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OSCILLATION CRITERIA OF KNESER-HILLE TYPE FOR SECOND-ORDER DIFFERENTIAL EQUATIONS WITH NONLINEAR PERTURBED TERMS

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ABSTRACT. This paper is concerned with the oscillation problem for nonlinear differential equations of Euler type, which are denoted by (E_n) with $n = 1, 2, 3, \dots$. Equation (E_n) consists of a linear main term and a nonlinear perturbed term. If the nonlinear perturbation vanishes, then all nontrivial solutions of (E_n) are nonoscillatory. A pair of sufficient and necessary conditions on the perturbed term for all nonlinear solutions of (E_n) to be oscillatory is given. It is also proved that all solutions of (E_n) tend to zero.

1. Introduction. The existence and number of the zeros of the solutions of ordinary differential equations are an important subject in the qualitative theory. By an *oscillatory solution* we mean one having an infinite number of zeros on $0 \le t < \infty$. Otherwise, the solution is called *nonoscillatory*.

For example, we consider the Euler differential equation with positive damping

$$(L_1) \qquad y'' + \frac{2}{t}\, y' + \frac{\delta}{t^2}\, y = 0,$$

where $' = d/dt$. Then we see that all nontrivial solutions of (L_1) are nonoscillatory if and only if $\delta \le 1/4$. In fact, equation (L_1) has the general solution

$$y(t) = \begin{cases} \dfrac{1}{\sqrt{t}}\,(K_1 t^{\zeta} + K_2 t^{-\zeta}) & \text{if } \delta \ne \dfrac{1}{4}, \\ \dfrac{1}{\sqrt{t}}\,(K_3 + K_4 \log t) & \text{if } \delta = \dfrac{1}{4}, \end{cases}$$

where K_i, $i = 1, 2, 3, 4$, are arbitrary constants and ζ is a number satisfying

$$(1.1) \qquad \frac{1}{4} - \zeta^2 = \delta.$$

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Hence, for equation (L_1) the critical value of δ is $1/4$. Such a number is generally called the *oscillation constant*.

Let us add a nonlinear perturbation to equation (L_1) with $\delta = 1/4$. Then the question arises: Do oscillatory solutions appear even when the nonlinear perturbation is very small? If not, what is the upper limit of the nonlinear perturbation for all nontrivial solutions to be nonoscillatory?

Recently the author et al. [**3**] gave an answer to this question. They discussed the oscillation problem for the nonlinear differential equation of Euler type

$$(E_2) \qquad x'' + \frac{2}{t}\,x' + \frac{1}{4t^2}\,x + \frac{1}{t^2}\,g(x) = 0,$$

where $g(x)$ is locally Lipschitz continuous on $\mathbf{R}$ and satisfies

$$(1.2) \qquad xg(x) > 0 \quad \text{if } x \neq 0,$$

and established the following oscillation theorem and nonoscillation theorem for equation (E_2).

Theorem A. *Assume* (1.2) *and suppose that there exists a* $\lambda > 1/4$ *such that*

$$\frac{g(x)}{x} \geq \frac{\lambda}{(2\log|x|)^2}$$

for $|x|$ *sufficiently small. Then all nontrivial solutions of* (E_2) *are oscillatory.*

Theorem B. *Assume* (1.2) *and suppose that*

$$\frac{g(x)}{x} \leq \frac{1}{4(2\log|x|)^2}$$

for $x > 0$ *or* $x < 0$, $|x|$ *sufficiently small. Then all nontrivial solutions of* (E_2) *are nonoscillatory.*

It is known that all nontrivial solutions of the linear differential equation

$$(L_2) \qquad y'' + \frac{2}{t}\,y' + \frac{1}{4t^2}\,y + \frac{\delta}{t^2(\log t)^2}\,y = 0$$

are (i) oscillatory if $\delta > 1/4$, (ii) nonoscillatory if $0 \le \delta \le 1/4$. In [**3**], the author et al. regarded the equation

$$(N_2) \qquad x'' + \frac{2}{t}\,x' + \frac{1}{4t^2}\,x + \frac{\lambda}{t^2(2\log|x|)^2}\,x = 0,$$

where $|x|$ is sufficiently small, as a typical case of (E_2). Using the facts (i) and (ii), they showed that all nontrivial solutions of (N_2) are (iii) oscillatory if $\lambda > 1/4$, (iv) nonoscillatory if $0 \le \lambda \le 1/4$, and compared orbits of (E_2) with those of (N_2) in order to prove the pair of Theorems A and B.

Now from the facts (i) and (ii) we see that the oscillation constant δ is $1/4$ for equation (L_2). This will lead us further into a consideration of a nonlinear perturbation which is added to equation (L_2) with $\delta = 1/4$. Let us add a nonlinear perturbation to equation (L_2) with $\delta = 1/4$. Then, judging from Theorems A and B, we can expect that all nontrivial solutions have a tendency to be oscillatory as the nonlinear perturbation grows larger. What is the lower limit of the nonlinear perturbation for all nontrivial solutions to be oscillatory? Of course, Theorems A and B cannot answer this question.

