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BOUNDARY VALUE PROBLEMS IN OSCILLATING CUSPIDAL WEDGES

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ABSTRACT. The paper is devoted to pseudodifferential boundary value problems in domains with cuspidal wedges. We show a criterion for the Fredholm property of a boundary value problem and derive estimates of solutions close to edges.

1. Introduction. Boundary value problems in domains (or on manifolds) with singular boundary appear in numerous models of applied sciences, in particular, in mechanics, crack theory, hydrodynamics, mathematical physics. Many authors contributed to the corresponding theory under different aspects, especially Kondrat'ev [**19**], Grisvard [**12**], Maz'ya and Plamenevskii [**23, 24**], Feigin [**8, 9**], Bagirov and Feigin [**1**], Maz'ya, Kozlov and Roßmann [**22**], Nazarov and Plamenevskii [**28**].

In recent years the interest in such problems increased enormously, and new structure insight was obtained by applying pseudodifferential methods, cf., in particular, Melrose and Mendoza [**27**], Rabinovich [**31, 32**], Schrohe and Schulze [**35, 36**], Mazzeo and Melrose [**26**].

Some general ideas are the same as in the classical theory for smooth domains, cf. Boutet de Monvel [**3, 4**], for instance, to embed the differential boundary value problems into an algebra of operators in which the parametrices of elliptic elements can be expressed.

A typical feature of these theories is that a given fixed, say differential, boundary value problem generates a hierarchy of symbols whose components are operator-valued and consist of parametrized operators in the corresponding algebras on spaces of lower order singularity. For the smooth case and pseudodifferential operators with the transmission property this is the interior symbolic structure and the boundary symbolic calculus on the half-axis. For operators in the sense of Vishik and Eskin [**6, 44**] there appear more general singular integral operators on

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the half-axis (modulo reductions of orders), cf. Gokhberg and Krupnik [**10, 11**]. This aspect was widely employed also in the book of Schulze [**38**].

Parametrices associated with the inverted symbols should be available in the corresponding algebras. This emphasizes the role of the values of operator-valued symbols as elements in the algebras on spaces of lower order singularity, where the inverses of those parameter-dependent families are to be expressed. Although these general aspects are rather clear at first glance, the investigation of boundary value problems in concrete singular configurations is far from being straightforward, in particular, for cuspidal singularities on the boundary. In many cases there are in fact no results at all. The reason is not only the wealth of new structures but also rather unexpected ambiguities in the choice of the approach.

Let us mention in this connection the paper [**40**] where methods of the noncommutative analysis are used to study operator algebras on manifolds with power-like cusps, in case the link of the local cone is closed. There are obtained parametrices and the Fredholm property in the corresponding weighted Sobolev spaces.

Another approach of the authors, modeled more in the spirit of the edge pseudodifferential calculus of Schulze [**39**], is presented in [**42**]. It relies on strongly continuous groups acting in Sobolev spaces along the model cuspidal cone, and "twisted" homogeneity of the edge symbols. Notice that in such a theory additional trace and potential conditions along the edge with an analogue of the Lopatinskii condition are quite natural, while our results here refer to weighted Sobolev spaces where no data of this kind are required.

As but one example of singular spaces studied in the present paper we show the canonical wedge

$$W = \{(\varphi(z)x, y, z) \in \mathbf{R}^3 : x \in [-1,1],\ y \in \mathbf{R},\ z \geq 0\}$$

in $\mathbf{R}^3$, where φ is a positive C^∞ function on the half-axis $z > 0$ which satisfies $\varphi(0) = 0$. The behavior of $\varphi(z)$ near $z = 0$ specifies the singularity of W along the edge $\mathbf{R}_y = \{(0, y, 0) \in \mathbf{R}^3 : y \in \mathbf{R}\}$. If $\varphi(z) = O(z)$, then the singularity of W along the y-axis is conical. If $\varphi(z) = o(z)$ this singularity is cuspidal. In other cases W has finite smoothness at the edge. In general the derivative $\varphi'(z)$ need not have any limit when $z \to 0$, hence W may oscillate close to the edge.

The results of the paper are based on a specific interplay between classes of operator-valued symbols and 'order reductions' which are also involved in the symbol estimates. In a simpler situation (isolated cuspidal singularities on the boundary), ideas of this kind are developed in [**33**]. There are applied weighted pseudodifferential operators which contain the local inverses to the operators of elliptic boundary value problems near singularities of the boundary. The property of being slowly varying is of great importance for the symbols of pseudodifferential operators near singularities. It means that the pseudodifferential operators may bear oscillating discontinuities in symbols which allows one to consider boundary value problems in domains with oscillating cuspidal singularities.

The behavior of symbols in [**33**] is controlled by an operator-valued function $\lambda(\tau)$ satisfying the condition

$$\|\lambda(\tau+\upsilon)\lambda^{-1}(\tau)\|_{\mathcal{L}(\tilde{H})} \le c\langle\upsilon\rangle^{\varepsilon} \quad \text{for all } \tau,\upsilon \in \mathbf{R},$$

with some $c,\varepsilon \in \mathbf{R}$, where $\tilde{H}$ is a Hilbert space and $\langle\upsilon\rangle = (1+|\upsilon|^2)^{1/2}$.

In the case of cuspidal wedges we need a calculus of pseudodifferential operators where the behavior of symbols is controlled by an operator-valued function $\lambda(t,\tau)$ depending on two variables $t,\tau \in \mathbf{R}$. It should satisfy

$$\|\lambda(t+\theta,\tau+\upsilon)\lambda^{-1}(t,\tau)\|_{\mathcal{L}(\tilde{H})} \le c\langle\theta\rangle^{\varepsilon_1}\langle\upsilon\rangle^{\varepsilon_2} \\ \text{for all } t,\tau,\theta,\upsilon \in \mathbf{R},$$

with some constants $c,\varepsilon_1,\varepsilon_2 \in \mathbf{R}$ independent of t,τ,θ,υ. Moreover, a standing condition on the symbols under study will be that they vary slowly close to singularities.

The typical differential operator on a manifold with cuspidal edges is of the form

$$A = (\delta'(r))^m \sum_{|\alpha|+j\le m} a_{\alpha,j}(y,r)\left(\frac{1}{\delta'(r)}D_y\right)^{\alpha}\left(\frac{1}{\delta'(r)}D_r\right)^{j}$$

where y stands for local coordinates along the edges, r is the distance to the edges and $a_{\alpha,j}(y,r)$ are C^∞ functions of y,r whose values are differential operators of order $m-|\alpha|-j$ on some compact C^∞ manifold B. Depending on the context, the manifold B is closed or has a boundary. Furthermore, $t = \delta(r)$ is a diffeomorphism of $\mathbf{R}_+$ onto the entire real axis $\mathbf{R}$ such that $\delta(0) = +\infty$ and $\delta(+\infty) = -\infty$. The derivative $\delta'(r)$ is a qualitative characteristic of the cuspidal degeneracy. For the canonical wedge in $\mathbf{R}^3$ described above we choose, e.g.,

$$\delta(r) = \int_r^\varepsilon \frac{dz}{\varphi(z)}$$

for $r > 0$ small enough, $\varepsilon > 0$ being any fixed number.

The typical weight function in the theory with edges is

$$\lambda(t,\tau) = ((1 + (\phi(t))^2\Delta_y + \tau^2)^{1/2} + \Lambda_B)^s,$$

where $\phi(t) = -1/\delta'(\delta^{-1}(t))$, $t \in \mathbf{R}$ and $\Lambda_B : H^s(B) \to H^{s-1}(B)$ is an order reduction.

The key property of the cuspidal degeneracy is that the function $\phi(t)$ meets the condition

$$\lim_{t\to+\infty} \frac{\phi'(t)}{\phi(t)} = 0. \tag{0.1}$$

It is easy to check that the property (0.1) holds in the case of power-like and exponential cuspidal degeneracies. Moreover, some higher-order cuspidal degeneracies obey (0.1). On the other hand, this property does not hold for usual conical wedges in which case we have $\phi(t) = e^{-t}$.

It was Feigin [**9**] who first studied general boundary value problems in domains with cuspidal wedges. However, this paper does not contain any proofs and, as far as we know, no proof has appeared until now. Moreover, Feigin [**9**] assumed merely power-like cuspidal degeneracy.

Our approach allows us to consider boundary value problems in domains with oscillating cuspidal wedges as well as pseudodifferential operators on closed manifolds with cuspidal edges. The boundary may oscillate near edges and the speed of this oscillation is connected with the degree of degeneracy.

Part I. A Class of Pseudodifferential Operators with Operator-Valued Symbols.

1. Weight operator-valued functions. Let H and $\tilde{H}$ be complex Hilbert spaces and $\mathcal{L}(H, \tilde{H})$ the space of all bounded linear operators from H to $\tilde{H}$.

Definition 1.1. We denote by $\Lambda(H, \tilde{H})$ the space of all functions $\lambda(t, \tau)$ on $\mathbf{R} \times \mathbf{R}$ with values in $\mathcal{L}(H, \tilde{H})$ such that, for each $(t, \tau) \in \mathbf{R} \times \mathbf{R}$ there exists an inverse $\lambda^{-1}(t, \tau)$ and

$$\|\lambda(t+\theta, \tau+\upsilon)\lambda^{-1}(t, \tau)\|_{\mathcal{L}(\tilde{H})} \leq c\langle\theta\rangle^{\varepsilon_1}\langle\upsilon\rangle^{\varepsilon_2} \tag{1.1}$$

for all $t, \tau, \theta, \upsilon \in \mathbf{R}$, where $c, \varepsilon_1, \varepsilon_2 \in \mathbf{R}$ are constants independent of $t, \tau, \theta, \upsilon$.

The elements of $\Lambda(H, \tilde{H})$ will be referred to as operator-valued weight functions on $\mathbf{R} \times \mathbf{R}$.

It is easily seen that an operator-valued function $\lambda(t, \tau)$ satisfies (1.1) if and only if

$$\begin{aligned} &\|\lambda(t+\theta, \tau)\lambda^{-1}(t, \tau)\|_{\mathcal{L}(\tilde{H})} \leq c\langle\theta\rangle^{\varepsilon_1}, \\ &\|\lambda(t, \tau+\upsilon)\lambda^{-1}(t, \tau)\|_{\mathcal{L}(\tilde{H})} \leq c\langle\upsilon\rangle^{\varepsilon_2}, \end{aligned} \tag{1.2}$$

the constants ε_1 and ε_2 being the same. Indeed, if (1.2) is fulfilled, then we get

$$\begin{aligned} &\|\lambda(t+\theta, \tau+\upsilon)\lambda^{-1}(t, \tau)\|_{\mathcal{L}(\tilde{H})} \\ &\quad \leq \|\lambda(t+\theta, \tau+\upsilon)\lambda^{-1}(t, \tau+\upsilon)\|_{\mathcal{L}(\tilde{H})}\|\lambda(t, \tau+\upsilon)\lambda^{-1}(t, \tau)\|_{\mathcal{L}(\tilde{H})} \\ &\quad \leq c^2\langle\theta\rangle^{\varepsilon_1}\langle\upsilon\rangle^{\varepsilon_2} \end{aligned}$$

showing (1.1). The reverse implication is obvious.

2. Symbol classes. Fix

$$\lambda_1(t, \tau) \in \Lambda(H_1, \tilde{H}_1), \qquad \lambda_2(t, \tau) \in \Lambda(H_2, \tilde{H}_2).$$

Definition 2.1. By $\mathcal{S}(\lambda_1, \lambda)$ is meant the class of C^∞ functions $a(t,\tau)$ on $\mathbf{R}\times\mathbf{R}$ with values in $\mathcal{L}(H_1, H_2)$ such that, for each $\alpha, \beta \in \mathbf{Z}_+$, there is a constant $c_{\alpha,\beta}(a)$ with the property that

$$\begin{aligned} \|\lambda_2(t,\tau)(D_t^\beta D_\tau^\alpha a(t,\tau))\lambda_1^{-1}(t,\tau)\|_{\mathcal{L}(\tilde{H}_1,\tilde{H}_2)} \le c_{\alpha,\beta}(a) \\ \text{for all } (t,\tau) \in \mathbf{R}\times\mathbf{R}. \end{aligned} \tag{2.1}$$

The best constants $c_{\alpha,\beta}(a)$ in (2.1) define a Fréchet topology in the space $\mathcal{S}(\lambda_1, \lambda_2)$. The elements of $\mathcal{S}(\lambda_1, \lambda_2)$ are called operator-valued symbols on $T^*\mathbf{R} \cong \mathbf{R}\times\mathbf{R}$.

To any symbol $a \in \mathcal{S}(\lambda_1, \lambda_2)$ there corresponds a pseudodifferential operator $A = \mathrm{op}\,(a)$ by

$$\mathrm{Au}\,(t) = \frac{1}{2\pi}\int_{\mathbf{R}} d\tau \int_{\mathbf{R}} e^{i(t-t')\tau} a(t,\tau)u(t')\,dt',$$

the operator A being first defined on functions $u \in C^\infty_{\mathrm{comp}}(\mathbf{R}, H_1)$.

Denote by $\mathcal{OPS}(\lambda_1, \lambda_2)$ the class of all operators $A = \mathrm{op}\,(a)$ with symbols $a \in \mathcal{S}(\lambda_1, \lambda_2)$.

Pseudodifferential operators with scalar-valued symbols whose behavior is controlled by scalar-valued weight functions $\lambda(t,\tau)$ were introduced by Kumano-go and Taniguchi [**20**]. The calculus of [**20**] was later generalized by Beals [**2**] (see also Hörmander [**18**]). The calculus of Beals and Hörmander was extended to operator-valued symbols in Levendorskii [**21**]. However, the calculus of [**21**] requires certain restrictions on symbols which are not fulfilled for the symbols arising in the study of differential operators on cuspidal wedges.

We introduce an analogue of the calculus of Kumano-go and Taniguchi [**20**] for operator-valued symbols, which relies on oscillatory integrals with operator-valued amplitude functions. For weight functions $\lambda(t,\tau)$ independent of t, a calculus of pseudodifferential operators with applications to boundary value problems in domains with singular boundary points was given by the first author [**29, 30, 32**] and in [**35**].

3. A composition formula for pseudodifferential operators. The following result gives rise to a calculus of pseudodifferential operators with symbols in $\mathcal{S}(\lambda_1, \lambda_2)$.

Proposition 3.1. *Suppose that* $A \in \mathcal{OPS}(\lambda_1, \lambda_2)$ *and* $B \in \mathcal{OPS}(\lambda_2, \lambda_3)$. *Then* $BA \in \mathcal{OPS}(\lambda_1, \lambda_3)$, *the symbol of* BA *is*

$$\sigma_{BA}(t,\tau) = \frac{1}{2\pi} \iint_{\mathbf{R}\times\mathbf{R}} e^{-i\theta\upsilon} \sigma_B(t, \tau+\upsilon)\sigma_A(t+\theta,\tau)\, d\theta\, d\upsilon \tag{3.1}$$

and the corresponding mapping $\mathcal{S}(\lambda_1, \lambda_2) \times \mathcal{S}(\lambda_2, \lambda_3) \to \mathcal{S}(\lambda_1, \lambda_3)$ *is continuous.*

Proof. The proof is actually the same as the proof of Proposition 1.4.1 in [**33**]. □

We emphasize that the double integral in (3.1) is regarded as an oscillatory integral. For a definition of oscillating integrals with operator-valued amplitude functions, we refer the reader to [**33**].

4. Formal adjoint. Let H be a Hilbert space. Denote by $\mathcal{S}(\mathbf{R}, H) = \mathcal{S}(\mathbf{R})\hat{\otimes}_\pi H$ the space of all rapidly decreasing C^∞ functions on $\mathbf{R}$ with values in H. We endow $\mathcal{S}(\mathbf{R}, H)$ with a Fréchet topology defined by the sequence of norms

$$\|u\|_{H,J} = \sup_{\substack{t\in\mathbf{R}\\ 0\le j\le J}} \langle t\rangle^J \|u^{(j)}(t)\|_H, \quad J = 0, 1, \dots .$$

Proposition 4.1. *If* $a \in \mathcal{S}(\lambda_1, \lambda_2)$, *then* $\operatorname{op}(a)$ *is a bounded operator from* $\mathcal{S}(\mathbf{R}, H_1)$ *to* $\mathcal{S}(\mathbf{R}, H_2)$.

Proof. Indeed (1.1) yields

$$\begin{aligned}\|\lambda(t,\tau)\|_{\mathcal{L}(H,\tilde{H})} &\le \|\lambda(t,\tau)\lambda^{-1}(0,0)\|_{\mathcal{L}(\tilde{H})} \|\lambda(0,0)\|_{\mathcal{L}(H,\tilde{H})} \\ &\le c\langle t\rangle^{\varepsilon_1} \langle\tau\rangle^{\varepsilon_2}\end{aligned}$$

as well as a similar estimate for the inverse $\lambda^{-1}(t,\tau)$. When combined with (2.1) these give

$$\|D_t^\beta D_\tau^\alpha a(t,\tau)\|_{\mathcal{L}(H_1,H_2)} \le c_{\alpha,\beta} \langle t\rangle^{\delta_1} \langle\tau\rangle^{\delta_2}$$

for all $\alpha, \beta \in \mathbf{Z}_+$, the constants $c_{\alpha,\beta}$, $\delta_1, \delta_2 \in \mathbf{R}$ being independent of t and τ. Now the desired assertion follows by differentiation and integration by parts just in the same way as for scalar-valued functions. $\square$

Let $A = \operatorname{op}(a)$ where $a \in \mathcal{S}(\lambda_1, \lambda_2)$. Then the *formal adjoint* A^* of A is defined by the equality

$$(Au, v)_{L^2(\mathbf{R}, H_2)} = (u, A^* v)_{L^2(\mathbf{R}, H_1)}$$

for any $u \in \mathcal{S}(\mathbf{R}, H_1)$ and $v \in \mathcal{S}(\mathbf{R}, H_2)$.

Proposition 4.2. *If $A \in \mathcal{OPS}(\lambda_1, \lambda_2)$, then $A^* \in \mathcal{OPS}((\lambda_2^{-1})^*, (\lambda_1^{-1})^*)$ and*

$$\sigma_{A^*}(t, \tau) = \frac{1}{2\pi} \iint_{\mathbf{R}\times\mathbf{R}} e^{-i\theta\upsilon} (\sigma_A(t+\theta, \tau+\upsilon))^* \, d\theta \, d\upsilon,$$

the corresponding mapping $\mathcal{S}(\lambda_1, \lambda_2) \to \mathcal{S}((\lambda_2^{-1})^*, (\lambda_1^{-1})^*)$ being continuous.

Proof. The proof is standard. $\square$

Applying Proposition 4.1 to the adjoint operator A^* and using a duality argument, we arrive at the following result.

Corollary 4.3. *Each operator $A \in \mathcal{OPS}(\lambda_1, \lambda_2)$ extends to a continuous mapping $\mathcal{S}'(\mathbf{R}, H_1) \to \mathcal{S}'(\mathbf{R}, H_2)$, where $\mathcal{S}'(\mathbf{R}, H)$ is the dual space of $\mathcal{S}(\mathbf{R}, H)$.*

Recall that the elements of $\mathcal{S}'(\mathbf{R}, H)$ are usually referred to as temperate distributions on $\mathbf{R}$ with values in H.

5. Boundedness of pseudodifferential operators in Sobolev spaces of distributions. Unless otherwise stated we assume that the operator-valued weight functions $\lambda(t, \tau)$ under consideration are of class C^∞ on $\mathbf{R} \times \mathbf{R}$ and satisfy

$$\begin{aligned} &\|(D_t^\beta D_\tau^\alpha \lambda(t,\tau)) \lambda^{-1}(t,\tau)\|_{\mathcal{L}(\tilde{H})} \le c_{\alpha,\beta}, \\ &\|\lambda^{-1}(t,\tau)(D_t^\beta D_\tau^\alpha \lambda(t,\tau))\|_{\mathcal{L}(H)} \le c_{\alpha,\beta} \end{aligned} \tag{5.1}$$

for all $\alpha, \beta \in \mathbf{Z}_+$, the constants $c_{\alpha,\beta}$ being independent of $(t,\tau) \in \mathbf{R}\times\mathbf{R}$.

We denote by $\Lambda'(H,\tilde{H})$ the subspace of $\Lambda(H,\tilde{H})$ consisting of operator-valued weight functions satisfying (5.1).

By the very definition, if $\lambda \in \Lambda'(H,\tilde{H})$, then λ belongs to both $\mathcal{S}(\lambda, 1_{\tilde{H}})$ and $\mathcal{S}(1_H, \lambda^{-1})$, and conversely.

Our next goal is to introduce, given any $\lambda \in \Lambda'(H,\tilde{H})$, a Sobolev space $H(\lambda)$ related to this weight function. If $\lambda(t,\tau)$ were independent of t, the operator $\mathrm{op}\,(\lambda)$ would be invertible and we might proceed just in the same way as in [**33**]. In order to adapt the definition of [**33**] to general weight functions, we need an auxiliary construction.

