2, r)$ is consistent, and one intersection is given by

$$p = \mathcal{A}(\mathcal{X}).$$

4. Let $\varphi_1, \dots, \varphi_{n+1}$ be given angles and let $a_i = \cos \varphi_i$, $i = 1, \dots, n+1$. The cycle, intersecting $n+1$ given nonpoint geometric cycles $\mathcal{C}_{x_1}, \dots, \mathcal{C}_{x_{n+1}}$ under these angles, is

$$y = \mathcal{A}(\tilde{\mathbf{H}}_{a_1}(x_1), \dots, \tilde{\mathbf{H}}_{a_{n+1}}(x_{n+1})).$$

5. Let $l_1, l_2, l_3 \in \mathbf{P}^4$ represent lines in the plane. The line \mathcal{C}_l , intersecting \mathcal{C}_{l_1} in the point represented by p under the angle of intersection of \mathcal{C}_{l_2} and \mathcal{C}_{l_3} , is given by

$$l = \mathcal{A}(\tilde{\mathbf{H}}[l_1, l_2](l_3), p, w)$$

6. Let q_1, \dots, q_{n+1} be given powers and x_1, \dots, x_{n+1} given nonplane cycles. The geometric cycle which has power q_i with respect to \mathcal{C}_{x_i} , $i = 1, \dots, n+1$ is given by

$$y = \mathcal{A}(\hat{\mathbf{H}}_{q_1}(x_1), \dots, \hat{\mathbf{H}}_{q_{n+1}}(x_{n+1})).$$

7. Figure 3 shows the solution of the following problem. Given a power and three cycles in the plane, we are looking for the cycle which

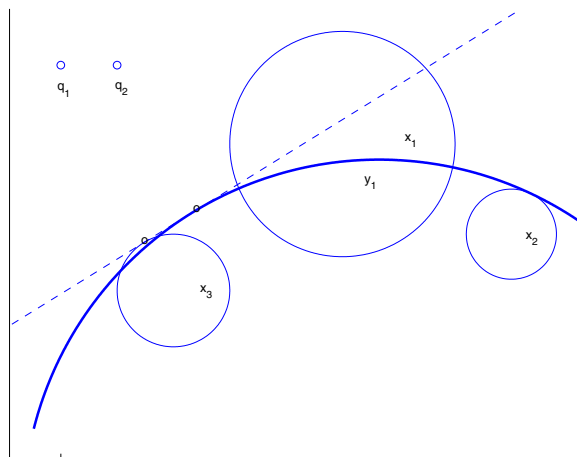


FIGURE 3. A general Apollonius problem.

has given power with respect to the first cycle, is tangent to the second, and orthogonal to the third cycle.

$$y_1 = \mathcal{A}(H[r, r, r]x_1, x_2, H([w, q_1, q_2])x_3)$$

Constructions with spheres. 1. For a given $\rho \in \mathbf{R}$, let

$$\Xi(\rho) = \left(\frac{\rho^2}{2}, \mathbf{0}, 0, \rho \right).$$

The cycle $\Xi(\rho)$ has the function of a *compass*, since it is used to construct the sphere \mathcal{C}_s with radius ρ and center in a given point \mathbf{p} . Let $x = (v, \mathbf{p}, 1, 0)$ be the cycle corresponding to \mathbf{p} . Then

$$s = \mathbf{L}[\Xi(\rho)](x).$$

2. For a given cycle s representing a sphere \mathcal{C}_s , the center c of this sphere is represented by

$$c = \mathbf{L}[s_0](w), \quad s_0 = \mathbf{H}[r, r, r](s).$$

3. A cycle s represents a sphere with radius ρ if s is Lie-orthogonal to $r(\rho) = \mathbf{L}[\Xi(\rho)]r$.

4. Let $x_1, x_2 \in \mathbf{P}^4$ represent two points in the plane. The circle with these two points as endpoints of a diameter is given by

$$s = \mathcal{A}(x_1, x_2, r(\rho)), \quad r(\rho) = \mathbf{L}[\Xi(\rho)](r)$$

where ρ is the distance between the points x_1 and x_2 .

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