The aim of this paper is to answer the above question and to discuss the nonlinear perturbation problem in more delicate cases. Our main theorems are stated in Section 2. For this purpose we introduce some notation. We also examine the asymptotic behavior of solutions of certain linear differential equations of Euler type. In Section 3, we show that all solutions of Euler differential equations with nonlinear perturbed terms tend to zero as $t \to \infty$. In Section 4, we complete the proof of our main theorems by means of the obtained results in the preceding section.

2. Statement of main theorems. Equations (L_1) has the simplest form in the set of second-order linear Euler differential equations with positive damping and equation (L_2) is more delicate for oscillation of solutions than (L_2). In this sense we may consider (L_1) and (L_2) to be the first and the second stages, respectively. Similarly, we can regard the equation

$$(E_1) \qquad x'' + \frac{2}{t}\,x' + \frac{1}{t^2}\,g(x) = 0$$

as the first stage in nonlinear differential equations of Euler type. Hence equation (E_2) corresponds to the second stage.

We next go on to the nth stage of linear and nonlinear differential equations of Euler type. To this end, we introduce the following notation. Define

$$l_1(w) = 1 \quad \text{and} \quad l_{k+1}(w) = l_k(w) \log_k w, \quad k \in \mathbf{N},$$

where $\log_0 w = w$, $\log_1 w = |\log w|$ and $\log_k w = \log(\log_{k-1} w)$, and let

$$S_0(w) = 0 \quad \text{and} \quad S_k(w) = \sum_{i=1}^{k} \frac{1}{\{l_i(w)\}^2}, \quad k \in \mathbf{N}.$$

Then we have

$$\log_k w \geq 1 \quad \text{for} \quad w \geq e_k \quad \text{and} \quad 0 < w \leq 1/e_k$$

where $e_0 = 1$ and $e_k = \exp(e_{k-1})$. Hence the sequences of the functions $\{l_k(w)\}$ and $\{S_k(w)\}$ are well-defined in these intervals. To be precise we enumerate the sequences $\{l_k(w)\}$ and $\{S_k(w)\}$:

$$\begin{aligned}
&l_2(w) = |\log w|, \quad l_3(w) = |\log w|(\log|\log w|),\\
&l_4(w) = |\log w|(\log|\log w|)(\log(\log|\log w|)), \dots ;\\
&S_1(w) = 1, \quad S_2(w) = 1 + \frac{1}{(\log w)^2},\\
&S_3(w) = 1 + \frac{1}{(\log w)^2} + \frac{1}{(\log w)^2(\log|\log w|)^2},\\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .
\end{aligned}$$

Using the above notation, we can represent the nth stage of linear and nonlinear differential equations of Euler type:

$$(L_n) \qquad y'' + \frac{2}{t}\, y' + \frac{1}{4t^2}\, S_{n-1}(t) y + \frac{\delta}{t^2\{l_n(t)\}^2}\, y = 0;$$

$$(E_n) \qquad x'' + \frac{2}{t}\, x' + \frac{1}{4t^2}\, S_{n-1}(t) x + \frac{1}{t^2}\, g(x) = 0.$$

Before we give the statement of main results concerning the oscillation and nonoscillation of solutions of (E_n), let us examine the oscillation constant for equation (L_n). By a straightforward calculation, we obtain the following formula, but we omit the details.

Lemma 2.1. *Equation* (L_n) *has the general solution*

$$y(t) = \begin{cases} \dfrac{\sqrt{l_n(t)}}{t}\,\{K_1(\log_{n-1} t)^{\zeta} + K_2(\log_{n-1} t)^{-\zeta}\} & \text{if } \delta \neq \dfrac{1}{4}, \\ \dfrac{\sqrt{l_n(t)}}{t}\,\{K_3 + K_4 \log_n t\} & \text{if } \delta = \dfrac{1}{4}, \end{cases}$$

where K_i, $i = 1, 2, 3, 4$, *are arbitrary constants and* ζ *is a number satisfying condition* (1.1).

From Lemma 2.1 we see that the facts (i) and (ii) in Section 1 also hold for equation (L_n), and therefore the oscillation constant δ is $1/4$ in this case.

Let $s = \log t$ and $u(s) = y(e^s) = y(t)$. Then equation (L_n) is transferred into the system

$$\begin{aligned} \dot{\xi} &= \eta - \xi, \\ \dot{\eta} &= -\frac{1}{4}\, S_{n-1}(e^s)\xi - \frac{\delta}{\{l_n(e^s)\}^2}\,\xi, \end{aligned}$$

where $\dot{} = d/ds$. In polar coordinates $(\xi, \eta) \to (\rho, \varphi)$, where $\xi = \rho\cos\varphi$ and $\eta = \rho\sin\varphi$, the above system becomes

(PL_n)

$$\begin{aligned} \dot{\rho} &= \rho\left[f_1(\varphi) - \frac{1}{4}(S_{n-1}(e^s) - 1)\sin\varphi\cos\varphi - \frac{\delta}{\{l_n(e^s)\}^2}\sin\varphi\cos\varphi\right], \\ \dot{\varphi} &= f_2(\varphi) - \frac{1}{4}(S_{n-1}(e^s) - 1)\cos^2\varphi - \frac{\delta}{\{l_n(e^s)\}^2}\cos^2\varphi, \end{aligned}$$

where

$$\begin{aligned} f_1(\varphi) &= (\sin\varphi - \cos\varphi)\cos\varphi - \frac{1}{4}\sin\varphi\cos\varphi, \\ f_2(\varphi) &= -(\sin\varphi - \cos\varphi)\sin\varphi - \frac{1}{4}\cos^2\varphi. \end{aligned}$$