Proposition 5.1. *Suppose $\lambda \in \Lambda'(H,\tilde{H})$. Then*

1) *the inverse λ^{-1} belongs to both $\mathcal{S}(1_{\tilde{H}}, \lambda)$ and $\mathcal{S}(\lambda^{-1}, 1_H)$;*

2) *setting $\lambda_\varepsilon(t,\tau) = \lambda(t, \varepsilon\tau)$, we get*

$$\begin{aligned} \mathrm{op}\,(\lambda_\varepsilon)\mathrm{op}\,(\lambda_\varepsilon^{-1}) &= 1_{L^2(\mathbf{R},\tilde{H})} + r'_\varepsilon, \\ \mathrm{op}\,(\lambda_\varepsilon^{-1})\mathrm{op}\,(\lambda_\varepsilon) &= 1_{L^2(\mathbf{R},H)} + r''_\varepsilon \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} \lim_{\varepsilon\to 0} \|r'_\varepsilon\|_{\mathcal{L}(L^2(\mathbf{R},\tilde{H}))} &= 0, \\ \lim_{\varepsilon\to 0} \|r''_\varepsilon\|_{\mathcal{L}(L^2(\mathbf{R},H))} &= 0. \end{aligned} \tag{5.3}$$

Proof. The first part follows by the rule of differentiation of the inverse of an operator-valued function, cf. [**33**, Proposition 1.6.1].

Let us prove the second part. Put

$$A_\varepsilon = \mathrm{op}\,(\lambda_\varepsilon), \qquad B_\varepsilon = \mathrm{op}\,(\lambda_\varepsilon^{-1}).$$

By formula (3.1), we get

$$\sigma_{A_\varepsilon,B_\varepsilon}(t,\tau) = \frac{1}{2\pi} \iint_{\mathbf{R}\times\mathbf{R}} e^{-i\theta\upsilon} \lambda_\varepsilon(t,\tau+\upsilon)\lambda_\varepsilon^{-1}(t+\theta,\tau)\, d\theta\, d\upsilon. \tag{5.4}$$

We now make use of the Lagrange formula to see that

$$\lambda_\varepsilon(t,\tau+\upsilon) = \lambda_\varepsilon(t,\tau) + \upsilon \int_0^1 \frac{\partial \lambda_\varepsilon}{\partial \tau}(t, \tau + \vartheta\upsilon)\, d\vartheta.$$

Substituting this into (5.4) and using a particular case of the Fourier inversion formula,

$$\frac{1}{2\pi}\int_{\mathbf{R}} d\upsilon \int_{\mathbf{R}} e^{-i\theta\upsilon}\lambda_\varepsilon^{-1}(t+\theta,\tau)\,d\theta = \lambda_\varepsilon^{-1}(t,\tau),$$

we get

$$\sigma_{A_\varepsilon,B_\varepsilon}(t,\tau) = 1 + \int_0^1 q_\varepsilon(t,\tau,\vartheta)\,d\vartheta,$$

where

$$q_\varepsilon(t,\tau,\vartheta) = \frac{1}{2\pi i}\iint_{\mathbf{R}\times\mathbf{R}} e^{-i\theta\upsilon}\,\frac{\partial\lambda_\varepsilon}{\partial\tau}(t,\tau+\vartheta\upsilon)\,\frac{\partial\lambda_\varepsilon^{-1}}{\partial t}(t+\theta,\tau)\,d\theta\,d\upsilon,$$

the double integrals on the right side being regarded as oscillatory ones.

From the first estimate (5.1) it follows that $q_\varepsilon(t,\tau,\vartheta)$ meets an estimate

$$\|D_t^\beta D_\tau^\alpha q_\varepsilon(t,\tau,\vartheta)\|_{\mathcal{L}(\tilde{H})} \le c_{\alpha,\beta}\varepsilon^{\alpha+1}$$

for all $\alpha,\beta\in\mathbf{Z}_+$ with $c_{\alpha,\beta}$ a constant independent of $(t,\tau)\in\mathbf{R}\times\mathbf{R}$ and $\vartheta\in[0,1]$. The Calderon-Vaillancourt theorem now shows that

$$\lim_{\varepsilon\to 0}\|\mathrm{op}\,(q_\varepsilon(t,\tau,\vartheta))\|_{\mathcal{L}(L^2(\mathbf{R},\tilde{H}))} = 0$$

uniformly with respect to $\vartheta\in[0,1]$. Thus we can assert that

$$\mathrm{op}\,(\lambda_\varepsilon)\mathrm{op}\,(\lambda_\varepsilon^{-1}) = 1_{L^2(\mathbf{R},\tilde{H})} + r'_\varepsilon,$$

with $r'_\varepsilon \in \mathcal{OPS}(1_{\tilde{H}},1_{\tilde{H}})$ satisfying

$$\lim_{\varepsilon\to 0}\|r'_\varepsilon\|_{\mathcal{L}(L^2(\mathbf{R},\tilde{H}))} = 0,$$

as required.

The second equality of (5.2) is proved in just the same way, using the second estimate of (5.1). This completes the proof. □

The interest of the proposition is that it allows one to construct so-called order reductions within the calculus.

Corollary 5.2. *For any $\lambda \in \Lambda'(H, \tilde{H})$ the operator $R_\varepsilon(\lambda) = \mathrm{op}\,(\lambda_\varepsilon)$ is invertible for $\varepsilon > 0$ small enough, and the inverse $R_\varepsilon^{-1}(\lambda)$ is in $\mathcal{OPS}(\lambda^{-1}, 1_H)$.*

Proof. By Proposition 5.1 $R_\varepsilon(\lambda) \in \mathcal{OPS}(\lambda, 1_{\tilde{H}})$ fulfills (5.2) with

$$r'_\varepsilon \in \mathcal{OPS}(1_{\tilde{H}}, 1_{\tilde{H}}), \qquad r''_\varepsilon \in \mathcal{OPS}(1_H, 1_H)$$

satisfying (5.3). Hence it follows that both $1+r'_\varepsilon$ and $1+r''_\varepsilon$ are invertible for sufficiently small $\varepsilon > 0$. Moreover, we can assert by a theorem of Beals [**2**], that

$$(1 + r'_\varepsilon)^{-1} \in \mathcal{OPS}(1_{\tilde{H}}, 1_{\tilde{H}}), \qquad (1 + r''_\varepsilon)^{-1} \in \mathcal{OPS}(1_H, 1_H)$$

for $\varepsilon > 0$ small enough. We deduce that

$$\begin{aligned} &\mathrm{op}\,(\lambda_\varepsilon^{-1})(1 + r'_\varepsilon)^{-1} \in \mathcal{OPS}(1_{\tilde{H}}, \lambda), \\ &(1 + r''_\varepsilon)^{-1}\mathrm{op}\,(\lambda_\varepsilon^{-1}) \in \mathcal{OPS}(\lambda^{-1}, 1_H) \end{aligned}$$

are the right inverse and the left inverse of $R_\varepsilon(\lambda)$, respectively. Hence they coincide, thus giving an inverse $R_\varepsilon^{-1}(\lambda) \in \mathcal{OPS}(\lambda^{-1}, 1_H)$ for $R_\varepsilon(\lambda)$, as required. $\square$

We make use of the operators $R_\varepsilon(\lambda)$ for $\varepsilon > 0$ small enough to introduce Sobolev spaces of operator-valued functions.

Definition 5.3. Let $\lambda \in \Lambda'(H, \tilde{H})$. We denote by $H(\lambda)$ the space of all distributions $u \in \mathcal{S}'(\mathbf{R}, H)$ with finite norm

$$\|u\|_{H(\lambda)} = \|R_\varepsilon(\lambda)u\|_{L^2(\mathbf{R}, \tilde{H})}.$$

Analysis similar to that in the proof of Proposition 5.1 shows that the composition $R_\delta(\lambda)R_\varepsilon^{-1}(\lambda)$ is a bounded operator in $L^2(\mathbf{R}, \tilde{H})$, provided that $\delta, \varepsilon > 0$ are sufficiently small. Hence the space $H(\lambda)$ is independent of the particular choice of $0 < \varepsilon \ll 1$.

Proposition 5.4. *Suppose $\lambda_1 \in \Lambda'(H_1, \tilde{H}_1)$ and $\lambda_2 \in \Lambda'(H_2, \tilde{H}_2)$. Every operator $A \in \mathcal{OPS}(\lambda_1, \lambda_2)$ extends to a continuous mapping*

$H(\lambda_1) \to H(\lambda_2)$. Moreover,

$$\|Au\|_{H(\lambda_2)} \le c\Big(\sum_{\alpha+\beta\le N} c_{\alpha,\beta}(\sigma_A)\Big)\|u\|_{H(\lambda_1)}, \quad u \in H(\lambda_1), \tag{5.5}$$

the constants $c > 0$ and $N \in \mathbf{Z}_+$ being independent of A.

Proof. The boundedness of $A : H(\lambda_1) \to H(\lambda_2)$ is equivalent to the boundedness of $\tilde{A} : L^2(\mathbf{R}, \tilde{H}_1) \to L^2(\mathbf{R}, \tilde{H}_2)$, where

$$\tilde{A} = R_\varepsilon(\lambda_2) A R_\varepsilon^{-1}(\lambda_1),$$

$R_\varepsilon(\lambda_2)$ and $R_\varepsilon^{-1}(\lambda_1)$ being given by Corollary 5.2.

By Proposition 3.1 we conclude that $\tilde{A} \in \mathcal{OPS}(1_{\tilde{H}_1}, 1_{\tilde{H}_1})$, and so

$$\sup_{x,\xi} \|D_t^\beta D_\tau^\alpha \sigma_{\tilde{A}}(t,\tau)\|_{\mathcal{L}(\tilde{H}_1,\tilde{H}_2)} \le c_{\alpha,\beta}(\sigma_{\tilde{A}})$$

for all $\alpha, \beta \in \mathbf{Z}_+$. According to the Calderon-Vaillancourt theorem, $\tilde{A}$ extends to a bounded operator $L^2(\mathbf{R}, \tilde{H}_1) \to L^2(\mathbf{R}, \tilde{H}_2)$ and

$$\|\tilde{A}\|_{\mathcal{L}(L^2(\mathbf{R},\tilde{H}_1), L^2(\mathbf{R},\tilde{H}_2))} \le \tilde{c} \sum_{\alpha+\beta\le\tilde{N}} c_{\alpha,\beta}(\sigma_{\tilde{A}}),$$

the constants $\tilde{c} > 0$ and $\tilde{N} \in \mathbf{Z}_+$ being independent of $\tilde{A}$. Combining this with Proposition 3.1, we arrive at estimate (5.5), as required. $\square$

We finish this section by yet another technical assertion whose proof is similar to the proof of Proposition 5.1.

Proposition 5.5. *Let $A \in \mathcal{OPS}(\lambda_1, \lambda_2)$. Suppose $\chi \in C^\infty(\mathbf{R})$ satisfies $\chi(t) = 0$ for $t \le 1$ and $\chi(t) = 1$ for $t \ge 2$. Then*

$$\lim_{R\to\infty} \|[A, \chi(\cdot/R)]\|_{\mathcal{L}(H(\lambda_1),H(\lambda_2))} = 0.$$

6. Pseudodifferential operators with symbols slowly varying at infinity. In studying pseudodifferential operators $A = \mathrm{op}\,(a(x,\xi))$

on $\mathbf{R}^n$ it is usually assumed that the symbol $a(x,\xi)$ stabilizes in some sense as $x \to \infty$. There are, however, a great number of problems which lead to differential or pseudodifferential operators without the condition of stabilization of the symbol at the point of infinity. A class of such operators was studied by Grushin [**13**] who extended a joint work with Vishik [**16**]. The following definition introduces this class in the case of operator-valued symbols on the real axis.

Definition 6.1. A symbol $a(t,\tau) \in \mathcal{S}(\lambda_1,\lambda_2)$ is called *slowly varying* as $t \to +\infty$ if

$$\lim_{t\to+\infty} \sup_{\tau\in\mathbf{R}} \|\lambda_2(t,\tau)(D_t^\beta D_\tau^\alpha a(t,\tau))\lambda_1^{-1}(t,\tau)\|_{\mathcal{L}(\tilde{H}_1,\tilde{H}_2)} = 0, \tag{6.1}$$

for each $\alpha \geq 0$ and $\beta \geq 1$. We write $\mathcal{S}_{\rm sv}(\lambda_1,\lambda_2)$ for the class of symbols slowly varying as $t \to +\infty$.

We also distinguish the subclass $\mathcal{S}_0(\lambda_1,\lambda_2)$ of $\mathcal{S}_{\rm sv}(\lambda_1,\lambda_2)$ consisting of the symbols $a(t,\tau)$ which obey (6.1) for all $\alpha,\beta \in \mathbf{Z}_+$.

Proposition 6.2. 1) *If $A \in \mathcal{OPS}_{\rm sv}(\lambda_1,\lambda_2)$ and $B \in \mathcal{OPS}_{\rm sv}(\lambda_2,\lambda_3)$, then $BA \in \mathcal{OPS}_{\rm sv}(\lambda_1,\lambda_3)$ and the symbol of BA is given by*

$$\sigma_{BA}(t,\tau) = \sigma_B(t,\tau)\sigma_A(t,\tau) + r(t,\tau),$$

where $r(t,\tau) \in \mathcal{S}_0(\lambda_1,\lambda_3)$.

2) *If $A \in \mathcal{OPS}_{\rm sv}(\lambda_1,\lambda_2)$, then $A^* \in \mathcal{OPS}_{\rm sv}((\lambda_2^{-1})^*,(\lambda_1^{-1})^*)$ and the symbol of A^* is given by*

$$\sigma_{A^*}(t,\tau) = (\sigma_A(t,\tau))^* + r(t,\tau),$$

where $r(t,\tau) \in \mathcal{S}_0((\lambda_2^{-1})^*,(\lambda_1^{-1})^*)$.

Proof. The proof is similar to the proof of Proposition 1.5.2 in [**33**]. □

7. Local invertibility of pseudodifferential operators at infinity. Let $\chi \in C^\infty(\mathbf{R})$ satisfy $\chi(t) = 0$, if $t \leq 1$ and $\chi(t) = 1$, if $t \geq 2$. Put $\chi_R(t) = \chi(t/R)$ for $R > 0$.

Definition 7.1. We say that an operator $A \in \mathcal{L}(H(\lambda_1), H(\lambda_2))$ is locally invertible from the left (right) at the point $+\infty$ if there exist $R > 0$ and an operator $B \in \mathcal{L}(H(\lambda_2), H(\lambda_1))$ such that $BA_{\chi_R} = \chi_R(\chi_R AB = \chi_R)$, respectively.

We call A *locally invertible* at the point $+\infty$ if it is locally invertible both from the left and from the right at this point.

Before formulating our next result, we note that the concept of being slowly varying is also applicable to the weight functions in $\Lambda'(H, \tilde{H})$. Namely, such a function $\lambda(t,\tau)$ is said to vary slowly as $t \to +\infty$ if

$$\lim_{t\to+\infty} \sup_{\tau\in\mathbf{R}} \|(D_t^\beta D_\tau^\alpha \lambda(t,\tau))\lambda_1^{-1}(t,\tau)\|_{\mathcal{L}(\tilde{H})} = 0$$

for all $\alpha \in \mathbf{Z}_+$ and $\beta = 1, 2, \dots$.

Theorem 7.2. *Suppose $\lambda_j \in \Lambda'(H_j, \tilde{H}_j)$, $j = 1, 2$, are slowly varying as $t \to +\infty$. Let $A = \mathrm{op}\,(a)$ where $a \in \mathcal{S}_{\mathrm{sv}}(\lambda_1, \lambda_2)$. Then $A : H(\lambda_1) \to H(\lambda_2)$ is locally invertible at the point $+\infty$ if and only if there exists a number $R > 0$ such that the symbol $a(t,\tau) : H_1 \to H_2$ is invertible for all $(t,\tau) \in (R, +\infty) \times \mathbf{R}$ and*

$$\sup_{(R,+\infty)\times\mathbf{R}} \|\lambda_1(t,\tau)a^{-1}(t,\tau)\lambda_2^{-1}(t,\tau)\|_{\mathcal{L}(\tilde{H}_2,\tilde{H}_1)} < \infty. \tag{7.1}$$

Proof. The proof of this theorem is actually the same as the proof of Theorem 1.7.4 in [**33**]. □

8. Exponential weighted estimates for pseudodifferential operators with analytic symbols. For $\gamma \in \mathbf{R}$, we denote by $H(\lambda; \gamma)$ the completion of $C^\infty_{\mathrm{comp}}(\mathbf{R}, H)$ with respect to the norm

$$\|u\|_{H(\lambda;\gamma)} = \|e^{\gamma t} u\|_{H(\lambda)}.$$

If $a(t, \tau + i\gamma) \in \mathcal{S}(\lambda_1, \lambda_2)$, then

$$\mathrm{op}\,(a(t, \tau + i\gamma)) = e^{\gamma t} \mathrm{op}_\gamma(a(t,z)) e^{-\gamma t},$$

where

$$\operatorname{op}_\gamma(a(t,z))u(t) = \frac{1}{2\pi}\int_{\mathbf{R}+i\gamma} dz \int_{\mathbf{R}} e^{i(t-t')z} a(t,z)u(t')\,dt', \quad t \in \mathbf{R},$$

for $u \in C^\infty_{\mathrm{comp}}(\mathbf{R}, H_1)$. Hence it follows that

$$\begin{aligned}\|\operatorname{op}_\gamma(a(t,z))u\|_{H(\lambda_2;\gamma)} &= \|e^{\gamma t}\operatorname{op}_\gamma(a(t,z))u\|_{H(\lambda_2)} \\ &= \|\operatorname{op}(a(t,\tau+i\gamma))e^{\gamma t}u\|_{H(\lambda_2)} \\ &\le c\|e^{\gamma t}u\|_{H(\lambda_1)} \\ &= c\|u\|_{H(\lambda_1;\gamma)},\end{aligned}$$

c being the norm of $\operatorname{op}(a(t,\tau+i\gamma))$ in $\mathcal{L}(H(\lambda_1), H(\lambda_2))$. Thus, $\operatorname{op}_\gamma(a(t,z))$ extends to a continuous mapping $H(\lambda_1;\gamma) \to H(\lambda_2;\gamma))$.

If $a(t,z)$ were a polynomial in z, the operator $\operatorname{op}_\gamma(a(t,z))$ would be differential and thus independent of the particular choice of $\gamma \in \mathbf{R}$. This still holds for those symbols $a(t,z)$ which extend analytically in z to some strip around $\mathbf{R}+i\gamma$. More precisely, assume that $a(t,z)$ is an analytic function of z in a horizontal strip $\mathbf{R}+i(a,b)$ such that $a(t,\tau+i\gamma) \in \mathcal{S}(\lambda_1,\lambda_2)$ uniformly in γ in compact intervals of (a,b). Then it is an easy consequence of the Cauchy theorem that the operator $\operatorname{op}_\gamma(a(t,z))$, when restricted to $C^\infty_{\mathrm{comp}}(\mathbf{R}, H_1)$, does not depend on $\gamma \in (a,b)$. We will denote it simply by $\operatorname{op}(a)$. As described above, $\operatorname{op}(a)$ extends to a continuous mapping $H(\lambda_1;\gamma) \to H(\lambda_2;\gamma)$ for each $\gamma \in (a,b)$ and this extension is given by $\operatorname{op}_\gamma(a(t,z))$.

Theorem 8.1. *Let $a(t,z)$ be an analytic function of $z \in \mathbf{R}+i(a,b)$ such that $a(t,\tau+i\gamma) \in \mathcal{S}_{\mathrm{sv}}(\lambda_1,\lambda_2)$ uniformly in $\gamma \in (a,b)$. Suppose there is an $R>0$ such that the symbol $a(t,z): H_1 \to H_2$ is invertible for all $t>R$ and $z \in \mathbf{R}+i(a,b)$ and*

$$\sup_{\substack{t>R \\ z\in\mathbf{R}+i(a,b)}} \|\lambda_1(t,\tau)a^{-1}(t,z)\lambda_2^{-1}(t,\tau)\|_{\mathcal{L}(\tilde{H}_2,\tilde{H}_1)} < \infty.$$

Then, if $a < \gamma' \le \gamma'' < b$ and $u \in H(\lambda_1;\gamma')$ satisfies $\operatorname{op}(a)u \in H(\lambda_2;\gamma'')$ near $t=+\infty$, then $u \in H(\lambda_1,\gamma'')$ near $t=+\infty$.

As usual, we say that $u \in \mathcal{D}'(\mathbf{R}, H)$ is of class $H(\lambda;\gamma)$ near $t=+\infty$ if $\varphi u \in H(\lambda;\gamma)$ for some function $\varphi \in C^\infty(\mathbf{R})$ equal to 0 near $t=-\infty$

and 1 near $t = +\infty$. Note that $\varphi H(\lambda;\gamma'') \hookrightarrow \varphi H(\lambda;\gamma')$ provided $\gamma' \le \gamma''$.

The proof of Theorem 8.1 is based on the following two lemmas proved in a more general context in [**31**].

Lemma 8.2. *Let $a(t,z)$ satisfy*

$$\sup_{\substack{t\in\mathbf{R}\\ z\in\mathbf{R}+i(a,b)}} \|\lambda_2(t,\tau)(D_t^\beta D_z^\alpha a(t,z))\lambda_1^{-1}(t,\tau)\|_{\mathcal{L}(\tilde{H}_1,\tilde{H}_2)} < \infty, \tag{8.1}$$
$$\alpha,\beta\in\mathbf{Z}_+,$$

and let $w(t) = \exp\gamma(t)$ with $\gamma\in C^\infty(\mathbf{R})$ such that $a < \inf\gamma' \le \sup\gamma' < b$. Then $\operatorname{op}(a)$ extends to a continuous mapping $H(\lambda_1;w(t)) \to H(\lambda_2;w(t))$.

Proof. See Theorem 3.1 (a) in [**31**]. In fact, Theorem 3.1 is proved in [**31**] for the weight functions $\lambda(t,\tau)$ that do not depend on $t\in\mathbf{R}$. However, the same proof still goes for arbitrary $\lambda(t,\tau)$ meeting our conditions. □

The spaces $H(\lambda;w(t))$ generalize $H(\lambda;\gamma)$ while the function $w(t) = \exp(\gamma t)$ is assigned to any $\gamma\in\mathbf{R}$. More precisely, by $H(\lambda;w(t))$ is meant the completion of $C^\infty_{\mathrm{comp}}(\mathbf{R},H)$ with respect to the norm $\|u\|_{H(\lambda;w(t))} = \|w(t)u\|_{H(\lambda)}$.

Lemma 8.3. *Suppose $a_R(t,z) = 1 + r_R(t,z)$, $R>0$, is a family of analytic functions of $z\in\mathbf{R}+i(a,b)$ with values in $\mathcal{L}(H)$ such that*

$$\lim_{R\to\infty}\sup_{\substack{t\in\mathbf{R}\\ z\in\mathbf{R}+i(a,b)}} \|\lambda(t,\tau)(D_t^\beta(D_z^\alpha r_R(t,z))\lambda^{-1}(t,\tau)\|_{\mathcal{L}(\tilde{H})} = 0, \tag{8.2}$$
$$\alpha,\beta\in\mathbf{Z}_+.$$

Then the operator $A_R = \operatorname{op}(a_R)$ on $H(\lambda;\gamma)$, $\gamma\in(a,b)$ is invertible for $R>0$ large enough and $A_R^{-1} = \operatorname{op}(s_R)$ where $s_R(t,z)$ is an analytic *operator-valued symbol satisfying*

$$\sup_{\substack{t\in\mathbf{R}\\ z\in\mathbf{R}+i(a,b)}} \|\lambda(t,\tau)(D_t^\beta D_z^\alpha s_R(t,z))\lambda^{-1}(t,\tau)\|_{\mathcal{L}(\tilde{H})} \le c_{\alpha,\beta}, \quad \alpha,\beta\in\mathbf{Z}_+. \tag{8.3}$$

Proof. See Theorems 2.2 and 2.3 in [**31**]. □

Proof of Theorem 8.1. Let $\chi_R(t)$ stand for a cut-off function at $t=+\infty$, as above. By assumption, the symbol $b_R(t,z)=\chi_R(t)a^{-1}(t,z)$ is an analytic operator-valued function satisfying an estimate of the type (8.1), provided that $R>0$ is large enough.

Set $B_R=\operatorname{op}(b_R)$. We get $B_RA=\chi_R+\operatorname{op}(r_R)$ where $r_R(t,z)$ is an analytic function of $z\in\mathbf{R}+i(a,b)$ with values in $\mathcal{L}(H_1)$ satisfying

$$\sup_{\substack{t\in\mathbf{R}\\ z\in\mathbf{R}+i(a,b)}} \|\lambda_1(t,\tau)(D_t^\beta D_z^\alpha r_R(t,z))\lambda_1^{-1}(t,\tau)\|_{\mathcal{L}(\tilde{H}_1)} < \infty$$

for all $\alpha,\beta\in\mathbf{Z}_+$. Moreover, we have

$$\lim_{t\to+\infty} \sup_{z\in\mathbf{R}+i(a,b)} \|\lambda_1(t,\tau)(D_t^\beta D_z^\alpha r_R(t,z))\lambda_1^{-1}(t,\tau)\|_{\mathcal{L}(\tilde{H}_1)} = 0, \tag{8.4}$$

for each $\alpha,\beta\in\mathbf{Z}_+$ because both $a(t,z)$ and $b_R(t,z)$ are slowly varying at $t=+\infty$.

Pick yet another cut-off function $\tilde{\chi}$ at $t=+\infty$ such that χ "covers" $\tilde{\chi}$, i.e., $\chi\tilde{\chi}=\tilde{\chi}$. By the above, we have

$$\begin{aligned}\chi_R B_R A\tilde{\chi}_R &= \tilde{\chi}_R+\chi_R\operatorname{op}(r_R)\tilde{\chi}_R\\ &= (1+\chi_R\operatorname{op}(r_R))\tilde{\chi}_R.\end{aligned}$$

It follows from (8.4) that the symbol $\chi_R(t)r_R(t,z)$ meets condition (8.2). By Lemma 8.3 the operator $1+\chi_R\operatorname{op}(r_R)$ on $H(\lambda_1;\gamma)$, $\gamma\in(a,b)$, is invertible for sufficiently large $R>0$ and the inverse has an analytic symbol $s_R(t,z)$ satisfying estimates (8.3), with λ replaced by λ_1. We thus obtain

$$(1+\chi_R\operatorname{op}(r_R))^{-1}\chi_R B_R A\tilde{\chi}_R=\tilde{\chi}_R$$

for all $R > 0$ large enough, where $P_R = (1 + \chi_R \operatorname{op}(r_R))^{-1} \chi_R B_R$ is a pseudodifferential operator with an analytic operator-valued symbol $p_R(t,z)$ satisfying

$$\sup_{\substack{t\in\mathbf{R}\\ z\in\mathbf{R}+i(a,b)}} \|\lambda_1(t,\tau)(D_t^\beta D_z^\alpha p_R(t,z))\lambda_2^{-1}(t,\tau)\|_{\mathcal{L}(\tilde{H}_2,\tilde{H}_1)} \le c_{\alpha,\beta},$$

for any $\alpha, \beta \in \mathbf{Z}_+$. These estimates ensure, by Lemma 8.2, the boundedness of the operator P_R acting as $H(\lambda_2; w(t)) \to H(\lambda_1; w(t))$, where $w(t) = \exp\gamma(t)$ and $\gamma \in C^\infty(\mathbf{R})$ is such that $a < \inf_t \gamma'(t) \le \sup_t \gamma'(t) < b$. In particular, we may take

$$\gamma(t) = \begin{cases} \gamma' t & \text{if } t \le 1; \\ \gamma'' t & \text{if } t \ge 2, \end{cases}$$

and extend it to a smooth function on the whole axis satisfying the above conditions.

Having disposed of this preliminary step, we are able to complete the proof of the theorem. Indeed, suppose $u \in H(\lambda_1, \gamma')$ satisfies $\operatorname{op}(a)u = f$ with $f \in H(\lambda_2; \gamma')$ bearing moreover the property that $f \in H(\lambda_2; \gamma'')$ near $t = +\infty$. Write

$$\tilde{\chi}_R u = -P_R A(1 - \tilde{\chi}_R)u + P_R f.$$

It is evident that $(1 - \tilde{\chi}_R)u \in H(\lambda_1; w(t))$ and $f \in H(\lambda_2; w(t))$. According to Lemma 8.2, $P_R A$ extends to a continuous mapping of $H(\lambda_1; w(t))$; hence, $\tilde{\chi}_R u \in H(\lambda_1; w(t))$. This yields $u \in H(\lambda_1; \gamma'')$ near $t = +\infty$ which is our claim. $\square$

9. Examples of weight functions. In this section we show several examples of operator-valued weight functions $\lambda(t,\tau)$ to be used in the calculus on manifolds with edges.

Example 9.1. Let

$$H = H^s(\mathbf{R}^q), \qquad \tilde{H} = L^2(\mathbf{R}^q)$$

and

$$\lambda(t,\tau) = (1 + (\phi(t))^2 \Delta_y + \tau^2)^{s/2},$$

where $\phi(t) > 0$ is a C^∞ function on the real axis satisfying an estimate

$$\frac{\phi(t)}{\phi(\theta)} \le c(1+|t-\theta|)^\varepsilon \quad \text{for all } t, \theta \in \mathbf{R}, \tag{9.1}$$

with c and $\varepsilon \ge 0$ independent of t and θ, $\Delta_y = D^2_{y_1} + \cdots + D^2_{y_q}$ is the nonnegative Laplace operator on $\mathbf{R}^q$ and $s \in \mathcal{R}$. We have

$$\lambda(t,\tau) = \mathrm{op}\,(1+(\phi(t))^2|\eta|^2+\tau^2)^{s/2},$$
$$\lambda^{-1}(t,\tau) = \mathrm{op}\,(1+(\phi(t))^2|\eta|^2+\tau^2)^{-s/2}$$

whence

$$\lambda(t+\theta,\tau+\upsilon)\lambda^{-1}(t,\tau) = \mathrm{op}\left(\frac{1+(\phi(t+\theta))^2|\eta|^2+(\tau+\upsilon)^2}{1+(\phi(t))^2|\eta|^2+\tau^2}\right)^{s/2},$$

for the symbols are independent of y. To verify (1.1) we need an elementary estimate.

Lemma 9.2. *If $q \ge 1$, then*

$$\frac{(q^2+\tau^2)^{s/2}}{(q^2+\upsilon^2)^{s/2}} \le 2^{|s|/2}(1+|\tau-\upsilon|^2)^{|s|/2}.$$

Proof. Indeed,

$$\begin{aligned}\frac{(q^2+\tau^2)^{s/2}}{(q^2+\upsilon^2)^{s/2}} &= \frac{(1+(\tau/q)^2)^{s/2}}{(1+(\upsilon/q)^2)^{s/2}}\\ &\le 2^{|s|/2}(1+((\tau/q)-(\upsilon/q))^2)^{|s|/2}\\ &\le 2^{|s|/2}(1+(\tau-\upsilon)^2)^{|s|/2},\end{aligned}$$

the first estimate being a consequence of the well-known Peetre inequality. This is our claim. □

Applying Lemma 9.2 we obtain

$$\frac{(1+(\phi(t))^2|\eta|^2+\tau^2)^{s/2}}{(1+(\phi(t))^2|\eta|^2+\upsilon^2)^{s/2}} \le 2^{|s|/2}(1+|\tau-\upsilon|^2)^{|s|/2} \tag{9.2}$$

for all $\tau, \upsilon \in \mathbf{R}$. Further,

$$\begin{aligned}1 + (\phi(t+\theta))^2|\eta|^2 + \tau^2 &\le 1 + c^2(1+|\theta|)^{2\varepsilon}(\phi(t))^2|\eta|^2 + \tau^2\\ &\le c^2(1+|\theta|)^{2\varepsilon}(1 + (\phi(t))^2|\eta|^2 + \tau^2)\end{aligned}$$

where $c \ge 1$ is the constant of (9.1). Hence it follows that

$$\frac{1}{c^2(1+|\theta|)^{2\varepsilon}} \le \frac{1 + (\phi(t+\theta))^2|\eta|^2 + \tau^2}{1 + (\phi(t))^2|\eta|^2 + \tau^2} \le c^2(1+|\theta|)^{2\varepsilon},$$

and so

$$\frac{(1 + (\phi(t+\theta))^2|\eta|^2 + \tau^2)^{s/2}}{(1 + (\phi(t))^2|\eta|^2 + \tau^2)^{s/2}} \le c^{|s|}(1+|\theta|)^{\varepsilon|s|} \tag{9.3}$$

for all $t, \theta \in \mathbf{R}$. As

$$\|\mathrm{op}\,(a(\eta))\|_{\mathcal{L}(L^2(\mathbf{R}^q))} \le \sup_{\eta \in \mathbf{R}^q} |a(\eta)|,$$

the estimates (9.2) and (9.3) imply the estimates (1.2) for $\lambda(t,\tau)$ with $\varepsilon_1 = \varepsilon|s|$ and $\varepsilon_2 = |s|$. Thus, we get $\lambda \in \Lambda(H^s(\mathbf{R}^q), L^2(\mathbf{R}^q))$, as required.

Example 9.3. Suppose B is a C^∞ compact closed manifold and $\mathbf{R}^q \times B$ a cylindrical manifold over B. For $s \in \mathbf{R}$, set

$$H = H^s(\mathbf{R}^q \times B), \qquad \tilde{H} = L^2(\mathbf{R}^q \times B)$$

and

$$\lambda(t,\tau) = (1 + (\phi(t))^2\Delta_y + \tau^2 + \Delta_B)^{s/2},$$

where $\phi(t) > 0$ is a C^∞ function on $\mathbf{R}$ satisfying (9.1) and $\Delta_B = \nabla^*\nabla$ is the Laplace operator associated with a connection ∇ on B. Let $(e_i)_{i=1,2,\dots}$ be a complete orthonormal system in $L^2(B)$ consisting of eigenfunctions of Δ_B, and let $(\mu_i)_{i=1,2,\dots}$ be the corresponding system of eigenvalues, each μ_i being nonnegative. If $u(y,x) \in L^2(\mathbf{R}^q \times B)$, then

$$\begin{aligned}&\lambda(t+\theta, \tau+\upsilon)\lambda^{-1}(t,\tau)u\\ &\qquad = \sum_{i=1}^{\infty} \mathcal{F}^{-1}_{\eta\mapsto y}\, \frac{\lambda_i(t+\theta,\eta,\tau+\upsilon)}{\lambda_i(t,\eta,\tau)}\, \mathcal{F}_{y\mapsto\eta}(u(y,\cdot),e_i)e_i\end{aligned}$$

where

$$\lambda_i(t,\eta,\tau) = (1 + (\phi(t))^2|\eta|^2 + \tau^2 + \mu_i)^{s/2}$$

and $(u(y,\cdot),e_i)$ is the scalar product of $u(y,x)$ and $e_i(x)$ in $L^2(B)$. Hence we deduce that

$$\begin{aligned}
&\|\lambda(t+\theta,\tau+\upsilon)\lambda^{-1}(t,\tau)u\|^2_{\tilde{H}} \\
&\quad= \sum_{i=1}^{\infty}\int_{\mathbf{R}^q}\left|\frac{\lambda_i(t+\theta,\eta,\tau+\upsilon)}{\lambda_i(t,\eta,\tau)}\right|^2 |\mathcal{F}_{y\mapsto\eta}(y(y,\cdot),e_i)|^2\,d\eta \\
&\quad\le \sup_{\substack{\eta\in\mathbf{R}^q\\ i=1,2,\dots}}\left|\frac{\lambda_i(t+\theta,\eta,\tau+\upsilon)}{\lambda_i(t,\eta,\tau)}\right|^2 \sum_{i=1}^{\infty}\int_{\mathbf{R}^q}|\mathcal{F}_{y\mapsto\eta}(u(y,\cdot),e_i)|^2\,d\eta \\
&\quad= \sup_{\substack{\eta\in\mathbf{R}^q\\ i=1,2,\dots}}\left|\frac{\lambda_i(t+\theta,\eta,\tau+\upsilon)}{\lambda_i(t,\eta,\tau)}\right|^2 \|u\|^2_{\tilde{H}},
\end{aligned}$$

and so

$$\|\lambda(t+\theta,\tau+\upsilon)\lambda^{-1}(t,\tau)\|_{\mathcal{L}(\tilde{H})} \le \sup_{\substack{\eta\in\mathbf{R}^q\\ i=1,2,\dots}}\left|\frac{\lambda_i(t+\theta,\eta,\tau+\upsilon)}{\lambda_i(t,\eta,\tau)}\right|$$

for all $t,\tau,\theta,\upsilon \in \mathbf{R}$. From what has already been proved in Example 9.1, it follows that

$$\left|\frac{\lambda_i(t+\theta,\eta,\tau+\upsilon)}{\lambda_i(t,\eta,\tau)}\right| \le 2^{(\varepsilon+1)|s|/2}c^{|s|}\langle\theta\rangle^{\varepsilon|s|}\langle\upsilon\rangle^{|s|}$$

with c a constant independent of t,τ,θ and υ. Thus we see that λ is of class $\Lambda(H^s(\mathbf{R}^q\times B), L^2(\mathbf{R}^q\times B))$.

Example 9.4. Let

$$\mathbf{R}^n_\pm = \{x = (x',x_n)\in\mathbf{R}^n : \pm x_n > 0\}$$

and let $\xi = (\xi',\xi_n)$ be the corresponding splitting of the covariables $\xi\in\mathbf{R}^n$. For $s\in\mathbf{R}$, we denote by $H^s(\overline{\mathbf{R}}^n_+)$ the space consisting of the restrictions to $\mathbf{R}^n_+$ of distributions in $H^s(\mathbf{R}^n)$. We have

$$H^s(\overline{\mathbf{R}}^n_+) = H^s(\mathbf{R}^n)/\mathring{H}^s(\overline{\mathbf{R}}^n_-)$$

where $\overset{\circ}{H}{}^s(\overline{\mathbf{R}}^n_-)$ is the subspace of $H^s(\mathbf{R}^n)$ consisting of distributions supported in $\overline{\mathbf{R}}^n_-$. Under the quotient norm, $H^s(\overline{\mathbf{R}}^n_+)$ is a Hilbert space.[1] As $\overset{\circ}{H}{}^s(\overline{\mathbf{R}}^n_-)$ coincides with the closure of $C^\infty_{\mathrm{comp}}(\mathbf{R}^n_-)$ in $H^s(\mathbf{R}^n)$, it follows that

$$(9.4)\qquad (H^s(\overline{\mathbf{R}}^n_+))' \overset{\mathrm{top}}{\cong} \overset{\circ}{H}{}^{-s}(\overline{\mathbf{R}}^n_+)$$

under the pairing induced by the scalar product of $L^2(\mathbf{R}^n)$. It is well known that the operator

$$\Lambda_+ = \mathrm{op}\,(\langle\xi'\rangle + i\xi_n)$$

restricts to a topological isomorphism $\overset{\circ}{H}{}^s(\overline{\mathbf{R}}^n_+) \to \overset{\circ}{H}{}^{s-1}(\overline{\mathbf{R}}^n_+)$ for any $s \in \mathbf{R}$ (see [**14**]). The formal adjoint to this mapping is given by $r_+\Lambda_-e_+$, where

$$\Lambda_- = \mathrm{op}\,(\langle\xi'\rangle - i\xi_n)$$

is preceded by extension e_+ by zero to $\mathbf{R}^n$ and followed by restriction r_+ to $\mathbf{R}^n_+$. From (9.4) we conclude that $r_+\Lambda_-e_+$ induces a topological isomorphism $H^s(\overline{\mathbf{R}}^n_+) \to H^{s-1}(\overline{\mathbf{R}}^n_+)$ for any $s > -1/2$. Put

$$H = \overset{\circ}{H}{}^s(\mathbf{R}^q \times \overline{\mathbf{R}}^n_+), \qquad \tilde{H} = L^2(\mathbf{R}^q \times \overline{\mathbf{R}}^n_+)$$

and

$$\lambda(t,\tau) = ((1 + (\phi(t))^2\Delta_y + \tau^2)^{1/2} + \Lambda_+)^s,$$

where $\phi \in C^\infty_{\mathrm{loc}}(\mathbf{R})$ and Δ_y are as above whereas $s \in \mathbf{Z}$. When passing to the Fourier images with respect to y and x, we reduce $\lambda(t,\tau)$ to multiplication by the scalar-valued weight function

$$((1 + (\phi(t))^2|\eta|^2 + \tau^2)^{1/2} + \langle\xi'\rangle + i\xi_n)^s,$$

and so estimate (1.1) for $\lambda(t,\tau)$ is verified in much the same way as in Example 9.1. Thus, $\lambda \in \Lambda(\overset{\circ}{H}{}^s(\mathbf{R}^q \times \overline{\mathbf{R}}^n_+), L^2(\mathbf{R}^q \times \overline{\mathbf{R}}^n_+))$. On the other hand, if

$$H = H^s(\mathbf{R}^q \times \overline{\mathbf{R}}^n_+), \qquad \tilde{H} = L^2(\mathbf{R}^q \times \overline{\mathbf{R}}^n_+)$$

and

$$\lambda(t,\tau) = ((1 + (\phi(t))^2\Delta_y + \tau^2)^{1/2} + r_+\Lambda_-e_+)^s,$$

where $s \in \mathbf{Z}_+$, then we make use of what has already been proved and a familiar duality argument to see that $\lambda \in \Lambda(H^s(\mathbf{R}^q \times \overline{\mathbf{R}}^n_+), L^2(\mathbf{R}^q \times \overline{\mathbf{R}}^n_+))$. This choice of the weight function is certainly more relevant to our theory than the preceding one.