Note that $f_2(\varphi) \leq 0$ for $\varphi \in \mathbf{R}$ and $f_2(\varphi) = 0$ if and only if $\tan\varphi = 1/2$. Let $(\rho(s), \varphi(s))$ be any nontrivial solution of (PL_n). Since

$$\dot{\varphi}(s) < f_2(\varphi(s)) \leq 0,$$

$\varphi(s)$ is decreasing. Hence, there are two possible cases: (i) $\varphi(s)$ tends to $-\infty$ as $s \to \infty$; (ii) there exists a φ^* such that

$$\varphi(s) \searrow \varphi^* \quad \text{as } s \to \infty.$$

In the latter case φ^* has to satisfy $\tan\varphi^* = 1/2$. If not, then there exists an $\varepsilon > 0$ such that

$$f_2(\varphi^*) < -\varepsilon.$$

Hence we have

$$\dot{\varphi}(s) < f_2(\varphi(s)) < -\varepsilon$$

for s sufficiently large, which leads to

$$\varphi(s) \longrightarrow -\infty \quad \text{as } s \to \infty,$$

a contradiction. Taking into account the facts (i) and (ii), we have the following result on the property of $\varphi(s)$.

Lemma 2.2. *If $\delta > 1/4$, then $\varphi(s)$ tends to $-\infty$ as $s \to \infty$, and otherwise it approaches φ^* as $s \to \infty$, where φ^* is an angle satisfying $\tan\varphi^* = 1/2$.*

We are ready to state our main results for equation (E_n).

Theorem 2.1. *Assume* (1.2) *and suppose that there exists a $\lambda > 1/4$ such that*

$$\frac{g(x)}{x} \geq \frac{\lambda}{\{l_n(x^2)\}^2} \tag{2.1}$$

for $|x|$ sufficiently small. Then all nontrivial solutions of (E_n) are oscillatory.

Theorem 2.2. *Assume* (1.2) *and suppose that*

$$\frac{g(x)}{x} \leq \frac{1}{4\{l_n(x^2)\}^2} \tag{2.2}$$

for $x > 0$ or $x < 0$, $|x|$ sufficiently small. Then equation (E_n) has a nonoscillatory solution.

We can represent our results in the style of Kneser-Hille [**1, 2**]; see also [**4**, Theorems 2.41–2.43].

Theorem 2.3. *Assume* (1.2). *Let*

$$\omega_n^* = \limsup_{x\to 0} \frac{g(x)\{l_n(x^2)\}^2}{x} \quad \textit{and} \quad \omega_{n^*} = \liminf_{x\to 0} \frac{g(x)\{l_n(x^2)\}^2}{x}.$$

Then equation (E_n) has a nontrivial solution which is nonoscillatory if $\omega_n^* < 1/4$ and it fails to have such a solution if $\omega_{n^*} > 1/4$.

Unfortunately, no conclusion can be drawn if either ω_n^* or ω_{n^*} equals $1/4$. However, if $\omega_n^* = \omega_{n^*} = 1/4$ and

$$\frac{g(x)\{l_n(x^2)\}^2}{x} \nearrow \frac{1}{4} \quad \text{as } x \to 0,$$

then, by Theorem 2.2 we can conclude that equation (E_n) has a nontrivial solution which is nonoscillatory.

Comparing the behavior of each solution of the equation

$$(E_1) \qquad x'' + \frac{2}{t}\,x' + \frac{1}{t^2}\,g(x) = 0$$

with that of (L_1), we see that all nontrivial solutions of (E_1) are oscillatory if there exists a $\lambda > 1/4$ such that

$$\frac{g(x)}{x} \geq \lambda$$

for $|x|$ sufficiently small; and they are nonoscillatory if

$$\frac{g(x)}{x} \leq \frac{1}{4}$$

for $x > 0$ or $x < 0$, $|x|$ sufficiently small. Since

$$l_2(x^2) = |\log x^2| = 2|\log |x||,$$

we also see that Theorems 2.1 and 2.2 coincide with Theorems A and B, respectively. Hence, Theorems 2.1 and 2.2 hold when $n = 1, 2$. For this reason we assume that $n \geq 3$ from now on.

3. Global asymptotic stability. We transform equation (E_n) into the equation

$$\ddot{u} + \dot{u} + \frac{1}{4} S_{n-1}(e^s)u + g(u) = 0,$$

which is equivalent to the system

$$(SE_n) \qquad \begin{aligned} \dot{u} &= v - u, \\ \dot{v} &= -\frac{1}{4} S_{n-1}(e^s)u - g(u), \end{aligned}$$

where $u(s) = x(e^s) = x(t)$. By assumption (1.2), system (SE_n) has the zero solution $(u(s), v(s)) \equiv (0, 0)$. We call the projection of a positive semi-trajectory of (SE_n) onto the phase plane a *positive orbit*. Taking the vector field (SE_n) into account, we see that if a positive orbit of (SE_n) crosses the positive, respectively negative, v-axis, then it moves from the left to the right, respectively from the right to the left.