Note that the "order-reducing" operators $\operatorname{op}(\langle \xi' \rangle \pm i\xi_n)^s$ for $s \in \mathbf{R}$ in the half-space have been used by Vishik and Eskin [**44**]. Their symbols are not in $\mathcal{S}^s_{1,0}(T^*\mathbf{R}^n)$ since $\langle \xi' \rangle$ does not satisfy all the estimates in terms of powers of $\langle \xi \rangle$ required for that. However, these operators are convenient for special purposes and sometimes allow simpler formulations. In the Boutet de Monvel calculus they are usually replaced by other operators with almost as convenient properties (see Grubb [**15**], Schrohe and Schulze [**35**], and so on).

Example 9.5. Finally suppose that B is a C^∞ compact manifold with boundary. Denote by $2B$ the "double" of B, i.e., a C^∞ compact closed manifold obtained by gluing together two copies of B along ∂B. For $s \in \mathbf{R}$, we define $H^s(B)$ to be the space formed by the restrictions of distributions in $H^s(2B)$ to the interior of B, with the standard quotient norm. There is an order-reducing operator

$$\Lambda_B : H^s(B) \longrightarrow H^{s-1}(B), \quad s > -1/2,$$

possessing the following properties:

- in local coordinates near the boundary, Λ_B is given by $r_+\Lambda_- e_+$ with ξ' replaced by (ξ', μ), $\mu \in \mathbf{R}$ being a sufficiently large parameter;
- the operator $\langle \tau \rangle + \Lambda_B$ is invertible for all $\tau \in \mathbf{R}$, the inverse being given by $(\langle \tau \rangle + r_+\Lambda_- e_+)^{-1}$ in local coordinates near ∂B; and
- the norm of $(\langle \tau + \upsilon \rangle + \Lambda_B)(\langle \tau \rangle + \Lambda_B)^{-1}$ in $\mathcal{L}(L^2(B))$ is dominated by $c \max(1, \langle \tau + \upsilon \rangle / \langle \tau \rangle)$, with c a constant independent of $\tau, \upsilon \in \mathbf{R}$,

cf. [**15**, Section 5]. We now set

$$H = H^s(\mathbf{R}^q \times B), \qquad \tilde{H} = L^2(\mathbf{R}^q \times B)$$

and

$$\lambda(t, \tau) = ((1 + (\phi(t))^2 \Delta_y + \tau^2)^{1/2} + \Lambda_B)^s,$$

for $s \in \mathbf{Z}_+$. We claim that $\lambda \in \Lambda(H^s(\mathbf{R}^q \times B), L^2(\mathbf{R}^q \times B))$. To prove this fix $u \in L^2(\mathbf{R}^q \times B)$. We have

$$\begin{aligned}
&\|\lambda(t+\theta,\tau+\upsilon)\lambda^{-1}(t,\tau)u\|^2_{\tilde{H}} \\
&\qquad = \int_{\mathbf{R}^q} \|F_{y\mapsto\eta}\lambda(t+\theta,\tau+\upsilon)\lambda^{-1}(t,\tau)u\|^2_{L^2(B)}\,d\eta \\
&\qquad \le \int_{\mathbf{R}^q} \|b(t+\theta,\eta,\tau+\upsilon)b^{-1}(t,\eta,\tau)\|^{2s}_{\mathcal{L}(L^2(B))}\|\mathcal{F}_{y\mapsto\eta}u\|^2_{L^2(B)}\,d\eta \\
&\qquad \le (\sup_{\eta\in\mathbf{R}^q} \|b(t+\theta,\eta,\tau+\upsilon)b^{-1}(t,\eta,\tau)\|^{2s}_{\mathcal{L}(L^2(B))})\|u\|^2_{\tilde{H}}
\end{aligned}$$

for all $t,\tau,\theta,\upsilon \in \mathbf{R}$, where

$$b(t,\eta,\tau) = (1+(\phi(t))^2|\eta|^2+\tau^2)^{1/2} + \Lambda_B.$$

This yields

$$\begin{aligned}
&\|\lambda(t+\theta,\tau+\upsilon)\lambda^{-1}(t,\tau)\|_{\mathcal{L}(\tilde{H})} \\
&\qquad \le \sup_{\eta\in\mathbf{R}^q} \|b(t+\theta,\eta,\tau+\upsilon)b^{-1}(t,\eta,\tau)\|^s_{\mathcal{L}(L^2(B))},
\end{aligned} \tag{9.5}$$

and so we are reduced to estimating the norm of $b(t+\theta,\eta,\tau+\upsilon)b^{-1}(t,\eta,\tau)$ in $\mathcal{L}(L^2(B))$. On the other hand, we get

$$\begin{aligned}
&\|b(t+\theta,\eta,\tau)b^{-1}(t,\eta,\tau)\|_{\mathcal{L}(L^2(B))} \\
&\qquad \le c\max\left(1, \frac{(1+(\phi(t+\theta))^2|\eta|^2+\tau^2)^{1/2}}{(1+(\phi(t))^2|\eta|^2+\tau^2)^{1/2}}\right) \\
&\qquad \le c\max\left(1, \frac{\phi(t+\theta)}{\phi(t)}\right)
\end{aligned}$$

hence, by (9.1),

$$\|b(t+\theta,\eta,\tau)b^{-1}(t,\eta,\tau)\|_{\mathcal{L}(L^2(B))} \le C\langle\theta\rangle^\varepsilon \tag{9.6}$$

where the constant C does not depend on t,τ,θ and η. On the other hand, we have

$$\begin{aligned}
&\|b(t,\eta,\tau+\upsilon)b^{-1}(t,\eta,\tau)\|_{\mathcal{L}(L^2(B))} \\
&\qquad \le c\max\left(1, \frac{(1+(\phi(t))^2|\eta|^2+(\tau+\upsilon)^2)^{1/2}}{(1+(\phi(t))^2|\eta|^2+\tau^2)^{1/2}}\right) \\
&\qquad \le \sqrt{2}c\langle\upsilon\rangle,
\end{aligned} \tag{9.7}$$

the last inequality being a consequence of Lemma 9.2. Combining (9.5), (9.6) and (9.7), we arrive at the desired estimate for $\lambda(t,\tau)$.

Part II. A Class of Weighted Pseudodifferential Operators with Operator-Valued Symbols.

10. Preliminaries. Let $t = \delta(r)$ be a diffeomorphism of $\mathbf{R}_+$ onto $\mathbf{R}$ such that $\delta'(r) < 0$ for all $r \in \mathbf{R}_+$.

Using this diffeomorphism we pull back the structure of an Abelian group from $\mathbf{R}$ to $\mathbf{R}_+$. More precisely, we introduce a group operation on the half-axis by

$$r \circ \theta = \delta^{-1}(\delta(r) + \delta(\theta)),$$

for $r, s \in \mathbf{R}_+$. It is easily seen that under this operation $\mathbf{R}_+$ is a locally compact Abelian group with an invariant measure $dm = |\delta'(r)|\,dr$.

Example 10.1. Set $\delta(r) = -\log r$ for $r \in \mathbf{R}_+$. Then $r \circ \theta = r\theta$ and so $\mathbf{R}_+$ with this operation is a multiplicative group whose invariant measure is $dm = dr/r$.

Example 10.2. For $p > 0$, take

$$\delta(r) = \begin{cases} 1/pr^p & r \in (0,1]; \\ -r & r \in [2,+\infty). \end{cases}$$

Then we can extend δ to the interval (1,2) in such a way that the extended mapping is a diffeomorphism of $\mathbf{R}_+$ onto $\mathbf{R}$.

Example 10.3. Set

$$\delta(r) = \begin{cases} \exp(1/r) & r \in (0,1]; \\ -r & r \in [2,+\infty) \end{cases}$$

and extend this to the entire half-axis to get a diffeomorphism of $\mathbf{R}_+$ onto $\mathbf{R}$.

11. Weighted pseudodifferential operators. Denote by $\Lambda_w(H,\tilde{H})$ the space formed by all functions $\lambda(r,\varrho)$ on $\mathbf{R}_+ \times \mathbf{R}$ with values in

$\mathcal{L}(H, \tilde{H})$ such that

$$\|\lambda(r \circ \theta, \varrho + \upsilon)\lambda^{-1}(r, \varrho)\|_{\mathcal{L}(\tilde{H})} \le c\langle\delta(\theta)\rangle^{\varepsilon_1}\langle\upsilon\rangle^{\varepsilon_2} \tag{11.1}$$

for any $r, \theta \in \mathbf{R}_+$ and $\varrho, \upsilon \in \mathbf{R}$, the constants $\varepsilon_1, \varepsilon_2, c \in \mathbf{R}$ being independent of r, ϱ, θ and υ.

The pull-back of the derivative $D_t = -i\partial/\partial t$ under the diffeomorphism $r = \delta(t)$ is

$$\mathbf{D}_r = \frac{1}{i}\frac{1}{\delta'(r)}\frac{\partial}{\partial r} \tag{11.2}$$

which degenerates at $r = 0$ because $\delta'(0) = -\infty$. As described in [**41**] and [**33**], this characteristic derivative is of great importance in the analysis on manifolds with singular points.

Suppose

$$\lambda_1(r, \varrho) \in \Lambda_w(H_1, \tilde{H}_1), \qquad \lambda_2(r, \varrho) \in \Lambda_w(H_2, \tilde{H}_2).$$

Definition 11.1. Let $\mathcal{S}_w(\lambda_1, \lambda_2)$ stand for the class of C^∞ functions $a(r, \varrho)$ on $\mathbf{R}_+ \times \mathbf{R}$ with values in $\mathcal{L}(H_1, H_2)$ such that, for any $\alpha, \beta \in \mathbf{Z}_+$, there is a constant $c_{\alpha,\beta}(a)$ with the property that

$$\|\lambda_2(r, \varrho)(\mathbf{D}_r^\beta D_\varrho^\alpha a(r, \varrho))\lambda_1^{-1}(r, \varrho)\|_{\mathcal{L}(\tilde{H}_1, \tilde{H}_2)} \le c_{\alpha,\beta}(a)$$
$$\text{for all } (r, \varrho) \in \mathbf{R}_+ \times \mathbf{R}.$$

To any symbol $a \in \mathcal{S}_w(\lambda_1, \lambda_2)$ we assign a "weighted" pseudodifferential operator $A = \mathrm{op}_w(a)$ by

$$Au(r) = \frac{1}{2\pi}\int_{\mathbf{R}} d\varrho \int_{\mathbf{R}_+} e^{i(\delta(r)-\delta(r'))\varrho} a(r, \varrho)u(r')\, dm(r'), \quad r \in \mathbf{R}_+, \tag{11.3}$$

for $u \in C^\infty_{\mathrm{comp}}(\mathbf{R}_+, H_1)$.

Note that $\mathrm{op}_w(a)$ is a Fourier integral operator on the half-axis with phase function $\varphi(r, r', \varrho) = (\delta(r) - \delta(r'))\varrho$.

Set

$$\delta_* u(t) = u(\delta^{-1}(t)), \quad t \in \mathbf{R};$$
$$\delta^* f(r) = f(\delta(r)), \quad r \in \mathbf{R}_+,$$

then

$$\delta_* : C^\infty_{\rm comp}(\mathbf{R}_+, H) \longrightarrow C^\infty_{\rm comp}(\mathbf{R}, H),$$
$$\delta^* : C^\infty_{\rm comp}(\mathbf{R}, H) \longrightarrow C^\infty_{\rm comp}(\mathbf{R}_+, H)$$

are the 'push-forward' and 'pull-back' operators induced by δ, respectively. If $a(t,\tau) \in \mathcal{S}(\lambda_1, \lambda_2)$, then a straightforward computation yields

$$\delta^\sharp {\rm op}\,(a) = \delta^* {\rm op}\,(a) \delta_* = {\rm op}_w(\delta^* a),$$

where $\delta^* a(r, \varrho) = a(\delta(r), \varrho)$. The operator $\delta^\sharp {\rm op}\,(a)$ is called the operator pull-back of ${\rm op}\,(a)$ under δ. In fact,

$$\delta^* : \Lambda(H, \tilde{H}) \longrightarrow \Lambda_w(H, \tilde{H}),$$
$$\delta^* : \mathcal{S}(\lambda_1, \lambda_2) \longrightarrow \mathcal{S}_w(\delta^* \lambda_1, \delta^* \lambda_2)$$

are easily verified to be isomorphisms, hence the calculus on $\mathbf{R}$ is pulled back to $\mathbf{R}_+$ under $t = \delta(r)$.

From what has been proved it follows that the weighted pseudodifferential operators ${\rm op}_w(a)$ behave in much the same way as the usual ones ${\rm op}(a)$. Thus, their properties can be deduced from those of usual pseudodifferential operators (cf. Chapter I). In [**33**], we gave an exposition of the theory for weight functions $\lambda(t,\tau)$ independent of t. The class of weighted pseudodifferential operators thereof is adapted for studying boundary value problems in domains with isolated singular points on the boundary. The class of pseudodifferential operators under consideration here is well adapted for treating boundary value problems in domains with cuspidal wedges.

Recall once again that condition (11.1) for a weight function λ just amounts to saying that $\delta_* \lambda$ satisfies estimate (1.1) where $\delta_* \lambda(t,\tau) = \lambda(\delta^{-1}(t), \tau)$.

12. Function spaces related to weighted pseudodifferential operators. We define $\Lambda'_w(H, \tilde{H})$ to consist of all weight functions

$\lambda(r, \varrho) \in \Lambda_w(H, \tilde{H})$ which are of class C^∞ on $\mathbf{R}_+ \times \mathbf{R}$ and satisfy

$$\begin{aligned} &\|(\mathbf{D}_r^\beta D_\varrho^\alpha \lambda(r, \varrho))\lambda^{-1}(r, \varrho)\|_{\mathcal{L}(\tilde{H})} \leq c_{\alpha,\beta}, \\ &\|\lambda^{-1}(r, \varrho)(\mathbf{D}_r^\beta D_\varrho^\alpha \lambda(r, \varrho))\|_{\mathcal{L}(H)} \leq c_{\alpha,\beta} \end{aligned} \tag{12.1}$$

for all $\alpha, \beta \in \mathbf{Z}_+$ where $c_{\alpha,\beta}$ do not depend on $(r, \varrho) \in \mathbf{R}_+ \times \mathbf{R}$.

Proposition 12.1. *For any $\lambda \in \Lambda'_w(H, \tilde{H})$ there are operators*

$$\begin{aligned} R_\varepsilon(\lambda) &\in \mathcal{OPS}_w(\lambda, 1_{\tilde{H}}), \\ R_\varepsilon^{-1}(\lambda) &\in \mathcal{OPS}_w(\lambda^{-1}, 1_H) \end{aligned}$$

such that

$$\begin{aligned} R_\varepsilon^{-1}(\lambda) R_\varepsilon(\lambda) &= 1_{L^2(\mathbf{R}_+, dm, H)}, \\ R_\varepsilon(\lambda) R_\varepsilon^{-1}(\lambda) &= 1_{L^2(\mathbf{R}_+, dm, \tilde{H})}. \end{aligned}$$

Proof. This follows from Corollary 5.2. □

Here by $L^2(\mathbf{R}_+, dm, H)$ we mean the space formed by all measurable functions u on $\mathbf{R}_+$ with values in H such that $\int_{\mathbf{R}_+} \|u\|_H^2 \, dm < \infty$. The square root of this integral provides a norm in $L^2(\mathbf{R}_+, dm, H)$.

Definition 12.2. Suppose $\lambda \in \Lambda'_w(H, \tilde{H})$. By $H_w(\lambda)$ is meant the completion of $C^\infty_{\mathrm{comp}}(\mathbf{R}_+, H)$ with respect to the norm

$$\|u\|_{H_w(\lambda)} = \|R_\varepsilon(\lambda) u\|_{\mathcal{L}^2(\mathbf{R}_+, dm, \tilde{H})}.$$

For analysis on manifolds with edges we need also two-parameter spaces $H_w(\lambda; \gamma, \mu)$ where γ and μ vary over $\mathbf{R}$. They consist of all distributions u on $\mathbf{R}_+$ with values in H such that $e^{\gamma\delta(r)}(\delta'(r))^\mu u \in H_w(\lambda)$. We equip $H_w(\lambda; \gamma, \mu)$ with the norm

$$\|u\|_{H_w(\lambda; \gamma, \mu)} = \|e^{\gamma\delta(r)}(\delta'(r))^\mu u\|_{H_w(\lambda)}. \tag{12.2}$$

If $\mu = 0$, we omit this index in the notation, i.e., we write $H_w(\lambda;\gamma,0)$ simply $H_w(\lambda;\gamma)$ when no confusion can arise.

Proposition 12.3. *Let $\lambda_1 \in \Lambda'_w(H_1,\tilde{H}_1)$ and $\lambda_2 \in \Lambda'_w(H_2,\tilde{H}_2)$. Suppose that $a(r,\varrho+i\gamma) \in \mathcal{S}_w(\lambda_1,\lambda_2)$ for some $\gamma \in \mathbf{R}$. Then $A = \mathrm{op}_{w,\gamma}(a(r,\zeta))$ extends to a continuous mapping $H_w(\lambda_1;\gamma) \to H_2(\lambda_2;\gamma)$ and*

$$\|Au\|_{H_w(\lambda_2;\gamma)} \leq c\bigg(\sum_{\alpha+\beta\leq N} c_{\alpha,\beta}(a(r,\varrho+i\gamma))\bigg)\|u\|_{H_w(\lambda_1;\gamma)}$$

for any $u \in H_w(\lambda_1;\gamma)$ where $c > 0$ and $N \in \mathbf{Z}_+$ do not depend on A.

Proof. This assertion is an immediate consequence of Proposition 5.4 if we apply the operator pull-back $\delta^\sharp$. □

Note that the operator $A = \mathrm{op}_{w,\gamma}(a(r,\zeta))$ is defined by

$$Au(r) = \frac{1}{2\pi}\int_{\mathbf{R}+i\gamma} d\zeta \int_{\mathbf{R}_+} e^{i(\delta(r)-\delta(r'))\zeta} a(r,\zeta)u(r')dm(r'), \quad r \in \mathbf{R}_+,$$

for $u \in C^\infty_{\mathrm{comp}}(\mathbf{R}_+,H_1)$.

Proposition 12.4. *Let $\lambda_1 \in \Lambda'_w(H_1,\tilde{H}_1)$ and $\lambda_2 \in \Lambda'_w(H_2,\tilde{H}_2)$. Suppose that $a(r,\varrho+i\gamma) \in \mathcal{S}_w(\lambda_1,\lambda_2)$ for some $\gamma \in \mathbf{R}$. If*

$$\begin{aligned} \delta'(r\circ\theta)/\delta'(r) &\leq c\langle\delta(\theta)\rangle^\varepsilon, \\ |(\mathbf{D}^\beta\delta'(r))/\delta'(r)| &\leq c_\beta, \quad \beta \in \mathbf{Z}_+, \end{aligned} \tag{12.3}$$

with ε, c and c_β independent of $r,\theta \in \mathbf{R}_+$, then $(\delta'(r))^m\mathrm{op}_{w,\gamma}(a(r,\zeta))$ extends to a continuous mapping $H_w(\lambda_1;\gamma,\mu) \to H_w(\lambda_2;\gamma,\mu-m)$ for each $\mu \in \mathbf{R}$.

Proof. The first condition in (12.3) implies, given any weight function $\lambda \in \Lambda_w(H,\tilde{H})$ that $(\delta'(r))^\mu\lambda(r,\varrho) \in \Lambda_w(H,\tilde{H})$ for each $\mu \in \mathbf{R}$. Indeed, letting $\Lambda(r,\varrho) = (\delta'(r))^{\mu}\lambda(r,\varrho)$, we get

$$\begin{aligned}
&\|\Lambda(r\circ\theta,\varrho+\upsilon)\Lambda^{-1}(r,\varrho)\|_{\mathcal{L}(\tilde{H})} \\
&\qquad = \left(\frac{\delta'(r\circ\theta)}{\delta'(r)}\right)^{\mu}\|\lambda(r\circ\theta,\varrho+\upsilon)\lambda^{-1}(r,\varrho)\|_{\mathcal{L}(\tilde{H})} \\
&\qquad \le c\langle\delta(\theta)\rangle^{\varepsilon_1}\langle\upsilon\rangle^{\varepsilon_2}\left(\frac{\delta'(r\circ\theta)}{\delta'(r)}\right)^{\mu}
\end{aligned}$$

with c a constant independent of $r,\theta\in\mathbf{R}_+$ and $\varrho,\upsilon\in\mathbf{R}$. Replacing r by $r\circ\theta^{-1}$ in the first estimate (12.3), where θ^{-1} is determined from the equality $\delta(\theta)+\delta(\theta^{-1})=0$, we see that

$$\delta'(r)/\delta'(r\circ\theta^{-1}) \le c\langle\delta(\theta^{-1})\rangle^{\varepsilon}$$

for all $r,\theta^{-1}\in\mathbf{R}_+$. Combining this with the first estimate of (12.3), we deduce easily that

$$\left(\frac{\delta'(r\circ\theta)}{\delta'(r)}\right)^{\mu} \le c^{|\mu|}\langle\delta(\theta)\rangle^{\varepsilon|\mu|}$$

for any $r,\theta\in\mathbf{R}_+$ showing $\Lambda(r,\varrho)\in\Lambda_w(H,\tilde{H})$, which is our claim.

On the other hand, the second condition of (12.3) means that $(\delta'(r))^m 1_H$ lies in $\Lambda'_w(H,H)$ for each $m\in\mathbf{R}$. We will prove more, namely, that $(\delta'(r))^m 1_H$ belongs to $\mathcal{S}_w((\delta'(r))^{\mu}\lambda,(\delta'(r))^{\mu-m}\lambda)$ for any weight function $\lambda\in\Lambda_w(H,\tilde{H})$ and $\mu\in\mathbf{R}$. To this end, consider

$$\begin{aligned}
&\|(\delta'(r))^{\mu-m}\lambda(r,\varrho))(\mathbf{D}^{\beta}(\delta'(r))^m 1_H)(\delta'(r))^{\mu}\lambda(r,\varrho)^{-1}\|_{\mathcal{L}(\tilde{H})} \\
&\qquad\qquad = |(\mathbf{D}^{\beta}(\delta'(r))^m)(\delta'(r))^{-m}|,
\end{aligned}$$

for $\beta\in\mathbf{Z}_+$. An easy computation shows that

$$\begin{aligned}
&\mathbf{D}^{\beta}(\delta'(r))^m \\
&\quad = \sum_{i_1+\dots+i_\beta\le\beta} c_{i_1\cdots i_\beta}(\delta'(r))^{m-i_1-\dots-i_\beta}(\mathbf{D}\delta'(r))^{i_1}\cdots(\mathbf{D}^{\beta}\delta'(r))^{i_\beta}
\end{aligned}$$

where the coefficients $c_{i_1\cdots i_\beta}$ depend only on m. We now invoke the second condition of (12.3) to obtain

$$\begin{aligned}
|(\mathbf{D}^{\beta}(\delta'(r))^m)(\delta'(r))^{-m}| &\le \sum_{i_1+\dots+i_\beta\le\beta}|c_{i_1\cdots i_\beta}|c_1^{i_1}\cdots c_\beta^{i_\beta} \\
&= \mathrm{const}\,(\beta)
\end{aligned}$$

for all $r \in \mathbf{R}$. Hence the desired symbol estimates for $(\delta'(r))^m 1_H$ follow.

We are now able to prove the boundedness of $A = (\delta'(r))^m \mathrm{op}_{w,\gamma}(a(r,\zeta))$. For this purpose fix $u \in H_w(\lambda_1;\gamma,\mu)$. We have

$$\begin{aligned}\|Au\|_{H_w(\lambda_1;\gamma,\mu-m)} &= \|(\delta'(r))^{\mu-m}Au\|_{H_w(\lambda_1;\gamma)}\\ &\le c\|(\delta'(r))^{\mu}\|_{H_w(\lambda_2;\gamma)}\\ &= c\|u\|_{H_w(\lambda_2;\gamma,\mu)},\end{aligned}$$

where c stands for the norm of

$$(\delta'(r))^{\mu}\mathrm{op}_{w,\gamma}(a(r,\zeta))(\delta'(r))^{-\mu} \tag{12.4}$$

in $\mathcal{L}(H_w(\lambda_1;\gamma), H_w(\lambda_2;\gamma))$. Thus we shall have established the proposition if we prove that c is finite. However, from what has already been proved it follows that

$$\begin{aligned}(\delta'(r))^{-\mu}1_{H_1} &\in \mathcal{S}_w(\lambda_1, (\delta'(r))^{\mu}\lambda_1),\\ a(r,\zeta+i\gamma) &\in \mathcal{S}_w((\delta'(r))^{\mu}\lambda_1, (\delta'(r))^{\mu}\lambda_2),\\ (\delta'(r))^{\mu}1_{H_2} &\in \mathcal{S}_w((\delta'(r))^{\mu}\lambda_2, \lambda_2),\end{aligned}$$

and so $(\delta'(r))^{\mu}a(r,\zeta+i\gamma)(\delta'(r))^{-\mu} \in \mathcal{S}_w(\lambda_1,\lambda_2)$. By Proposition 12.3 the operator (12.4) extends to a continuous mapping $H_w(\lambda_1;\gamma) \to H_w(\lambda_2;\gamma)$. Hence c is finite, as required. □

13. Local invertibility at the origin. A weight function $\lambda(r,\varrho) \in \Lambda'_w(H,\tilde{H})$ is said to be *slowly varying* at the point $r=0$ if

$$\lim_{r\to 0}\sup_{\varrho\in\mathbf{R}} \|(\mathbf{D}_r^{\beta}D_{\varrho}^{\alpha}\lambda(r,\varrho))\lambda^{-1}(r,\varrho)\|_{\mathcal{L}(\tilde{H})} = 0$$

for all $\alpha \in \mathbf{Z}_+$ and $\beta = 1,2,\dots$.

The concept of being slowly varying at $r=0$ is applicable as well to the scalar-valued function $\delta'(r)$ if we think of $\delta'(r)$ as multiplication operator in a Hilbert space. This means simply that

$$\lim_{r\to 0}(\mathbf{D}^{\beta}\delta'(r))/\delta'(r) = 0, \tag{13.1}$$

for every $\beta = 1,2,\dots$.

In this section we will be concerned with the problem of local invertibility at the singular point for an operator

$$A = (\delta'(r))^m \mathrm{op}_w(a(r,\zeta)),$$

where $a(r,\varrho+i\gamma) \in \mathcal{S}_w(\lambda_1,\lambda_2)$. We restrict our attention to those diffeomorphisms $t=\delta(r)$ which fulfill (12.3). Then Proposition 12.4 enables us to conclude that A maps $H_w(\lambda_1;\gamma,\mu)$ to $H_w(\lambda_1;\gamma,\mu-m)$ for any $\mu \in \mathbf{R}$. Moreover, we assume that the symbol $a(r,\varrho+i\gamma)$ is slowly varying at the point $r=0$, i.e.,

$$\lim_{r\to 0} \sup_{\varrho\in\mathbf{R}} \|\lambda_2(r,\varrho)(\mathbf{D}_r^\beta D_\varrho^\alpha a(r,\varrho+i\gamma))\lambda_1^{-1}(r,\varrho)\|_{\mathcal{L}(\tilde{H}_1,\tilde{H}_2)} = 0$$

for any $\alpha \in \mathbf{Z}_+$ and $\beta = 1,2,\dots$.

The definition of local invertibility of

$$A : H_w(\lambda_1;\gamma,\mu) \to H_w(\lambda_2;\gamma,\mu-m)$$

at $r=0$ is an evident change of Definition 7.1 with the cut-off function $\chi_R(t)$ at $t=+\infty$ replaced by the cut-off function $\delta^*\chi_R(r)$ at $r=0$.

Theorem 13.1. *Suppose both $\lambda_j \in \Lambda'_w(H_j,\tilde{H}_j)$, $j=1,2$ and δ' are slowly varying at $r=0$. Let $A=(\delta'(r))^m\mathrm{op}_w(a(r,\zeta))$ with $a(r,\varrho+i\gamma) \in \mathcal{S}_{w,sv}(\lambda_1,\lambda_2)$. Then $A : H_w(\lambda_1;\gamma,\mu) \to H_w(\lambda_2;\gamma,\mu-m)$ is locally invertible at $r=0$ if and only if there exists $\varepsilon>0$ such that the symbol $a(r,\varrho+i\gamma) : H_1 \to H_2$ is invertible for all $(r,\varrho) \in (0,\varepsilon)\times\mathbf{R}$ and*

$$\sup_{(0,\varepsilon)\times\mathbf{R}} \|\lambda_1(r,\varrho)a^{-1}(r,\varrho+i\gamma)\lambda_2^{-1}(r,\varrho)\|_{\mathcal{L}(\tilde{H}_2,\tilde{H}_1)} < \infty. \tag{13.2}$$

Proof. The proof of this theorem is similar to the proof of Theorem 1.7.4 in [**33**]. Proposition 12.4 yields all the additional information we need. □

The important point to note here is the form of the invertibility condition (13.2) which is independent of $\mu \in \mathbf{R}$. This is explained by the fact that under condition (13.1) the weight function $\exp\delta(r)$ dominates the weight function $\delta'(r)$ near $r = 0$. Thus, the case $\delta(r) = -\log r$ corresponding to conical singularities is automatically excluded from consideration.

We finish this chapter by a weighted estimate for pseudodifferential operators with analytic symbols.

Theorem 13.2. *Let $a(r,\zeta)$ extend to an analytic function of ζ in the strip $\mathbf{R} + i(a,b)$ such that $a(r,\varrho + i\gamma) \in \mathcal{S}_{w,sv}(\lambda_1,\lambda_2)$ uniformly in $\gamma \in (a,b)$. Suppose there is an $\varepsilon > 0$ such that $a(r,\zeta) : H_1 \to H_2$ is invertible for all $r \in (0,\varepsilon)$ and $\zeta \in \mathbf{R} + i(a,b)$, and*

$$\sup_{\substack{r\in(0,\varepsilon)\\ \zeta\in\mathbf{R}+i(a,b)}} \|\lambda_1(r,\varrho)a^{-1}(r,\zeta)\lambda_2^{-1}(r,\varrho)\|_{\mathcal{L}(\tilde{H}_2,\tilde{H}_1)} < \infty.$$

Then, if $a < \gamma' \le \gamma'' < b$ and $u \in H_w(\lambda_1;\gamma',\mu)$ satisfies $(\delta'(r))^m \mathrm{op}_w(a)\, u = f$ with $f \in H_w(\lambda_2;\gamma'',\mu - m)$ near $r = 0$, then $u \in H_w(\lambda_1;\gamma'',\mu)$ near $r = 0$.

Proof. This theorem is a reformulation of Theorem 8.1 in terms of weighted pseudodifferential operators on the half-axis. □

Part III. Differential Operators on Manifolds with Cuspidal Edges.

14. Canonical cuspidal wedge. We say that (r,ω) is a *polar system of coordinates* in $\mathbf{R}^{n+1}$ with center at the origin if each point $x \in \mathbf{R}^{n+1} \setminus \{0\}$ can be written in the form

$$x = rS(\omega), \quad (r,\omega) \in \mathbf{R}_+ \times \mathbf{R}^n,$$

where S is a smooth periodic mapping of $\mathbf{R}^n$ to the unit sphere $\mathbf{S}^n$ in $\mathbf{R}^{n+1}$.

Note that the periodicity of S is irrelevant as far as the oscillations of a singular surface are concerned. They come from other elements of the construction.

A well-known example of polar coordinates in the space $\mathbf{R}^{n+1}$ is given by the mapping

$$S(w) = \begin{cases} \cos\omega_1, \\ \sin\omega_1 \cos\omega_2, \\ \sin\omega_1 \sin\omega_2 \cos\omega_3, \\ \dots\dots\dots\dots\dots\dots \\ \sin\omega_1 \sin\omega_2 \sin\omega_3 \cdots \sin\omega_{n-1} \cos\omega_n, \\ \sin\omega_1 \sin\omega_2 \sin\omega_3 \cdots \sin\omega_{n-1} \sin\omega_n, \end{cases} \tag{14.1}$$

where $\omega = (\omega_1, \dots, \omega_n)$. This mapping fails to be one-to-one on the planes $\{\omega \in \mathbf{R}^n : \omega_j = \pi k\}$ for $j = 1, \dots, n-1$ and $k \in \mathbf{Z}$. This in turn results in degeneracy of the Jacobian matrix,

$$\det \frac{\partial x}{\partial(r,\omega)} = r^n \sin^{n-1}\omega_1 \sin^{n-2}\omega_1 \cdots \sin\omega_{n-1}.$$

To cope with this difficulty one often uses the so-called *stereographic projection* of $\mathbf{R}^n$ onto $\mathbf{S}^n$ with the north pole removed, given by

$$\mathrm{SP}\,(\omega) = \frac{(2\omega, |\omega|^2 - 1)}{|\omega|^2 + 1}, \quad \omega \in \mathbf{R}^n,$$

in which case

$$\det \frac{\partial x}{\partial(r,\omega)} = (-1)^{n+1} \frac{(2r)^n}{(|\omega|^2+1)^n}$$

does not vanish but for $r = 0$.

Let f be a C^∞ function on $\mathbf{R}_+$, with the following properties:

1) $f(r) < 0$ for all $r \in \mathbf{R}_+$;

2) $\int_0^\varepsilon dr/rf(r) = -\infty$; and

3) $|r^j f^{(j)}(r)| \le c_j$ near $r = 0$ for each $j \in \mathbf{Z}_+$.

Modifying f away from a finite interval if necessary, we may actually assume that $\int_R^\infty dr/rf(r) = \infty$. Thus, setting

$$\delta(r) = \int_\varepsilon^r \frac{d\theta}{\theta f(\theta)}, \quad \text{for } r \in \mathbf{R}_+, \tag{14.2}$$

we get a monotone decreasing function $\delta \in C^\infty_{\mathrm{loc}}(\mathbf{R}_+)$ such that $\delta'(r) = 1/rf(r)$ and

$$\lim_{r\to 0} \delta(r) = +\infty, \qquad \lim_{r\to\infty} \delta(r) = -\infty.$$

In the analysis on manifolds with singularities, only the germ of $f(r)$ at $r = 0$ is prescribed by the geometry of singularities. Hence we will restrict our attention to the behavior of $f(r)$ near $r = 0$, keeping in mind the construction above.

Example 14.1. Let

$$f(r) = \begin{cases} -r^p & r \in (0,1]; \\ -1/r & r \in [2,+\infty), \end{cases}$$

where $p \geq 0$. When appropriately extended to the interval (1,2), this function meets all the conditions above. In this case $\delta(r)$ differs by a constant from the diffeomorphism of Example 10.2.

Example 14.2. Consider

$$f(r) = \begin{cases} -(\log 1/r)^p & r \in (0,1/2]; \\ -1/r & r \in [2,+\infty), \end{cases}$$

where $p \leq 1$. If appropriately extended to the interval $(1/2, 2)$, the function $f(r)$ fulfills all the above conditions. Indeed, we have

$$\int_0^\varepsilon \frac{dr}{rf(r)} = -\int_{\log 2}^\infty t^{-p}\, dt + \int_{1/2}^\varepsilon \frac{dr}{rf(r)} = \infty$$

and

$$\lim_{r\to 0} r^j f^{(j)}(r) = (-1)^{j-1}(j-1)!p \lim_{r\to 0} (\log 1/r)^{p-1} < \infty$$

for all $j \in \mathbf{Z}_+$.

Note that, for $p > 0$, the function $f(r)$ of Example 14.2 tends to $-\infty$ as $r \to 0$. This corresponds to the case where the canonical surface C_{x^0} given by (14.3) is of finite smoothness at x^0. Such "singularities" require another pseudodifferential calculus on the half-line different from that of Chapter II. In fact, the function $f(r)$ of Example 14.2 fails to satisfy the first estimate of (12.3) for all $p \le 1$, hence Theorem 13.1 is not applicable.

Example 14.3. Set

$$f(r) = \begin{cases} -r\exp(-1/r) & r \in (0,1]; \\ -1/r & r \in [2,+\infty), \end{cases}$$

and extend $f(r)$ to the interval (1,2) so that the extension is negative and smooth. Then $f(r)$ bears all the above properties. Moreover, $\delta(r)$ differs by a constant from the diffeomorphism of Example 10.3.

Example 14.4. For $f(r) \equiv -1$, we have $\delta(r) = -\log r$ up to a constant term, which vanishes if $\varepsilon = 1$.

We now return to the conditions on the function f to show that they are not independent. In fact, the last condition for either of $j = 0$ and $j = 1$ implies the second one.

Lemma 14.5. *Suppose f is a C^1 function of singular signs on $\mathbf{R}_+$ such that $|rf'(r)| \le c$ for all $r \in (0,\varepsilon]$. Then*

$$\int_0^\varepsilon \frac{dr}{rf(r)} = \infty.$$

Proof. We can assume without loss of generality that f is nonnegative everywhere in $\mathbf{R}_+$.

Write

$$\int_0^\varepsilon \frac{dr}{rf(r)} = -\int_0^\varepsilon \frac{1}{f(r)} d(-\log r) = \int_T^\infty \frac{1}{F(t)}\, dt$$

where $F(t) = f(e^{-t})$ and $T = -\log \varepsilon$. If t varies over $[T,\infty)$, then $r = \exp(-t)$ varies over $(0,\varepsilon]$, hence

$$\sup_{t\in[T,\infty)} |F'(t)| = \sup_{r\in(0,\varepsilon]} |f'(r)r| \le c$$

by assumption. The Lagrange formula now yields

$$\begin{aligned} F(t) &= F(T) + (t-T)\int_0^1 F'(T+\vartheta(t-T))\,d\vartheta \\ &\le F(T) + c(t-T) \end{aligned}$$

for all $t \geq T$. Hence it follows that

$$\int_0^{\varepsilon} \frac{dr}{rf(r)} \geq \int_T^{\infty} \frac{1}{F(T)+c(t-T)}\,dt = \infty,$$

which is our claim. □

We also mention that condition 3) on f just amounts to saying that each derivative $(rD_r)^j f(r)$, $j \in \mathbf{Z}_+$, is bounded close to $r=0$.

By a *canonical surface* with an oscillating cusp at a point $x^0 \in \mathbf{R}^{n+1}$ we mean

$$C_{x^0} = \{x^0 + rS(\phi(r)f(r)\theta) : r \in \mathbf{R}_+, \theta \in B\}, \tag{14.3}$$

where B is a C^∞ compact closed submanifold of $\mathbf{R}^n$. Here $f \in C^\infty_{\mathrm{loc}}(\mathbf{R}_+)$ is a function with properties 1)–3) above. We shall say that $f(r)$ specifies the degeneracy of C_{x^0} at the cusp x^0. On the other hand, $\phi \in C^\infty_{\mathrm{loc}}(\mathbf{R}_+)$ is required to meet the following conditions:

a) $\inf_{r\in\mathbf{R}_+} \phi(r) > 0$;

b) $|\mathbf{D}^\beta \phi(r)| \leq c_\beta$ for every $\beta \in \mathbf{Z}_+$; and

c) $\lim_{r\to 0} \mathbf{D}\phi(r) = 0$.

We say that $\phi(r)$ specifies the oscillation of the surface C_{x^0} at the cusp x^0. A typical example of $\phi(r)$ satisfying a)–c) is as follows.

Example 14.6. For $\varepsilon \in [0,1)$, consider

$$\phi(r) = 1 - \frac{1}{2}\sin(\delta(r))^{\varepsilon}\omega(r),$$

where ω is a cut-off function on $\overline{\mathbf{R}}_+$ such that δ does not vanish on the support of ω. Then ϕ meets a)–c) as is easy to check.

Applying the Hardy-Littlewood inequality (cf. [**17**]), we deduce that conditions b) and c) actually imply

$$\lim_{r\to 0} \mathbf{D}^j \phi(r) = 0, \tag{14.4}$$

for each $j = 1, 2, \dots$. In fact, we have the following lemma.

Lemma 14.7. *If* $\phi \in C^2_{\mathrm{loc}}(\mathbf{R}_+)$ *satisfies*

$$|\mathbf{D}^2\phi(r)| \le c, \quad r \in (0, \varepsilon], \qquad \lim_{r\to 0} \phi(r) = 0,$$

then $\lim_{r\to 0} \mathbf{D}\phi(r) = 0$.

Proof. Set $\Phi(t) = \phi(\delta^{-1}(t))$, thus obtaining a C^2 function on the entire real axis. Since

$$\Phi'(t) = \mathbf{D}\phi(\delta^{-1}(t)), \qquad \Phi''(t) = \mathbf{D}^2\phi(\delta^{-1}(t)),$$

we get

$$\begin{aligned} |\Phi''(t)| &\le c, \quad t \in [\delta(\varepsilon), +\infty), \\ \lim_{t\to+\infty} \Phi(t) &= 0. \end{aligned}$$

Combining the Hardy-Littlewood inequality

$$|\Phi'(t)| \le \sqrt{2}\sqrt{\sup|\Phi| \sup|\Phi''|}$$

on $\mathbf{R}$ with a suitable extension operator from the half-line, we arrive at an estimate

$$|\Phi'(t)|^2 \le C \sup_{\theta\in[T,\infty)} |\Phi(\theta)| \sup_{\theta\in[T,\infty)} |\Phi''(\theta)|, \quad t \in [T, \infty),$$

with C an absolute constant. For example, $C = 32\pi$ fills the bill, cf. Remark 11 in [**5**].[2] Hence it follows that $\lim_{t\to+\infty} \Phi'(t) = 0$, which is the desired conclusion. □

It is worth pointing out that the product ϕf fails to fulfill 1)–3) in general for f and ϕ possessing the properties 1–3) and a)–c) respectively. To see this take $f(r) = -r^2$ and $\phi(r) = 1 - 1/2\sin(\delta(r))^\varepsilon$ for

$r > 0$ small enough, where $\varepsilon > 1/2$. Thus, introducing ϕ into the definition of a canonical surface with a cusp enriches the class of surfaces under consideration.

For f given in Example 14.1, we have a canonical surface with a power-like cusp. If f is given in Example 14.3, we get a canonical surface with an exponential cusp. Finally, for f of Example 14.4, we obtain a canonical surface with a conical point.