We next transform to polar coordinates $(u, v) \to (r, \theta)$ by $u = r\cos\theta$ and $v = r\sin\theta$ to obtain the system

$$(PE_n) \qquad \begin{aligned} \dot{r} &= r\left[f_1(\theta) - \frac{1}{4}(S_{n-1}(e^s) - 1)\sin\theta\cos\theta - \frac{g(r\cos\theta)}{r}\sin\theta\right], \\ \dot{\theta} &= f_2(\theta) - \frac{1}{4}(S_{n-1}(e^s) - 1)\cos^2\theta - \frac{g(r\cos\theta)}{r}\cos\theta. \end{aligned}$$

Let $(u(s), v(s))$ be any nontrivial solution of (SE_n) and let $(r(s), \theta(s))$ be the solution of (PE_n) which corresponds to $(u(s), v(s))$. Then by (1.2) we obtain

$$\begin{aligned} (1 + \tan^2\theta(s))\dot{\theta}(s) &= \frac{d}{ds}\tan\theta(s) \\ &= \frac{\dot{v}(s)u(s) - v(s)\dot{u}(s)}{u^2(s)} \\ &= -\frac{1}{4}(S_{n-1}(e^s) - 1) - \frac{g(u(s))}{u(s)} - \left(\frac{1}{2} - \frac{v(s)}{u(s)}\right)^2 \\ &= -\frac{1}{4}(S_{n-1}(e^s) - 1) - \frac{g(u(s))}{u(s)} - \left(\frac{1}{2} - \tan\theta(s)\right)^2 < 0 \end{aligned}$$

as long as $u(s) \neq 0$. Hence $\theta(s)$ is decreasing. This means that the positive orbit of (SE_n) corresponding to $(u(s), v(s))$ rotates around the origin clockwise. To be precise, the situation falls into two cases: $\theta(s)$ tends to $-\infty$ as $s \to \infty$ or $\theta(s)$ approaches φ^* as $s \to \infty$, where φ^* is the constant given in Lemma 2.2. The former gives an account of oscillatory solutions of (E_n) and the latter describes a property of nonoscillatory solutions of (E_n). To sum up, we have the following result.

Lemma 3.1. *Under the assumption* (1.2), *positive orbits of* (SE_n) *corresponding to oscillatory solutions of* (E_n) *rotate around the origin clockwise, on the other hand, positive orbits of* (SE_n) *corresponding to nonoscillatory solutions of* (E_n) *approach the line* $v = u/2$.

By means of Lemma 3.1, we can guarantee the global asymptotic stability of the zero solution of (SE_n). Hence all solutions of (SE_n) tend to the origin as $s \to \infty$.

Lemma 3.2. *Assume* (1.2). *Then the zero solution of* (SE_n) *is globally asymptotically stable.*

Proof. Let $(u(s), v(s))$ be any nontrivial solution of (SE_n) initiating at $s = s_0$. Recall that $n \geq 3$. Since $S_{n-1}(e^s)$ is greater than 1 and decreasing for $s \geq s_0$, we have

$$1 < S_{n-1}(e^s) \leq S_{n-1}(e^{s_0}). \tag{3.1}$$

The positive orbit of (SE_n) corresponding to $(u(s), v(s))$ does not converge to any interior point in the phase plane except the origin. In fact, if there exists a point (α, β) such that

$$(u(s), v(s)) \longrightarrow (\alpha, \beta) \quad \text{as } s \to \infty,$$

then

$$\begin{aligned} v(s) - u(s) = \dot{u}(s) \to 0, \\ -\frac{1}{4} S_{n-1}(e^s) u(s) - g(u(s)) = \dot{v}(s) \to 0 \end{aligned}$$

as $s \to \infty$. Hence, by (3.1), both $u(s)$ and $v(s)$ tend to zero, that is, $(\alpha, \beta) = (0, 0)$.

Let $x(t)$ be the solution of (E_n) corresponding to $(u(s), v(s))$. We divide the proof into two cases: (i) $x(t)$ is oscillatory; (ii) $x(t)$ is nonoscillatory.

Case (i). Let

$$U(s, u, v) = \frac{1}{2}\, v^2 + \frac{1}{8}\, S_{n-1}(e^s)u^2 + \int_0^u g(\xi)\, d\xi$$

and consider the curve defined by

$$U(s, u, v) = c,$$

where $c = U(s_0, u(s_0), v(s_0))$. Then the curve is an oval surrounding the origin and the oval expands as s increases. Since

$$U(s, 0, \pm\sqrt{2c}\,) = c \quad \text{for } s \ge s_0,$$

the curve passes through two points $(0, \sqrt{2c})$ and $(0, -\sqrt{2c})$ for each fixed $s \ge s_0$. The curve is in the rectangle

$$R_0 = \{(u, v) : |u| \le 2\sqrt{2c} \text{ and } |v| \le \sqrt{2c}\}.$$

In fact, by (1.2) and (3.1),

$$\frac{1}{8}\, u^2 \le U(s, u, v) = c,$$
$$\frac{1}{2}\, v^2 \le U(s, u, v) = c$$

for $s \ge s_0$. By (1.2) and (3.1) again, the derivative of U along a solution of (SE_n) satisfies

$$\text{(3.2)}\quad \dot{U}_{(SE_n)}(s, u, v) = \frac{1}{8}\, \frac{d}{ds}\, (S_{n-1}(e^s))u^2 - \frac{1}{4}\, S_{n-1}(e^s)u^2 - ug(u) \le 0.$$

This means that $(u(s), v(s))$ stays in R_0 for $s \ge s_0$.