Definition 14.8. 1) If C_{x^0} is given by (14.3) with B being a domain in $\mathbf{R}^n$, then we call C_{x^0} a canonical domain with an oscillating cusp at the boundary point x^0.

2) Let C_0 be a canonical domain with an oscillating cusp at the origin. Then $W = \mathbf{R}^q \times C_0$ is said to be a canonical oscillating cuspidal wedge.

Clearly, the boundary of a canonical oscillating cuspidal wedge is of the form $W = \mathbf{R}^q \times C_0$ where C_0 is a canonical surface with an oscillating cusp at the origin. In this way we obtain what will be referred to as a *canonical surface with an oscillating cuspidal edge* $\mathbf{R}^q$.

15. Differential operators. If $W = \mathbf{R}^q \times C_0$ is a canonical oscillating cuspidal wedge or a canonical surface with an oscillating cuspidal edge, then one has distinguished local coordinates in W. These are given by (y, r, θ) where $y \in \mathbf{R}^q$, $r \in \mathbf{R}_+$ and θ stands for local coordinates on B. Using the coordinates (y, r, θ) actually leads to desingularization of W, for $\mathcal{W} = \mathbf{R}^q \times \mathbf{R}_+ \times B$ bears a cylindrical structure and one has a blow-down mapping $\mathcal{W} \to W$ which is a diffeomorphism away from $r = 0$ and restricts to a diffeomorphism of $\mathbf{R}^q$. Under this desingularization, differential operators near $\mathcal{W}$ in $\mathbf{R}^{q+n+1}$ are pulled back to $\mathcal{W}$. The pull-backs give rise to typical differential operators in the calculus on manifolds with oscillating cuspidal edges.

To illustrate this, we confine ourselves to the case where W is a canonical oscillating cuspidal wedge. Similar arguments apply to the case of canonical surfaces with oscillating cuspidal edges.

Let

$$A = \sum_{|\beta|+|\gamma|\le m} a_{\beta,\gamma}(y,x) D_y^\beta D_x^\gamma$$

be a differential operator with C^∞ coefficients on W. We assume that the coefficients $a_{\beta,\gamma}(y,x)$ with $|\beta|+|\gamma|\le m$ satisfy the conditions

$$\begin{aligned} |D_y^B D_x^G a_{\beta,\gamma}(y,x)| &\le c_{B,G}(a_{\beta,\gamma})(-\delta'(|x|))^{|G|}, \\ \lim_{x\to 0} \sup_{y\in\mathbf{R}^q} (D_{x_j} a_{\beta,\gamma}(y,x))/\delta'(|x|) &= 0 \end{aligned} \tag{15.1}$$

for all multi-indices $B\in\mathbf{Z}^q$, $G\in\mathbf{Z}^{n+1}$ and for every $j=1,\dots,n+1$, the constants $c_{B,G}(a_{\beta,\gamma})$ being independent of $(y,x)\in W$.

Denote by $\pi:\mathcal{W}\to W$ the mapping of passage to the "cylindrical" coordinates (y,r,θ) via

$$\begin{cases} y = y, \\ x = rS(\phi(r)f(r)\theta), \end{cases} \tag{15.2}$$

for $y\in\mathbf{R}^q$, $r\in\mathbf{R}_+$ and $\theta\in B$. Thus $\pi:(y,r,\theta)\mapsto(y,rS(\phi(r)f(r)\theta))$ is a diffeomorphism provided the dilatations tB, $t>0$, do not meet the set where $S(\omega)$ fails to be a diffeomorphism.

Applying Proposition 3.2.1 of [**33**] yields

(15.3)

$$D_x = \delta'(S(\phi f\theta)\mathbf{D}_r + ((\partial S/\partial\omega)^{-1}(\phi f\theta))^T D_\theta/\phi - r(\phi f)' S(\phi f\theta)\theta D_\theta/\phi),$$

where $(\partial S/\partial\omega)^{-1}$ is a left inverse for the Jacobian matrix of S, the superscript 'T' indicates the transposed matrix and $\theta D_\theta = \sum_{\iota=1}^n \theta_\iota D_{\theta_\iota}$. As

$$\begin{aligned} \mathbf{D}(\delta')^\mu &= i\mu(\delta')^\mu(f+rf'), \\ \mathbf{D}(r^j f^{(j)}) &= -if(jr^j f^{(j)} + r^{j+1} f^{(j+1)}), \\ \mathbf{D}\phi^\mu &= \mu\phi^{\mu-1}\mathbf{D}\phi, \\ r(\phi f)' &= i\mathbf{D}\phi + \phi r f' \end{aligned} \tag{15.4}$$

and

$$\begin{aligned} \mathbf{D}_r S(\phi f\theta) &= \sum_{\iota=1}^n (\omega_\iota \partial S/\partial\omega_\iota)|_{\omega=\phi f\theta}(\mathbf{D}\phi/\phi + rDf), \\ D_\theta S(\phi f\theta) &= D_\omega S|_{\omega=\phi f\theta}(\phi f), \end{aligned} \tag{15.5}$$

we conclude that

$$D_x^\gamma = (\delta')^{|\gamma|} \sum_{j+|\alpha|\le|\gamma|} \phi^{j-|\gamma|} p_{j,\alpha}^{(\gamma)} \mathbf{D}_r^j D_\theta^\alpha,$$

for any $\gamma \in \mathbf{Z}_+^{n+1}$ where $p_{j,\alpha}^{(\gamma)}$ are polynomials with integer coefficients of $r^\iota f^{(\iota)}$ and $\mathbf{D}^\iota\phi$, $\iota = 0, 1, \dots, |\gamma| - j$, θ and elements of the matrices $D_\omega^I S$ and $D_\omega^I(\partial S/\partial\omega)^{-1}$, $|I| \le |\gamma| - j - |\alpha|$ with $\omega = \phi f \theta$ substituted. It follows that, under the change of variables (15.2), A transforms into a differential operator

$$\pi^\sharp A = (\delta'(r))^m \sum_{|\beta|+j+|\alpha|\le m} a_{\beta,j,\alpha}(y,r,\theta) \mathbf{D}_y^\beta \mathbf{D}_r^j D_\theta^\alpha \tag{15.6}$$

on the stretched wedge $\mathcal{W}$, where $\mathbf{D}_y$ is a new "totally characteristic" derivative in the calculus, given by

$$\mathbf{D}_y = \frac{1}{\delta'} D_y = r f(r) D_y,$$

and

$$a_{\beta,j,\alpha}(y,r,\theta) = \sum_{j+|\alpha|\le|\gamma|\le m-|\beta|} (\delta')^{|\beta|+|\gamma|-m} \phi^{j-|\gamma|} p_{j,\alpha}^{(\gamma)} \pi^* a_{\beta,\gamma}(y,r,\theta).$$

Note that the pull-backs $\pi^* a_{\beta,\gamma}(y,r,\theta)$ behave "well" near the base $r = 0$ of $\mathcal{W}$. Indeed, applying (15.4), (15.5) and the chain rule yields

$$\begin{aligned}
\mathbf{D}_y^\beta &= (\delta')^{-|\beta|} D_y^\beta, \\
\mathbf{D}_r^j &= (\delta')^{-j} \sum_{|G|\le j} p_G^{(j)}((r^\iota f^{(\iota)})_{\iota\le j}, (\mathbf{D}^\iota\phi)_{\iota\le j}, \theta, (D_\omega^I S)_{|I|\le j}) D_x^G, \\
D_\theta^\alpha &= (\delta')^{-|\alpha|} \phi^{|\alpha|} \sum_{|G|\le|\alpha|} p_G^{(\alpha)}((D_\omega^I S)_{|I|\le|\alpha|}) D_x^G,
\end{aligned}$$

$p_G^{(j)}$ and $p_G^{(\alpha)}$ being polynomials with integer coefficients of the variables indicated in the parentheses. We now invoke the first estimate of (15.1) and the property 3) of f to see that

$$\begin{gathered}
|\mathbf{D}_r^j D_y^B D_\theta^A \pi^* a_{\beta,\gamma}(y,r,\theta)| \le c_{j,B,A}(a_{\beta,\gamma}), \\
j \in \mathbf{Z}_+,\ B \in \mathbf{Z}_+^q,\ A \in \mathbf{Z}_+^n,
\end{gathered}$$

uniformly in $(y, r, \theta) \in \mathcal{W}$. Combining these estimates with the explicit formulas for the coefficients of $\pi^\sharp A$ given above, we get

$$\sup_{r \in (0,\varepsilon]} |\mathbf{D}_r^k D_y^B D_\theta^A a_{\beta,j,\alpha}(y, r, \theta)| \le c_{k,B,A}, \quad k \in \mathbf{Z}_+,\ B \in \mathbf{Z}_+^q,\ A \in \mathbf{Z}_+^n, \tag{15.7}$$

uniformly in $(y, \theta) \in \mathbf{R}^q \times B$.

Estimates (15.7) may be summarized by saying that $(\delta')^{-m}\pi^\sharp A$ is a weighted differential operator in the sense of Section 11, with δ given by (14.2). Moreover, $(\delta')^{-m}\pi^\sharp A$ is a singular differential operator with a symbol slowly varying as $r \to 0$ if, in addition to (15.7), the coefficients $a_{\beta,j,\alpha}$ bear

$$\lim_{r\to 0} \mathbf{D}_r a_{\beta,j,\alpha}(y, r, \theta) = 0 \tag{15.8}$$

uniformly in $(y, \theta) \in \mathbf{R}^q \times B$ (cf. Lemma 14.7). Our next result highlights conditions on f under which the second condition of (15.1) implies (15.8).

Proposition 15.1. *Suppose that*

$$\lim_{r\to 0+} r f'(r) = 0. \tag{15.9}$$

Let (15.1) *hold. Then* $a_{\beta,j,\alpha}$ *satisfies* (15.8) *for each* $\beta \in \mathbf{Z}_+^q$, $j \in \mathbf{Z}_+$ *and* $\alpha \in \mathbf{Z}_+^n$ *satisfying* $|\beta| + j + |\alpha| \le m$.

Proof. We first observe, by Lemma 14.7, that equality (15.9) actually implies

$$\lim_{r\to 0+} r^j D^j f(r) = 0,$$

for every $j = 1, 2, \dots$. Moreover, the second condition of (15.1) just amounts to the fact that

$$\lim_{r\to 0} \mathbf{D}_r \pi^* a_{\beta,\gamma}(y, r, \theta) = 0$$

uniformly in $(y, \theta) \in \mathbf{R}^q \times B$ for all β and γ with $|\beta| + |\gamma| \le m$. Since both $\mathbf{D}_r(\delta')^{|\beta|+|\gamma|-m}$ and $\mathbf{D}_r \phi^{j-|\gamma|}$ vanish as $r \to 0$ (cf. (15.4)), it remains to evaluate the derivative $\mathbf{D}_r p_{j,\alpha}^{(\gamma)}$ when $r \to 0$. To this end, set

$$\begin{aligned} v_\iota &= r^\iota f^{(\iota)}, \quad \iota = 0, 1, \dots, |\gamma| - j; \\ w_\iota &= \mathbf{D}^\iota \phi_\iota, \quad \iota = 0, 1, \dots, |\gamma| - j, \end{aligned}$$

and let z_κ, $\kappa = 1, \dots, K$ be an indexing of the elements of both matrices $D_\omega^I S$ and $D_\omega^I (\partial S/\partial \omega)^{-1}$, $|I| \le |\gamma| - j - |\alpha|$ where $\omega = \phi f \theta$. By the chain rule, we get

$$\begin{aligned}\mathbf{D}_r p_{j,\alpha}^{(\gamma)} = \sum_{\iota=0}^{|\gamma|-j} \frac{\partial p_{j,\alpha}^{(\gamma)}}{\partial v_\iota} \mathbf{D}(r^\iota f^{(\iota)}) + \sum_{\iota=0}^{|\gamma|-j} \frac{\partial p_{j,\alpha}^{(\gamma)}}{\partial w_\iota} \mathbf{D}(\mathbf{D}^\iota \phi) \\ + \sum_{\kappa=1}^{K} \frac{\partial p_{j,\alpha}^{(\gamma)}}{\partial z_\kappa} \mathbf{D}_r z_k(\phi f \theta)\end{aligned}$$

whence

$$\lim_{r \to 0} \mathbf{D}_r P_{j,\alpha}^{(\gamma)} = 0$$

uniformly in $\theta \in B$, which is due to (15.4), (15.5) and (15.9). This completes the proof. $\square$

The choice of f meeting (15.9) seems to be the best adapted to our theory. Recall that (15.9) strengthens condition 3) on the functions f under consideration.

In case f satisfies (15.9) we can distinguish in a natural way a proper part of $\pi^\sharp A$ responsible for the local invertibility of this operator near $r = 0$. To this end, denote by $\tilde{p}_{j,\alpha}^{(\gamma)}$ the polynomial obtained from $p_{j,\alpha}^{(\gamma)}$ by replacing $rf', \dots, r^{|\gamma|-j} f^{(|\gamma|-j)}$ and $\mathbf{D}\phi, \dots, \mathbf{D}^{|\gamma|-j}\phi$ via zeros. It is easy to see that $\tilde{p}_{j,\alpha}^{(\gamma)}$ is of the form $\tilde{p}_{j,\alpha}^{(\gamma)} = \phi^{|\gamma|-j-|\alpha|} q_{j,\alpha}^{(\gamma)}$ where $q_{j,\alpha}^{(\gamma)}$ is a polynomial with integer coefficients of f and elements of the matrices $D_\omega^I S$ and $D_\omega^I (\partial S/\partial \omega)^{-1}$, $|I| < |\gamma|$, with $\omega = \phi f \theta$. Write

(15.10)
$$\begin{aligned}\pi^\sharp A = (\delta')^m \sum_{|\beta|+j+|\alpha| \le m} \left(\phi^{-|\alpha|} \sum_{|\gamma| = m - |\beta|} q_{j,\alpha}^{(\gamma)} \pi^* a_{\beta,\gamma} \right) \mathbf{D}_y^\beta \mathbf{D}_r^j D_\theta^\alpha \\ + (\delta')^m S.\end{aligned}$$

Proposition 15.2. *Under condition* (15.9) *if, moreover,* $a_{\beta,\gamma}$ *fulfill* (15.1)*, then the coefficients of the differential operator* S *in* (15.10) *are infinitesimal as* $r \to 0$*.*

Proof. Indeed we have

$$S = \sum_{|\beta|+j+|\alpha|\le m} \delta_{\beta,j,\alpha} \mathbf{D}_y^\beta \mathbf{D}_r^j D_\theta^\alpha,$$

with

$$\begin{aligned}\delta_{\beta,j,\alpha} = & \sum_{|\gamma|=m-|\beta|} \phi^{j-|\gamma|} (p_{j,\alpha}^{(\gamma)} - \tilde{p}_{j,\alpha}^{(\gamma)}) \pi^* a_{\beta,\gamma} \\ & + \sum_{j+|a|\le|\gamma|<m-|\beta|} (\delta')^{|\beta|+|\gamma|-m} p_{j,\alpha}^{(\gamma)} \pi^* a_{\beta,\gamma}.\end{aligned}$$

If $j = |\gamma|$, then $p_{j,\alpha}^{(\gamma)} - \tilde{p}_{j,\alpha}^{(\gamma)} = 0$ by the very definition. For fixed j, α and γ with $j < |\gamma| = m - |\beta|$, set $N = |\gamma| - j$. Using Taylor's expansion for the polynomial $p_{j,\alpha}^{(\gamma)}$ yields

$$\begin{aligned} p_{j,\alpha}^{(\gamma)} - \tilde{p}_{j,\alpha}^{(\gamma)} = & \sum_{\substack{I \in \mathbf{Z}_+^{2N} \\ I \neq 0}} \frac{1}{I!} \partial_{v',w'}^I p_{j,\alpha}^{(\gamma)} \Big|_{\substack{v'=0 \\ w'=0}} (rf')^{i_1} \\ & \cdots (r^N f^{(N)})^{i_N} (\mathbf{D}\phi)^{i_{N+1}} \cdots (\mathbf{D}^N \phi)^{i_{2N}}, \end{aligned}$$

where $I = (i_1, \ldots, i_{2N})$ and $v' = (v_1, \ldots, v_N)$, $w' = (w_1, \ldots, w_N)$. Combining this with (15.9) and taking into account the properties of ϕ, we deduce that the first sum in the expression for $\delta_{\beta,j,\alpha}$ vanishes when $r \to 0$.

On the other hand, if $|\gamma| < m - |\beta|$, then $(\delta')^{|\beta|+|\gamma|-m} \to 0$ as $r \to 0$. This shows that the second term of $\delta_{\beta,j,\alpha}$ also vanishes when $r \to 0$. Hence the desired conclusion follows. □

We show below that the operator $(\delta')^m S$ has a small local norm in suitable function spaces and is thus immaterial in the problem of local invertibility at the point $r = 0$.

The class of coefficients meeting (15.7) and (15.8) contains some functions rapidly oscillating near the edges, i.e., close to $r = 0$.

Example 15.3. For each $0 < p < 1$ and any $c \in C^\infty(\mathbf{R}^q \times B)$ with bounded derivatives, the function $a(y, r, \theta) = e^{i(\delta(r))^p} c(y, \theta)$ satisfies both (15.7) and (15.8).

16. Local invertibility of differential operators on a surface with oscillating cuspidal edges. Let A be a differential operator on a canonical surface $W = \mathbf{R}^q \times C_0$ with an oscillating cuspidal edge, C_0 being of the form (14.3). The cross-section of C_0 close to 0 is identified with B, a compact closed submanifold of $\mathbf{R}^n$.

When written in the cylindrical coordinates (15.2), the operator A takes the form

$$A = (\delta'(r))^m \sum_{|\beta|+j\le m} a_{\beta,j}(y,r)\mathbf{D}_y^\beta \mathbf{D}_r^j, \quad (y,r)\in \mathbf{R}^q\times\mathbf{R}_+, \tag{16.1}$$

where $a_{\beta,j}$ is a C^∞ function on $\mathbf{R}^q\times\mathbf{R}_+$ taking its values in $\mathrm{Diff}^{m-|\beta|-j}(B)$. We can thus regard $a_{\beta,j}$ as an operator-valued function on $\mathbf{R}^q\times\mathbf{R}_+$ with values in $\mathcal{L}(H^s(B), H^{s-(m-|\beta|-j)}(B))$ for any $s\in\mathbf{R}$. Moreover, from (15.7) and (15.8) it follows that

$$\begin{aligned} \|\mathbf{D}_r^k D_y^b a_{j,\beta}(y,r)\|_{\mathcal{L}(H^s(B),H^{s-(m-|\beta|-j)}(B))} &\le c_{k,b}(a_{j,\beta}),\\ \lim_{r\to 0}\sup_{y\in\mathbf{R}^q}\|\mathbf{D}_r a_{j,\beta}(y,r)\|_{\mathcal{L}(H^s(B),H^{s-(m-|\beta|-j)}(B))} &= 0 \end{aligned} \tag{16.2}$$

for all $k\in\mathbf{Z}_+$ and $b\in\mathbf{Z}_+^q$, the constants $c_{k,b}(a_{j,\beta})$ depending on s but not on y and r.

We next introduce appropriate function spaces to be domains of A. Namely, given any $s\in\mathbf{Z}_+$ and $\gamma,\mu\in\mathbf{R}$, we define $H^{s,\gamma,\mu}(W)$ to consist of all distributions u on W with finite norm

$$\|u\|_{H^{s,\gamma,\mu}(W)} = \left(\iint_{\mathbf{R}^q\times\mathbf{R}_+} e^{2\gamma\delta}(\delta')^{2\mu}\left(\sum_{|\beta|+j\le s}\|\mathbf{D}_y^\beta\mathbf{D}_r^j\pi^* u\|^2_{H^{s-|\beta|-j}(B)}\right)|\delta'|^q\,dy\,dm\right)^{1/2}, \tag{16.3}$$

cf. [**33**]. For integer $s<0$ and noninteger $s\in\mathbf{R}$, these spaces are defined by duality and interpolation.

Note that the factor $|\delta'|^q$ is included by purely aesthetic reasoning. In fact, under the change of coordinates

$$\begin{cases} z = \delta'(r)y,\\ t = \delta(r),\end{cases}$$

the norm (16.3) transforms into an equivalent norm

$$\|u\|_{H^{s,\gamma,\mu}(W)} \sim \left(\iint_{\mathbf{R}^{q+1}} e^{2\gamma t} (\delta' \circ \delta^{-1})^{2\mu} \left(\sum_{|\beta|+j \le s} \|D_z^\beta D_t^j \tilde{u}\|^2_{H^{s-|\beta|-j}(B)} \right) dz\, dt \right)^{1/2},$$

where $\tilde{u}(z,t,\theta) = u(z/\delta', rS(\phi f \theta))$ for $r = \delta^{-1}(t)$. To prove the equivalence of the norms, it suffices to use the equality

$$[\mathbf{D}_y, \mathbf{D}_r] = i(f + rf')\mathbf{D}_y \tag{16.4}$$

and the property 3) of f.

We think of operator (16.1) as acting from $H^{s,\gamma,\mu}(W)$ to $H^{s-m,\gamma,\mu-m}(W)$. By (15.4) and the property 3) of f, this is really the case.

Note that the space $H^{s,\gamma,\mu}(W)$ coincides, modulo equivalent norms, with the space $H_w(\lambda_s; \gamma, \mu + q/2)$ of (12.2), where

$$\lambda_s(r, \varrho) = (1 + (\delta'(r))^{-2}\Delta_y + \varrho^2 + \Delta_B)^{s/2}, \tag{16.5}$$

Δ_B being a nonnegative Laplacian on the manifold B, cf. Example 9.3.

As described in Section 9, the function

$$\phi(t) = \frac{-1}{\delta'(\delta^{-1}(t))}$$

should satisfy (9.1). This is equivalent to the first condition of (12.3). Moreover, we require $\phi(t)$ to be slowly varying as $t \to +\infty$, i.e.,

$$\lim_{t \to +\infty} (D^\beta \phi(t))/\phi(t) = 0 \tag{16.6}$$

for each $\beta = 1, 2, \dots$. It is a simple matter to see that (16.6) just amounts to (13.1). Indeed, we have

$$\frac{D^j \phi(t)}{\phi(t)}\Big|_{t=\delta(r)} = -\frac{\mathbf{D}^j \delta'(r)}{\delta'(r)} + p_j \left(\frac{\mathbf{D}\delta'(r)}{\delta'(r)}, \dots, \frac{\mathbf{D}^{j-1}\delta'(r)}{\delta'(r)} \right)$$

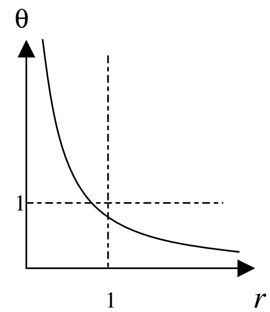

FIGURE 1. The domain $\mathcal{S}_{\mathcal{R}}$, $R > 0$.

for any $j \in \mathbf{Z}_+$, where p_j is a polynomial with integer coefficients of the variables indicated in the parentheses, such that $p_j(0) = 0$. Thus, under the assumptions on δ just imposed, Proposition 12.4 is applicable.

Example 16.1. Let $\delta(r) = -\log r$ be the diffeomorphism of Example 10.1. Then

$$\frac{\delta'(r \circ \theta)}{\delta'(r)} = \frac{1}{\theta}$$

for any $r, \theta \in \mathbf{R}_+$; hence, the first estimate of (12.3) fails to hold. On the other hand, we have

$$\mathbf{D}^\beta \delta'(r) = (-i)^\beta \delta'(r)$$

for $\beta \in \mathbf{Z}_+$ and so condition (16.6) is violated, too.

Example 16.2. Suppose $t = \delta(r)$ is the diffeomorphism of Example 10.2. To show that δ fulfills the first condition (12.3), set

$$\mathcal{S}_R = \{(r, \theta) \in \mathbf{R}_+ \times \mathbf{R}_+ : \delta^{-1}(\delta(r) + \delta(\theta)) \leq R\},$$

for $R > 0$, cf. Figure 1. We can choose $R > 0$ small enough so that

$$\mathcal{S}_R \subset ((0, 1] \times \mathbf{R}_+) \cup (\mathbf{R}_+ \times (0, 1]).$$

If $(r, \theta) \notin \mathcal{S}_R$, then

$$\frac{\delta'(r \circ \theta)}{\delta'(r)} \leq \left(\sup_{\rho > 0} \frac{1}{|\delta'(\rho)|} \right) \left(\sup_{\rho > R} |\delta'(\rho)| \right) < \infty,$$

the last estimate being a consequence of the properties of δ. Thus, for (r,θ) away from $\mathcal{S}_R$, the first estimate (12.3) holds with $\varepsilon = 0$. We are left with the task of establishing the estimate for $(r,\theta) \in \mathcal{S}_R$. To this end it suffices to examine the following three cases:

1) $(r,\theta) \in (0,1] \times (0,1]$;
2) $(r,\theta) \in (1,\infty) \times (0,1]$; and
3) $(r,\theta) \in (0,1] \times (1,\infty)$.

In the case 1) we make use of the explicit formula for δ on the interval $(0,1]$ to obtain

$$\frac{\delta'(r \circ \theta)}{\delta'(r)} \leq (2p)^{p+1/p} \langle \delta(\theta) \rangle^{p+1/p}.$$

We have used the condition $p > 0$. In the case 2), we have $r \geq 1$ whence

$$|r \circ \theta| \geq 1 \circ \theta.$$

As $r \circ \theta \leq 1$, it follows that

$$\begin{aligned}
\frac{\delta'(r \circ \theta)}{\delta'(r)} &\leq \left(\sup_{\rho \in (0,1]} \frac{1}{|\delta'(\rho)|} \right) |\delta'(1 \circ \theta)| \\
&= \left(\sup_{\rho \in (0,1])} \frac{1}{|\delta'(\rho)|} \right) (1 + \theta^p)^{(p+1)/p} \left(\frac{1}{\theta} \right)^{p+1} \\
&\leq \left(\sup_{\rho \in (0,1]} \frac{1}{|\delta'(\rho)|} \right) (2p)^{(p+1)/p} \langle \delta(\theta) \rangle^{(p+1)/p}.
\end{aligned}$$

Finally, in the case 3), we have $\theta \geq 1$ implying $r \circ \theta \geq r \circ 1$. Since $r \geq 1$ and $r \circ \theta \leq 1$, we get

$$\frac{\delta'(r \circ \theta)}{\delta'(r)} \leq \frac{\delta'(r \circ 1)}{|\delta'(r)|} = (1 + r^p)^{(p+1)/p} \leq (2p)^{(p+1)/p}$$

independently of θ. Combining the above estimates, we arrive at the first estimate of (12.3). Moreover,

$$\mathbf{D}^\beta \delta'(r) = \operatorname{const}(j) r^{pj} \delta'(r)$$

for $r \in (0,1]$, hence δ meets (16.6).

Example 16.3. The diffeomorphism of Example 10.3 satisfies both (12.3) and (16.6). This is verified as in the preceding example.

The conditions (9.1) and (16.6) guarantee that $\lambda_s(r,\varrho)$ is a weight function slowly varying at the point $r=0$.

From what has already been proved it follows that the operator (16.1) can be thought of as acting from $H_w(\lambda_s;\gamma,\mu)$ to $H_w(\lambda_{s-m};\gamma,\mu-m)$.

In this section we indicate how the results of Section 13 highlight the problem of local invertibility of A. To this end we first treat this problem for A with coefficients "frozen" at any point y^0 along the edge $\mathbf{R}^q$. Namely, consider the operator

$$A_{y^0}=(\delta'(r))^m\sum_{|\beta|+j\le m}a_{\beta,j}(y^0,r)\mathbf{D}_y^\beta\mathbf{D}_r^j$$

acting from $H^{s,\gamma,\mu}(W)$ to $H^{s-m,\gamma,\mu-m}(W)$ for any $s,\gamma,\mu\in\mathbf{R}$. By the above, A_{y^0} maps $H_w(\lambda_s;\gamma,\mu)$ to $H_w(\lambda_{s-m};\gamma,\mu-m)$; hence, it can be specified within the class $\mathcal{OPS}_w(\lambda_s,\lambda_{s-m})$. The symbol of this operator is easily seen to be

$$\sigma_{A_{y^0}}(r,\varrho)=(\delta'(r))^m\sum_{|\beta|+j\le m}a_{\beta,j}(y^0,r)\varrho^j\mathbf{D}_y^\beta,$$

$(r,\varrho)\in T^*\mathbf{R}_+$. Moreover, the estimates (16.2) imply that $\sigma_{A_{y^0}}$ varies slowly as $r\to 0$.

In what follows, a so-called "compressed" symbol of A with respect to action in both y and r variables proves to be of great importance. It is given by

$$\sigma(A)(y,r;\eta,\varrho)=\sum_{|\beta|+j\le m}a_{\beta,j}(y,r)\eta^\beta\varrho^j, \tag{16.7}$$

for $(y,r;\eta,\varrho)\in T^*(\mathbf{R}^q\times\mathbf{R}_+)$. Thus η and ϱ substitute the totally characteristic derivatives $\mathbf{D}_y$ and $\mathbf{D}_r$, respectively.

By the very construction $\sigma(A)$ is a C^∞ function on $T^*(\mathbf{R}^q\times\mathbf{R}_+)$ taking its values in the space of differential operators of order m on B. Note that $\sigma(A)$ is actually C^∞ up to $r=0$ if also are the coefficients of A.

We also apply the symbol mapping σ to our weight functions $\lambda_s(r,\varrho)$ by $\sigma(\lambda_s)(\eta,\varrho) = (1+|\eta|^2+\varrho^2+\Delta_B)^{s/2}$.

Theorem 16.4. *In order that $A_{y^0} : H^{s,\gamma,\mu}(W) \to H^{s-m,\gamma,\mu-m}(W)$ is locally invertible at $r=0$, it is necessary and sufficient that there be an $\varepsilon > 0$ such that the symbol $\sigma(A)(y^0,r;\eta,\varrho+i\gamma) : H^s(B) \to H^{s-m}(B)$ is invertible for all $(r,\eta,\varrho) \in (0,\varepsilon)\times \mathbf{R}^{q+1}$ and*

(16.8)
$$\sup_{(0,\varepsilon)\times\mathbf{R}^{q+1}} \|\sigma(\lambda_s)(\eta,\varrho)\sigma(A)^{-1}(y^0,r;\eta,\varrho+i\gamma)\sigma(\lambda_{-s+m})(\eta,\varrho)\|_{\mathcal{L}(L^2(B))} < \infty.$$

Proof. Indeed, the estimate (16.8) is equivalent to the estimate

$$\sup_{(0,\varepsilon)\times\mathbf{R}} \|\lambda_s(r,\varrho)\sigma^{-1}_{A_{y^0}}(r,\varrho+i\gamma)\lambda^{-1}_{s-m}(r,\varrho)\|_{\mathcal{L}(L^2(\mathbf{R}^q\times B))} < \infty,$$

as is easy to see by applying the Fourier transform in $y\in\mathbf{R}^q$. Thus, Theorem 16.4 is a direct consequence of Theorem 13.1. □

The condition (16.8) implies that $\sigma(A)(y^0,r;\eta,\varrho+i\gamma)$ is an elliptic operator on the manifold B, for any $(\eta,\varrho)\in\mathbf{R}^{q+1}$, uniformly in $r\in(0,\varepsilon)$. Under a stronger condition on A, Theorem 16.4 can be reformulated without any weight functions.

Corollary 16.5. *Suppose $\sigma(A)(y^0,r;\eta,\varrho+i\gamma)$ is an elliptic operator on B with parameter $(\eta,\varrho)\in\mathbf{R}^{q+1}$, uniformly in $r>0$ small enough. Then $A_{y^0} : H^{s,\gamma,\mu}(W)\to H^{s-m,\gamma,\mu-m}(W)$ is locally invertible at $r=0$ if and only if there exists $\varepsilon>0$ such that $\sigma(A)(y^0,r;\eta,\varrho+i\gamma)$: $H^s(B)\to H^{s-m}(B)$ is invertible for all $(r,\eta,\varrho)\in(0,\varepsilon)\times\mathbf{R}^{q+1}$ and*

$$\sup_{r\in(0,\varepsilon)} \|\sigma(A)^{-1}(y^0,r;\eta,\varrho+i\gamma)\|_{\mathcal{L}(H^{s-m}(B),H^s(B))} < \infty. \tag{16.9}$$

Proof. Since

$$\|\sigma(A)^{-1}(y^0,r;\eta,\varrho+i\gamma)\|_{\mathcal{L}(H^{s-m}(B),H^s(B))}$$
$$\le c_1c_2\|\sigma(\lambda_s)(\eta,\varrho)\sigma(A)^{-1}(y^0,r;\eta,\varrho+i\gamma)\sigma(\lambda_{-s+m})(\eta,\varrho)\|_{\mathcal{L}(L^2(B))},$$

with

$$c_1 = \|\sigma(\lambda_{-s})(\eta, \varrho)\|_{\mathcal{L}(L^2(B), H^s(B))},$$
$$c_2 = \|\sigma(\lambda_{s-m})(\eta, \varrho)\|_{\mathcal{L}(H^{s-m}(B), L^2(B))},$$

estimate (16.8) implies estimate (16.9).

Conversely, $\sigma(\lambda_{s-m})(\eta, \varrho)\sigma(A)(y^0, r; \eta, \varrho + i\gamma)\sigma(\lambda_{-s})(\eta, \varrho)$ is an elliptic pseudodifferential operator of order zero with parameter $(\eta, \varrho) \in \mathbf{R}^{q+1}$ on B, uniformly in $r \in (0, \varepsilon)$. From standard composition formulas for parameter-dependent pseudodifferential operators (cf. Shubin [**43**]), it follows that there is an $R > 0$ such that $\sigma(A)(y^0, r; \eta, \varrho + i\gamma)$ is invertible for all $r \in (0, \varepsilon)$ and $(\eta, \varrho) \in \mathbf{R}^{q+1}$ with $|(\eta, \varrho)| > R$ and

$$\sup_{\substack{r\in(0,\varepsilon)\\ |(\eta,\varrho)|>R}} \|\sigma(\lambda_s)(\eta, \varrho)\sigma(A)^{-1}(y^0, r; \eta, \varrho+i\gamma)\sigma(\lambda_{-s+m})(\eta, \varrho)\|_{\mathcal{L}(L^2(B))} < \infty.$$

This gives (16.8) for $(\eta, \varrho) \in \mathbf{R}^{q+1}$ large enough. On the other hand, for those $(\eta, \varrho) \in \mathbf{R}^{q+1}$ which meet $|(\eta, \varrho)| \le R$, the estimate (16.8) follows from (16.9). $\square$

It is clear that the exponential estimate of Theorem 13.2 holds for solutions of $A_{y^0}u = f$, too. We skip the formulation because the result is actually valid for solutions of the "perturbed" equation $Au = f$. The proof of this takes, however, much more effort including a localization procedure (cf. Section 17).

Example 16.6. Let us endow the surface W with the Riemannian metric induced by the embedding $W \hookrightarrow \mathbf{R}^q \times \mathbf{R}^{n+1}$. We require δ to satisfy (15.9). When combined with (16.6) for $\beta = 1$, this gives $f(0) = 0$ because

$$\frac{\mathbf{D}\delta'}{\delta'} = if - rDf.$$

Hence it follows, by Proposition 15.2, that the Laplace operator Δ on W takes the form

$$\Delta = (\delta'(r))^2\left(\left(\frac{1}{\delta'(r)}\right)^2\Delta_y + \left(\frac{1}{\delta'(r)}D_r\right)^2 + \left(\frac{1}{\phi(r)}\right)^2\Delta_B\right)$$

in the coordinates $(y, r, \theta) \in \mathbf{R}^q \times \mathbf{R}_+ \times B$, modulo operators of small local norm near the edge $r = 0$. Here $f(r)$ specifies the *degeneracy* of W along the edge $\mathbf{R}^q$ whereas $\phi(r)$ specifies the *oscillation* of W near the edge. We regard Δ as acting from $H^{s,\gamma,\mu}(W)$ to $H^{s-2,\gamma,\mu-2}(W)$ for $s, \gamma, \mu \in \mathbf{R}$. The compressed symbol of the Laplace operator is

$$\sigma(\Delta)(y, r; \eta, \varrho + i\gamma) = |\eta|^2 + (\varrho + i\gamma)^2 + \left(\frac{1}{\phi(r)}\right)^2 \Delta_B,$$

where $(\eta, \varrho) \in \mathbf{R}^{q+1}$. As 0 is a point of the spectrum of Δ_B, the condition of Theorem 16.4 is satisfied for no $\gamma \in \mathbf{R}$, because $|\eta|^2 + (\varrho + i\gamma)^2$ vanishes whenever $|\eta| = |\gamma|$ and $\varrho = 0$. Thus, Theorem 16.4 shows that the Laplace operator on W is never locally invertible near the edge in the scale of weighted Sobolev spaces $(H^{s,\gamma,\mu}(W))_{s,\gamma,\mu \in \mathbf{R}}$.

Example 16.7. On the other hand, let us consider the Schrödinger operator on the surface W,

$$A = \Delta + p(r),$$

the potential p being of the form $p(r) = (\delta'(r))^2 a(r)$ with $a(r)$ a C^∞ function on $\mathbf{R}$, satisfying

$$\begin{aligned} &|\mathbf{D}^j a(r)| \le c_j, \quad j \in \mathbf{Z}_+; \\ &\lim_{r \to 0} \mathbf{D} a = 0. \end{aligned}$$

In this case we get

$$\sigma(A)(y, r; \eta, \varrho + i\gamma) = |\eta|^2 + (\varrho + i\gamma)^2 + \left(\frac{1}{\phi(r)}\right)^2 \Delta_B + a(r),$$

for $(\eta, \varrho) \in \mathbf{R}^{q+1}$. If

$$\liminf_{r \to 0} a(r) > 0, \tag{16.10}$$

then it is evident that the conditions of Theorem 16.4 are satisfied with $\gamma = 0$, for each $s \in \mathbf{R}$. It follows that the Schrödinger operator, when acting from $H^{s,0,\mu}(W)$ to $H^{s-2,0,\mu-2}(W)$, is locally invertible at $r = 0$. We now assume that a stronger condition than (16.10) is fulfilled, namely

$$\inf_{r\in(0,\varepsilon)} a(r) \geq k^2 \tag{16.11}$$

where $k > 0$. Then the operator-valued function $\sigma(A)(y;r;\eta,\varrho+i\gamma)$ is invertible for all $(\eta,\varrho) \in \mathbf{R}^{q+1}$ uniformly in $r \in (0,\varepsilon)$ provided that $\gamma \in (-k,k)$. Consequently, for any $s,\mu \in \mathbf{R}$ and $\gamma \in (-k,k)$, the Schrödinger operator, if acting from $H^{s,\gamma,\mu}(W)$ to $H^{s-2,\gamma,\mu-2}(W)$, is locally invertible near the edge $r = 0$. Furthermore, as the coefficients of A are independent of y, we can make efficient use of Theorem 13.2. Thus, if $-k < \gamma' \leq \gamma'' < k$ and $u \in H^{s,\gamma',\mu}(W)$ satisfies $Au = f$ with $f \in H^{s-2,\gamma'',\mu-2}(W)$ near $r = 0$, then $u \in H^{s,\gamma'',\mu}(W)$ near $r = 0$.

17. Fredholm property of differential operators on manifolds with oscillating cuspidal edges. When discussing pseudodifferential operators on manifolds with singularities, we will confine ourselves to those manifolds which are embedded into an Euclidean space. The same arguments still go for general manifolds where we should take more care of rigorous definitions.

Let M be a compact closed topological manifold in $\mathbf{R}^N$ and S a submanifold of M. We assume that

1) $M \setminus S$ is a C^∞ submanifold of $\mathbf{R}^n \setminus S$;

2) S is a C^∞ submanifold of $\mathbf{R}^n$ and

3) for each point $p \in S$ there are a neighborhood O in $\mathbf{R}^N$ and a diffeomorphism h of O to an open set in $\mathbf{R}^N$ such that $h((M\setminus S)\cap O) = B(y^0,\varepsilon)\times C_0^{(\varepsilon)}$ where

$$\begin{aligned} B(y^0,\varepsilon) &= \{y \in \mathbf{R}^q : |y-y^0| < \varepsilon\},\\ C_0^{(\varepsilon)} &= \{rS(\phi(r)f(r)\theta) \in \mathbf{R}^{n+1} : r \in (0,\varepsilon), \theta \in B\}. \end{aligned}$$

Using the cylindrical coordinates (y,r,θ) near the edge S actually leads to a compact C^∞ manifold with boundary, $\mathcal{M}$. Roughly speaking, it is obtained from M by identifying any neighborhood $(M\setminus S)\cap O$ in 3) with its image in $B(y^0,\varepsilon)\times(0,\varepsilon)\times B$ under the diffeomorphism h. By the very construction, there is a blow-down mapping $b : \mathcal{M} \to M$

which restricts to a diffeomorphism $\mathcal{M} \setminus \partial\mathcal{M} \stackrel{\cong}{\to} M \setminus S$. Moreover, b gives $\partial\mathcal{M}$ the structure of a fibered bundle over S, the fiber being B. Note that M bears a C^∞ structure with edges induced from $\mathbf{R}^N$, C^∞ functions on M being the restrictions of C^∞ near M in $\mathbf{R}^N$. Under the blow-down mapping, this structure is pulled back to $\mathcal{M}$. Thus, various degeneracies of M along S may be specified within various C^∞ structures on a compact closed manifold with boundary. We have therefore arrived at the slogan, the analysis on a closed manifold with edges reduced to that on a C^∞ manifold with boundary.