Define $U(s) = U(s, u(s), v(s))$. Then from (3.2) we conclude that $U(s)$ is decreasing for $s \geq s_0$. To complete the proof it suffices to show that $U(s)$ tends to zero as $s \to \infty$. If the assertion is not true, then there exists an $H_0 > 0$ such that

$$U(s) \searrow H_0 \quad \text{as } s \to \infty. \tag{3.3}$$

Let $S_{H_0} = \{(u, v) : U(s_0, u, v) < H_0\}$. Then, by Lemma 3.1, we see that the positive orbit of (SE_n) corresponding to $(u(s), v(s))$ rotates around the region S_{H_0} clockwise but does not enter S_{H_0}. Let ε_0 be a small constant satisfying

$$\{(u, v) : |u| < \varepsilon_0 \text{ and } |v| < \varepsilon_0\} \subset S_{H_0}.$$

Then there is a pair of sequences $\{\tau_k\}$ and $\{\sigma_k\}$ with $s_0 < \tau_k < \sigma_k < \tau_{k+1}$ and $\tau_k \to \infty$ as $k \to \infty$ such that

$$u(\tau_k) > \varepsilon_0, \quad v(\tau_k) = 0; \quad u(\sigma_k) = \varepsilon_0, \quad v(\sigma_k) < -\varepsilon_0; \tag{3.4}$$

$$u(s) > \varepsilon_0 \quad \text{and} \quad v(s) < 0 \quad \text{for } \tau_k < s < \sigma_k. \tag{3.5}$$

Since $(u(s), v(s))$ stays in R_0, we see that

$$u(s) \leq 2\sqrt{2c} \quad \text{for } s \geq s_0. \tag{3.6}$$

Put $M = \max\{g(u) : \varepsilon_0 \leq u \leq 2\sqrt{2c}\}$. Then, by (3.1) and (3.4)–(3.6), we have

$$\begin{aligned} -\varepsilon_0 > v(\sigma_k) - v(\tau_k) &= -\int_{\tau_k}^{\sigma_k} \left\{ \frac{1}{4}\, S_{n-1}(e^s)u(s) + g(u(s)) \right\} ds \\ &\geq -\left\{ \frac{\sqrt{2c}}{2}\, S_{n-1}(e^{s_0}) + M \right\} (\sigma_k - \tau_k). \end{aligned}$$

Hence, together with (3.1), (3.2) and (3.5), we get

$$\begin{aligned} U(\sigma_k) - U(\tau_1) &= \int_{\tau_1}^{\sigma_k} \dot{U}_{(SE_n)}(s, u(s), v(s))\, ds \\ &\leq \sum_{i=1}^{k} \int_{\tau_i}^{\sigma_i} \dot{U}_{(SE_n)}(s, u(s), v(s))\, ds \\ &\leq -\sum_{i=1}^{k} \int_{\tau_i}^{\sigma_i} \frac{1}{4}\, S_{n-1}(e^s) u^2(s)\, ds < -\frac{1}{4} \sum_{i=1}^{k} \int_{\tau_i}^{\sigma_i} u^2(s)\, ds \\ &< -\frac{\varepsilon_0^2}{4} \sum_{i=1}^{k} (\sigma_i - \tau_i) < -\frac{\varepsilon_0^3 k}{2\sqrt{2c}\, S_{n-1}(e^{s_0}) + 4M}. \end{aligned}$$

We therefore conclude that $U(\sigma_k)$ tends to $-\infty$ as $k \to \infty$. This is a contradiction to (3.3).

Case (ii). Let $(r(s), \theta(s))$ be the solution of (PE_n) corresponding to $(u(s), v(s))$. Then, from Lemma 3.1 we see that there exist an $s_1 \geq s_0$ and an $m \in \mathbf{N}$ such that

$$m\pi < \varphi^* < \theta(s) < \left(\frac{1}{4} + m\right)\pi$$

for $s \geq s_1$. Since $\tan \varphi^* = 1/2$, we have

$$\frac{2}{5} = \sin \varphi^* \cos \varphi^* < \sin \theta(s) \cos \theta(s) < \frac{1}{2},$$
$$-\frac{1}{2} = f_1(\varphi^*) < f_1(\theta(s)) < f_1(\pi/4) = -\frac{1}{8}$$

for $s \geq s_1$. Hence, by (1.2) and (3.1), we get

$$\dot{r}(s) < -\frac{1}{8}\, r(s) \quad \text{for } s \geq s_1.$$

From this inequality we see that the positive orbit of (SE_n) corresponding to $(u(s), v(s))$ passes through the region $R_+ = \{(u, v) : u > 0 \text{ and } u/2 < v < u\}$ or $R_- = \{(u, v) : u < 0 \text{ and } u < v < u/2\}$, and then approaches the origin as $s \to \infty$.

Thus, in both cases $(u(s), v(s))$ tends to (0,0) as $s \to \infty$. The proof is now complete. □

4. Proof of main theorems. We are now able to prove our main results.

Proof of Theorem 2.1. By way of contradiction, we suppose that equation (E_n) has a nonoscillatory solution $\zeta(t)$. Then, without loss of generality, we may assume that there exists a $T > 0$ such that

$$\zeta(t) > 0 \quad \text{for } t \geq T.$$

Let $(u(s), v(s))$ and $(r(s), \theta(s))$ be the solutions of (SE_n) and (PE_n) corresponding to $\zeta(t)$, respectively. Then it follows from Lemma 3.1 that there exist an $s_1 \geq \log T$ and an $m \in \mathbf{N}$ such that

$$(u(s), v(s)) \in R_+$$

and

$$m\pi < \varphi^* < \theta(s) \le \theta(s_1) < \left(\frac{1}{4} + m\right)\pi \tag{4.1}$$

for $s \ge s_1$. From Lemma 3.2 it also turns out that

$$(u(s), v(s)) \longrightarrow (0,0) \quad \text{as } s \to \infty.$$

Hence, by (2.1), we have

(4.2)

$$\begin{aligned}\dot{\theta}(s) &= f_2(\theta(s)) - \frac{1}{4}\left(S_{n-1}(e^s) - 1\right)\cos^2\theta(s) - \frac{g(u(s))}{u(s)}\cos^2\theta(s) \\ &\le f_2(\theta(s)) - \frac{1}{4}\left(S_{n-1}(e^s) - 1\right)\cos^2\theta(s) - \frac{\lambda}{\{l_n(u^2(s))\}^2}\cos^2\theta(s).\end{aligned}$$

Since $\lambda > 1/4$, we can choose an $\varepsilon > 0$ so small that

$$\frac{1}{4} < \frac{1}{4}(1+\varepsilon)^{2(n-1)} < \lambda. \tag{4.3}$$

From (4.1) we see that

$$\frac{1}{2} < \tan\theta(s) = \frac{v(s)}{u(s)} < 1$$

for $s \ge s_1$, which implies that

$$\dot{u}(s) = v(s) - u(s) > -\frac{1}{2}u(s).$$

Hence, there exists an $s_2 \ge s_1$ such that

$$0 > \log u(s) > \log u(s_1) - \frac{1}{2}(s - s_1) > -\frac{1}{2}(1+\varepsilon)s$$

for $s \ge s_2$ and, therefore,

$$\log_1 u^2(s) = |2\log u(s)| < (1+\varepsilon)s = (1+\varepsilon)\log_0 s \tag{4.4}$$

for $s \ge s_2$.

Let X_0 be so large that

$$(1+\varepsilon)X < X^{1+\varepsilon} \quad \text{for } X \geq X_0. \tag{4.5}$$

We here estimate $\log_k u^2(s)$ where $k = 2, 3, \dots, n$. By (4.4) we obtain

$$\log_2 u^2(s) = \log(\log_1 u^2(s)) < \log((1+\varepsilon)s).$$

Using (4.5) with $X = s$, we can find an $s_3 \geq s_2$ such that

$$\log((1+\varepsilon)s) < (1+\varepsilon)\log s = (1+\varepsilon)\log_1 s$$

for $s \geq s_3$ and, therefore,

$$\log_2 u^2(s) < (1+\varepsilon)\log_1 s \quad \text{for } s \geq s_3.$$

From this inequality and (4.5) with $X = \log s$, we get

$$\begin{aligned}\log_3 u^2(s) = \log(\log_2 u^2(s)) &< \log((1+\varepsilon)\log s)\\ &< (1+\varepsilon)\log(\log s) = (1+\varepsilon)\log_2 s\end{aligned}$$

for $s \geq s_4$, where s_4 is larger than s_3. Repeating the same argument, we can select a finite sequence $\{s_k\}$, $2 \leq k \leq n$, with $s_1 \leq s_2 \leq \cdots \leq s_n$, such that

$$\log_{k-1} u^2(s) < (1+\varepsilon)\log_{k-2} s \quad \text{for } s \geq s_k.$$

From this estimation we see that

$$\begin{aligned}l_n(u^2(s)) = \prod_{k=1}^{n-1} \log_k u^2(s) &< \prod_{k=1}^{n-1} (1+\varepsilon)\log_{k-1} s\\ &= (1+\varepsilon)^{n-1} \prod_{k=1}^{n-1} \log_k t = (1+\varepsilon)^{n-1} l_n(t)\\ &= (1+\varepsilon)^{n-1} l_n(e^s)\end{aligned}$$

for $s \geq s_n$. Hence, by (4.2) we have

$$\begin{aligned}\dot\theta(s) < f_2(\theta(s)) &- \frac{1}{4}(S_{n-1}(e^s) - 1)\cos^2\theta(s)\\ &- \frac{\lambda}{(1+\varepsilon)^{2(n-1)}\{l_n(e^s)\}^2}\cos^2\theta(s)\end{aligned} \tag{4.6}$$

for $s \geq s_n$.

Now we consider the first order differential equation

$$\begin{aligned}\dot{\varphi} = f_2(\varphi) - \frac{1}{4}(S_{n-1}(e^s) - 1)\cos^2\varphi \\ - \frac{\lambda}{(1+\varepsilon)^{2(n-1)}\{l_n(e^s)\}^2}\cos^2\varphi\end{aligned} \tag{4.7}$$

and let $\varphi(s)$ be the solution of (4.7) satisfying $\varphi(s_n) = \theta(s_n)$. Note that (4.7) coincides with the second equation in system (PL_n) when $\delta = \lambda/(1+\varepsilon)^{2(n-1)}$. Comparing (4.6) with (4.7) and using a simple comparison theorem, we have

$$\theta(s) \leq \varphi(s) \quad \text{for } s \geq s_n.$$

From (4.3) we get

$$\frac{1}{4} < \frac{\lambda}{(1+\varepsilon)^{2(n-1)}}.$$

Hence, by means of Lemma 2.2, we see that $\varphi(s)$ tends to $-\infty$ as $s \to \infty$, and so does $\theta(s)$. This is a contradiction to (4.1). We therefore conclude that all nontrivial solutions of (E_n) are oscillatory. The proof is complete. □

Proof of Theorem 2.2. The proof is by contradiction. Suppose that all nontrivial solutions of (E_n) are oscillatory. Then by Lemma 3.1 all nontrivial positive orbits of (SE_n) rotate around the origin in a clockwise direction. In particular, we consider the positive orbit of (SE_n) passing through the point

$$A = \left(y_0, \frac{y_0}{2}\sum_{k=1}^{n}\frac{1}{l_k(e^{s_0})}\right)$$

at $s = s_0$, where s_0 is sufficiently large and

$$y_0 = \frac{1}{e^{1+s_0/2}}.$$

Since the positive orbit rotates around the origin, it crosses the line $v = u/2$ and the positive u-axis infinitely many times. Let s_1 and s_2 be the first intersecting points of the positive orbit with the line $v = u/2$ and the positive u-axis, respectively. It is clear that $s_0 < s_1 < s_2$. Also, it may safely be assumed that the point A is near the line $v = u/2$ and s_1 is arbitrarily close to s_0, say

$$s_1 - s_0 < 1,$$

because s_0 is sufficiently large.

Let $(u(s), v(s))$ and $(r(s), \theta(s))$ be the solutions of (SE_n) and (PE_n) corresponding to the positive orbit, respectively. Then from the vector field of (SE_n), we see that

$$u(s_1) < u(s_0) = y_0 \quad \text{and} \quad 0 \le \tan\theta(s) = \frac{v(s)}{u(s)} \le \frac{1}{2} \quad \text{for } s_1 \le s \le s_2.$$

Hence we have

$$\dot{u}(s) = v(s) - u(s) \le -\frac{1}{2}\,u(s)$$

for $s_1 \le s \le s_2$ and, therefore,

$$\begin{aligned}\log u(s) &\le \log u(s_1) - \frac{1}{2}(s - s_1) < \log y_0 - \frac{1}{2}(s - s_1)\\ &= -\frac{1}{2}\,s - \frac{1}{2} + \frac{1}{2}(s_1 - s_0 - 1) < -\frac{1}{2}\,s - \frac{1}{2}.\end{aligned}$$

Since s_1 depends on s_0 and approaches s_0 as $s_0 \to \infty$, we may estimate that

$$\log u(s) < -\frac{1}{2}\,s \quad \text{for } s_0 \le s \le s_2.$$

Hence we get

$$\begin{aligned}\log_1 u^2(s) &= |2\log u(s)| > s = \log_0 s,\\ \log_2 u^2(s) &= \log(\log_1 u^2(s)) > \log s = \log_1 s,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \log_{n-1} u^2(s) &= \log(\log_{n-2} u^2(s)) > \log(\log_{n-3} s) = \log_{n-2} s\end{aligned}$$

for $s_0 \le s \le s_2$. From these inequalities we obtain

$$l_n(u^2(s)) = \prod_{k=1}^{n-1} \log_k u^2(s) > \prod_{k=1}^{n-1} \log_{k-1} s = l_n(e^s).$$

Hence, together with (2.2), we have

$$\begin{aligned}
\dot\theta(s) &= f_2(\theta(s)) - \frac{1}{4}(S_{n-1}(e^s)-1)\cos^2\theta(s) - \frac{g(u(s))}{u(s)}\cos^2\theta(s)\\
&\ge f_2(\theta(s)) - \frac{1}{4}(S_{n-1}(e^s)-1)\cos^2\theta(s)\\
&\quad - \frac{1}{4\{l_n(u^2(s))\}^2}\cos^2\theta(s)\\
&> f_2(\theta(s)) - \frac{1}{4}(S_{n-1}(e^s)-1)\cos^2\theta(s)\\
&\quad - \frac{1}{4\{l_n(e^s)\}^2}\cos^2\theta(s)\\
&= f_2(\theta(s)) - \frac{1}{4}(S_n(e^s)-1)\cos^2\theta(s)
\end{aligned} \tag{4.8}$$

for $s_0 \le s \le s_2$.

We here consider the linear differential equation

$$y'' + \frac{2}{t}\,y' + \frac{1}{4t^2}\,S_n(t)y = 0, \tag{4.9}$$

which coincides with equation (L_n) when $\delta = 1/4$. From Lemma 2.1 we see that all nontrivial solutions of (4.9) are nonoscillatory. Letting $t_0 = e^{s_0}$ and putting $K_3 = y_0\sqrt{t_0/l_n(t_0)}$ and $K_4 = 0$ in Lemma 2.1, we can choose the solution

$$y(t) = y_0\sqrt{\frac{t_0}{l_n(t_0)}}\sqrt{\frac{l_n(t)}{t}}$$

which satisfies the initial conditions $y(t_0) = y_0$ and

$$y'(t_0) = \frac{y_0}{2t_0}\left(\sum_{k=2}^{n}\frac{1}{l_k(t_0)} - 1\right).$$

In fact, noticing that

$$(tl_n(t))' = l_n(t)\sum_{k=1}^{n}\frac{1}{l_k(t)},$$

we get

$$
\begin{aligned}
y'(t) &= y_0 \sqrt{\frac{t_0}{l_n(t_0)}} \sqrt{\frac{t}{l_n(t)}} \frac{(tl_n(t))' - 2l_n(t)}{2t^2} \\
&= \frac{y_0}{2t^2} \sqrt{\frac{t_0}{l_n(t_0)}} \sqrt{\frac{t}{l_n(t)}}\, l_n(t) \left(\sum_{k=2}^{n} \frac{1}{l_k(t)} - 1 \right).
\end{aligned}
$$

Making the change of variable $t = e^s$, we can transform equation (4.9) into the system

$$
\begin{aligned}
\dot{\xi} &= \eta - \xi, \\
\dot{\eta} &= -\frac{1}{4}\, S_n(e^s)\xi.
\end{aligned} \tag{4.10}
$$

Let $(\xi(s), \eta(s))$ be the solution of (4.10) corresponding to the solution $y(t)$. Then we have

$$
(\xi(s), \eta(s)) = (y(e^s), y'(e^s)e^s + y(e^s)).
$$

Since

$$
\begin{aligned}
\xi(s_0) &= y(t_0) = y_0, \\
\eta(s_0) &= y'(t_0)t_0 + y_0 = \frac{y_0}{2} \left(\sum_{k=2}^{n} \frac{1}{l_k(t_0)} - 1 \right) + y_0 \\
&= \frac{y_0}{2} \sum_{k=1}^{n} \frac{1}{l_k(t_0)},
\end{aligned}
$$

we see that $(\xi(s_0), \eta(s_0))$ coincides with the point A. Also, we have

$$
\frac{\eta(s)}{\xi(s)} = \frac{y'(t)t + y(t)}{y(t)} = \frac{1}{2} \sum_{k=1}^{n} \frac{1}{l_k(e^s)} \searrow \frac{1}{2} \quad \text{as } s \to \infty.
$$

Hence, we conclude that

$$
(\xi(s), \eta(s)) \in R_+ \quad \text{for } s \geq s_0, \tag{4.11}
$$

where R_+ is the region defined in the proof of Lemma 3.2.

In polar coordinates system (4.10) takes the form

$$
\begin{aligned}
\dot{\rho} &= \rho\left[f_1(\varphi) - \frac{1}{4}(S_n(e^s) - 1)\sin\varphi\cos\varphi\right], \\
\dot{\varphi} &= f_2(\varphi) - \frac{1}{4}(S_n(e^s) - 1)\cos^2\varphi.
\end{aligned} \tag{4.12}
$$

Let $(\rho(s), \varphi(s))$ be the solution of (4.12) which corresponds to $(\xi(s), \eta(s))$. Then by (4.11) we see that

$$\varphi^* < \varphi(s) < \frac{\pi}{4} \quad \text{for } s \geq s_0. \tag{4.13}$$

Let us compare $\varphi(s)$ with $\theta(s)$. Recall that $\theta(s)$ satisfies the inequality (4.8). On the other hand, $\varphi(s)$ satisfies the second equation in system (4.12). As shown above,

$$(\xi(s_0), \eta(s_0)) = A = (u(s_0), v(s_0)).$$

Hence by a simple comparison theorem, we obtain

$$\varphi(s) \leq \theta(s) \quad \text{for } s_0 \leq s \leq s_2.$$

From this inequality and (4.13), we conclude that

$$\varphi^* < \theta(s) \quad \text{for } s_0 \leq s \leq s_2.$$

However, by the definition of s_1 and s_2, we have

$$s_0 < s_1 \quad \text{and} \quad \theta(s) \leq \varphi^* \quad \text{for } s_1 \leq s \leq s_2.$$

This is a contradiction. We have completed the proof of Theorem 2.2. □

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