As described above, 'typical' differential operators A on M are those differential operators on the smooth part $M \setminus S$ of M which take the form (16.1) in the coordinates $(y, r, \theta) \in B(y^0, \varepsilon) \times (0, \varepsilon) \times B$ near S with coefficients $a_{\beta,j}$ meeting (16.2). More precisely, $a_{\beta,j}$ are required to fulfill

$$
\begin{aligned}
&\|\mathbf{D}_r^k D_y^B a_{j,\beta}(y,r)\|_{\mathcal{L}(H^s(B), H^{s-(m-|\beta|-j)}(B))} \le c_{k,\beta}(a_{j,\beta}),\\
&\lim_{r\to 0} \|\mathbf{D}_r a_{j,\beta}(y,r)\|_{\mathcal{L}(H^s(B), H^{s-(m-|\beta|-j)}(B))} = 0
\end{aligned}
$$

for all $k \in \mathbf{Z}_+$ and $B \in \mathbf{Z}_+^q$ uniformly in y on compact subsets of $B(y^0, \mathcal{E})$ and $r \in (0, \varepsilon]$, $\varepsilon < \mathcal{E}$.

For $s, \gamma, \mu \in \mathbf{R}$, the weighted Sobolev spaces $H^{s,\gamma,\mu}(M)$ on M are introduced in a familiar way by gluing together the usual Sobolev spaces $H^s_{\mathrm{loc}}(M \setminus S)$ on the smooth part of M with the weighted Sobolev spaces $H^{s,\gamma,\mu}(W)$ of (16.3) near S. Namely, fix a finite covering (O_ν) of M by open sets in $\mathbf{R}^N$ such that every $M \cap O_\nu$ lies in the domain of some chart on M. These charts are of two types: either $O_\nu \cap S = \varnothing$ and the local coordinates in $M \cap O_\nu$ are those on an open set in $\mathbf{R}^{\dim M}$ or $O_\nu \cap S \neq \varnothing$ and the local coordinates in $M \cap O_\nu$ are $(y, r, \theta) \in B(y^0, \varepsilon) \times (0, \varepsilon) \times B$. Pick a C^∞ partition of unity on M, (φ_ν), subordinate to the covering (O_ν). Then a distribution u on $M \setminus S$ is said to belong to $H^{s,\gamma,\mu}(M)$ if $\varphi_\nu u \in H^s(\mathbf{R}^{\dim M})$ for the charts away from S and $\varphi_\nu u \in H^{s,\gamma,\nu}(W)$ for the charts intersecting S. It is immaterial which charts and partition of unity on M we choose to define $H^{s,\gamma,\mu}(M)$ as long as transition diffeomorphisms obey the structure of M. Moreover, the space $H^{s,\gamma,\mu}(M)$ can be given a Hilbert structure in an evident way.

Obviously, the operator A maps $H^{s,\gamma,\mu}(M)$ to $H^{s-m,\gamma,\mu-m}(M)$ for any $s, \gamma, \mu \in \mathbf{R}$.

Given any point $p \in S$, the operator A possesses an operator-valued symbol $\sigma(A)(y,r;\eta,\varrho)$ in local coordinates $(y,r,\theta) \in B(y^0,\varepsilon) \times (0,\varepsilon) \times B$ near p. Here $y^0 = h(p)$. The changes of local coordinates on M obeying the structure of M are of the form

$$\begin{cases} Y = Y_0(y) + rY_1(y,r,\theta), \\ R = r \exp R_1(y,r,\theta), \\ \Theta = \Theta(y,r,\theta), \end{cases}$$

where Y_1, R_1 and Θ are smooth up to $r = 0$ and $\Theta(y,r,\theta)$ is a diffeomorphism of B for any fixed y and r. Under such a change we have

$$\begin{aligned} \mathbf{D}_y &= (\delta'(R)/\delta'(r))(\partial Y_0/\partial y)^T \mathbf{D}_Y, \\ \mathbf{D}_r &= (\delta'(R)/\delta'(r))((Y_1)^T \mathbf{D}_Y + (\exp R_1)\mathbf{D}_R), \\ D_\theta &= (\partial \Theta/\partial \theta)^T D_\Theta \end{aligned}$$

modulo operators of infinitesimal local norm as $r \to 0$. Hence it follows immediately that

$$\sigma((Y,R,\Theta)_* A)(Y,R;\eta,\varrho) = \Theta_* \sigma(A)(y,r;T(\eta,\varrho)) \tag{17.1}$$

modulo operators of small local norm at $R = 0$ where Θ_* means the push-forward operator on B under the diffeomorphism Θ for fixed y and r, and

$$T(\eta,\varrho) = (\delta'(R)/\delta'(r)) \begin{pmatrix} (\partial Y_0/\partial y)^T & 0 \\ (Y_1)^T & \exp R_1 \end{pmatrix} \begin{pmatrix} \eta \\ \varrho \end{pmatrix}.$$

The equality (17.1) shows that whether or not the compressed symbol $\sigma(A)$ is invertible at a point $p \in S$ does not depend on the choice of local coordinates on M to evaluate it.

Theorem 17.1. *Let $s, \gamma, \mu \in \mathbf{R}$. Then $A : H^{s,\gamma,\mu}(M) \to H^{s-m,\gamma,\mu-m}(M)$ is Fredholm if*

1) *A is an elliptic operator on $M \setminus S$ and*

2) *for each $p \in S$ there exists $\varepsilon > 0$ such that $\sigma(A)(p,r;\eta,\zeta)$ is an invertible operator on B for any $(r,\eta,\zeta) \in (0,\varepsilon) \times \mathbf{R}^q \times (\mathbf{R} + i\gamma)$ and the inverse satisfies* (16.8).

Proof. Let the condition 2) be fulfilled. We claim that there exist $\varepsilon^0 > 0$ and operators $B^{(L)}_{\varepsilon^0}$ and $B^{(R)}_{\varepsilon^0}$ such that

$$B^{(L)}_{\varepsilon^0} A \chi_\varepsilon = \chi_\varepsilon, \qquad \chi_\varepsilon A B^{(R)}_{\varepsilon^0} = \chi_\varepsilon \tag{17.2}$$

for all $\varepsilon \in (0, \varepsilon^0)$ where $\chi_\varepsilon(r)$ is a cut-off function of a collar neighborhood of the manifolds S.

Indeed, it follows from 2) that, given any point $p \in S$ with local coordinates $(y^0, 0)$, the operator A_{y^0} with coefficients frozen at y^0 is locally invertible. Hence there are operators $T^{(L)}_{y^0,\varepsilon}$ and $T^{(R)}_{y^0,\varepsilon}$ such that

$$T^{(L)}_{y^0,\varepsilon} A_{y^0} \chi_\varepsilon = \chi_\varepsilon, \qquad \chi_\varepsilon A_{y^0} T^{(R)}_{y^0,\varepsilon} = \chi_\varepsilon$$

for all $\varepsilon > 0$ small enough, in accordance with y^0.

A familiar argument based on smoothness of the coefficients along the edge shows that we can find a neighborhood O_p of p on S and operators $B^{(L)}_{y^0,\varepsilon}$ and $B^{(R)}_{y^0,\varepsilon}$ such that

$$\begin{aligned} B^{(L)}_{y^0,\varepsilon} A (\psi_p \otimes \chi_\varepsilon) &= \psi_p \otimes \chi_\varepsilon, \\ (\psi_p \otimes \chi_\varepsilon) A B^{(R)}_{y^0,\varepsilon} &= \psi_p \otimes \chi_\varepsilon, \end{aligned} \tag{17.3}$$

where $\psi_p \in C^\infty_{\mathrm{comp}}(O_p)$ is a cut-off function at the point p. Note that the domain of ε in (17.3) depends on y^0.

When p varies over S, the neighborhoods O_p cover S. Since S is compact, there is a finite subcovering O_{p_ν}, $\nu = 1, \ldots, N$. Fix a partition of unity $(\psi_\nu)_{\nu=1,\ldots,N}$ on S subordinate to this covering.

Let us choose $\varepsilon^0 > 0$ small enough so that $\chi_{\varepsilon^\nu}(r)\chi_{\varepsilon^0}(r) = \chi_{\varepsilon^0}(r)$ for each $\nu = 1, \ldots, N$. Then

$$\begin{aligned} B^{(L)}_{y^\nu,\varepsilon^\nu} A (\psi_\nu \otimes \chi_{\varepsilon^0}) &= \psi_\nu \otimes \chi_{\varepsilon^0}, \\ (\psi_\nu \otimes \chi_{\varepsilon^0}) A B^{(R)}_{y^\nu,\varepsilon^\nu} &= \psi_\nu \otimes \chi_{\varepsilon^0} \end{aligned}$$

for any $\nu = 1, \ldots, N$.

Set $\varphi_{\nu,\varepsilon} = \psi_\nu \otimes \chi_\varepsilon$. As $\sum_{\nu=1}^N \psi_\nu = 1$, we obtain $\sum_{\nu=1}^N \varphi_{\nu,\varepsilon} = \chi_\varepsilon$ for $\varepsilon > 0$ small enough.

Introducing the operator

$$P_{\varepsilon}^{(L)} = \sum_{\nu=1}^{N} B_{y^{\nu},\varepsilon^{\nu}}^{(L)} \varphi_{\nu,\varepsilon},$$

we get

$$P_{\varepsilon}^{(L)} A = \sum_{\nu=1}^{N} B_{y^{\nu},\varepsilon^{\nu}}^{(L)} A_{\varphi_{\nu,\varepsilon}} + \sum_{\nu=1}^{N} B_{y^{\nu},\varepsilon^{\nu}}^{(L)} [\varphi_{\nu,\varepsilon}, A].$$

We keep $\varepsilon < \varepsilon^0$ small enough such that $\varphi_{\nu,\varepsilon^0}\varphi_{\nu,\varepsilon} = \varphi_{\nu,\varepsilon}$, whence

$$B_{y^{\nu},\varepsilon^{\nu}}^{(L)} A\varphi_{\nu,\varepsilon} = B_{y^{\nu},\varepsilon^{\nu}}^{(L)} A(\varphi_{\nu,\varepsilon^0}\varphi_{\nu,\varepsilon}) = \varphi_{\nu,\varepsilon}.$$

On the other hand,

$$\lim_{\varepsilon\to 0} \|[\varphi_{\nu,\varepsilon}, A]\|_{\mathcal{L}(H^{s,\gamma,\mu}(M), H^{s-m,\gamma,\mu-m}(M))} = 0$$

for every $\nu = 1, \dots, N$, as is easy to check. Therefore,

$$P_{\varepsilon}^{(L)} A = \chi_{\varepsilon} + R_{\varepsilon}$$

where

$$\lim_{\varepsilon\to 0} \|R_{\varepsilon}\|_{\mathcal{L}(H^{s,\gamma,\mu}(M))} = 0.$$

Pick an $\varepsilon^0 > 0$ such that $\|R_{\varepsilon^0}\|_{\mathcal{L}(H^{s,\gamma,\mu}(M))} < 1$. Then the operator $1+R_{\varepsilon^0}$ is invertible within the calculus. If moreover $\varepsilon < \varepsilon^0$ is sufficiently small so that $\chi_{\varepsilon^0}\chi_{\varepsilon} = \chi_{\varepsilon}$, then

$$(1+R_{\varepsilon^0})^{-1} P_{\varepsilon^0}^{(L)} A_{\chi^{\varepsilon}} = (1+R_{\varepsilon^0})^{-1}(\chi_{\varepsilon^0} + R_{\varepsilon^0})\chi_{\varepsilon} = \chi_{\varepsilon},$$

i.e., $B_{\varepsilon^0}^{(L)} = (1+R_{\varepsilon^0})^{-1} P_{\varepsilon^0}^{(L)}$ is a local left inverse of A, as is required in (17.2).

In the same way we can construct a local right inverse of A satisfying the second equality in (17.2).

We now proceed by pasting together these local inverses with a parametrix of A on the smooth part of M. Namely, the condition 1) makes it legitimate to apply the usual elliptic theory away from the edge on M to deduce that, for any $\varepsilon > 0$, there are classical pseudodifferential operators $Q_\varepsilon^{(L)}$ and $Q_\varepsilon^{(R)}$ such that

$$\begin{aligned} Q_\varepsilon^{(L)} A(1-\chi_\varepsilon) &= (1-\chi_\varepsilon) + K'_\varepsilon, \\ (1-\chi_\varepsilon) A Q_\varepsilon^{(R)} &= (1-\chi_\varepsilon) + K''_\varepsilon, \end{aligned}$$

both R'_ε and R''_ε being compact operators. Set

$$R_\varepsilon^{(L)} = B_{\varepsilon^0}^{(L)} \chi_\varepsilon + Q_\varepsilon^{(L)}(1-\chi_\varepsilon),$$

then

$$\begin{aligned} R_\varepsilon^{(L)} A &= B_{\varepsilon^0}^{(L)} A \chi_\varepsilon + Q_\varepsilon^{(L)} A(1-\chi_\varepsilon) + B_{\varepsilon^0}^{(L)}[\chi_\varepsilon, A] + Q_\varepsilon^{(L)}[1-\chi_\varepsilon, A] \\ &= 1 + K_\varepsilon + S_\varepsilon, \end{aligned}$$

where

$$\begin{aligned} K_\varepsilon &= K'_\varepsilon + Q_\varepsilon^{(L)}[1-\chi_\varepsilon, A], \\ S_\varepsilon &= B_{\varepsilon^0}^{(L)}[\chi_\varepsilon, A]. \end{aligned}$$

It is clear that K_ε is a compact operator for each $\varepsilon > 0$ small enough. Furthermore, the operator $(1+S_\varepsilon)$ is invertible for any sufficiently small $\varepsilon > 0$ because

$$\lim_{\varepsilon \to 0} \|[\chi_\varepsilon, A]\|_{\mathcal{L}(H^{s,\gamma,\mu}(M), H^{s-m,\gamma,\mu-m}(M))} = 0.$$

Hence it follows that $(1+S_\varepsilon)^{-1} R_\varepsilon^{(L)}$ for $\varepsilon \ll 1$ is a left regularizer of A.

The same reasoning applies to prove the existence of a right regularizer, which completes the proof. $\square$

Note that if the coefficients $a_{\beta,j}(y,r)$ of A are C^∞ up to $r = 0$, then the condition 2) just amounts to saying that $\sigma(A)(p,0;\eta,\mathbf{Z})$ is an invertible operator on B for any $(\eta,\zeta) \in \mathbf{R}^q \times (\mathbf{R} + i\gamma)$. Indeed, $\sigma(A)(p,r;\eta,\zeta)$ is a perturbation of $\sigma(A)(p,0;\eta,\zeta)$ by an operator of small local norm as $r \to 0$.

Theorem 17.2. *Let the condition* 2) *of Theorem* 17.1 *hold uniformly in* γ *on compact intervals in* (a,b). *If* $a < \gamma' \le \gamma'' < b$ *and* $u \in H^{s,\gamma',\mu}(M)$ *satisfies* $Au = f$ *with* $f \in H^{s-m,\gamma'',\mu-m}(M)$, *then* $u \in H^{s,\gamma'',\mu}(M)$.

Proof. In the proof of Theorem 17.1 we have constructed a local left inverse for A near the singular manifold S. This operator $B^{(L)}_{\varepsilon^0}$ bears a symbol analytic in the strip $\mathbf{R}+i(a,b)$ and thus extends to a continuous mapping $H^{s-m,\gamma,\mu-m}(M) \to H^{s,\gamma,\mu}(M)$ for each $\gamma \in (a,b)$. If $\varepsilon > 0$ is small enough, then

$$\chi_\varepsilon u = -B^{(L)}_{\varepsilon^0} A(1-\chi_\varepsilon)u + B^{(L)}_{\varepsilon^0} f.$$

We have $(1-\chi_\varepsilon)u \in H^{s,\gamma'',\mu}(M)$ and $f \in H^{s-m,\gamma'',\mu-m}(M)$. By the mapping properties of A and $B^{(L)}_{\varepsilon^0}$, we deduce that $u \in H^{s,\gamma'',\mu}(M)$. The proof is complete. □

Example 17.3. Let us equip (the smooth part of) the manifold M with the Riemannian metric induced by the embedding $M \hookrightarrow \mathbf{R}^N$. Consider the Schrödinger operator on $M \setminus S$

$$A = \Delta + p,$$

where p is a C^∞ function on $M \setminus S$. We require the potential to be of the form $p(y,r,\theta) = (\delta'(r))^2 a(y,r,\theta)$ in local coordinates $(y,r,\theta) \in B(y^0,\varepsilon) \times (0,\varepsilon) \times B$ near S, with $a(y,r,\theta)$ satisfying

$$|D_y^\beta \mathbf{D}_r^j D_\theta^\alpha a(y,r,\theta)| \le c_{\beta,j,\alpha}, \qquad \lim_{r\to 0} \mathbf{D}_r a(y,r,\theta) = 0$$

for all multi-indices β, j and α uniformly in y on compact subsets of $B(y^0,\mathcal{E})$, $r \in (0,\varepsilon]$, $\varepsilon < \mathcal{E}$, and θ on compact subsets of the domains of local charts on B. If moreover

$$\liminf_{r\to 0} a(y,r,\theta) \ge k^2$$

where $k > 0$ (cf. (16.11)), then $A : H^{s,\gamma,\mu}(M) \to H^{s-2,\gamma,\mu-2}(M)$ is a Fredholm operator for all $s,\mu \in \mathbf{R}$ and $\gamma \in (-k,k)$. This follows from Theorem 17.1 and what has already been proved in Example 16.7.

Furthermore, any solution $u \in H^{s-k+0,\mu}(M)$ of the homogeneous equation $Au = 0$ actually belongs to $H^{\infty,k-0,\mu}(M)$.

Part IV. Boundary Value Problems in Domains with Cuspidal Wedges.

18. Domains with cuspidal wedges. Let $\mathcal{D}$ be a domain in $\mathbf{R}^{q+n+1}$ with a compact closure $\overline{\mathcal{D}}$ and S a closed subset of the boundary of $\mathcal{D}$. We assume that

1) $\partial\mathcal{D} \setminus S$ is a C^∞ submanifold of $\mathbf{R}^{q+n+1} \setminus S$;

2) S is a C^∞ submanifold of $\mathbf{R}^{q+n+1}$ of dimension q; and

3) for each $p \in S$ there are a neighborhood O in $\mathbf{R}^{q+n+1}$ and a diffeomorphism h of O to an open set in $\mathbf{R}^{q+n+1}$ such that $h(\mathcal{D} \cap O) = B(0,\varepsilon) \times C_0^{(\varepsilon)}$, where

$$\begin{aligned} B(0,\varepsilon) &= \{y \in \mathbf{R}^q : |y| < \varepsilon\}, \\ C_0^{(\varepsilon)} &= \{rS(\phi(r)f(r)\theta) \in \mathbf{R}^{n+1} : r \in (0,\varepsilon), \theta \in B\}, \end{aligned}$$

B being a relatively compact domain in $\mathbf{R}^n$ with smooth boundary.

It is worth mentioning that, in contrast to the definition of a closed manifold with edges (cf. Section 17), B has here a nonempty boundary.

The functions $f(r)$ and $\phi(r)$ have been introduced in Section 14. They control the degeneracy of $\mathcal{D}$ along the edge S and the oscillation of $\mathcal{D}$ near the edge S, respectively.

19. Boundary value problems in a canonical domain. Recall that $W = \mathbf{R}^q \times C_0$ is referred to as a canonical oscillating cuspidal wedge, C_0 being given by (14.3) with B a bounded domain in $\mathbf{R}^n$ with C^∞ boundary. We also call W the *canonical domain*.

Consider a boundary value problem in the canonical domain W with boundary data given on the smooth part of ∂W. Namely,

$$\begin{cases} Au = f & \text{in } W, \\ B_i u = u_i & \text{on } \partial W \setminus (\mathbf{R}^q \times \{0\}), \end{cases} \tag{19.1}$$

where A is a differential operator in W and (B_i) is a system of differential operators defined in a neighborhood of $\partial W \setminus (\mathbf{R}^q \times \{0\})$.

The pull-backs of A and (B_i) under cylindrical coordinates (15.2) in W are of the form

$$\pi^\sharp A = (\delta'(r))^m \sum_{|\beta|+j\le m} a_{\beta,j}(y,r)\mathbf{D}_y^\beta \mathbf{D}_r^j,$$
$$\pi^\sharp B_i = (\delta'(r))^{m_i} \sum_{|\beta|+j\le m_i} b_{i,\beta,j}(y,r)\mathbf{D}_y^\beta \mathbf{D}_r^j,$$

where $a_{\beta,j}$ is a C^∞ function of $(y,r) \in \mathbf{R}^q \times \mathbf{R}_+$ whose values are differential operators of order $m-|\beta|-j$ on B and $b_{i,\beta,j}$ is a C^∞ function of $(y,r)\in \mathbf{R}^q \times \mathbf{R}_+$ whose values are differential operators of order $m_i - |\beta| - j$ in a neighborhood of ∂B. When passing to the cylindrical coordinates, we are actually lifted to the infinite “stretched” wedge $\mathcal{W} = \mathbf{R}^q \times \mathbf{R}_+ \times B$.

In order to apply the theory of Chapter II, we require the coefficients $a_{j,\alpha}$ and $b_{i,j,\alpha}$ to satisfy (16.2) uniformly in $(y,r)\in \mathbf{R}^q \times \mathbf{R}_+$.

For $s \ge 0$ and $\gamma,\mu \in \mathbf{R}$, we introduce weighted Sobolev spaces $H^{s,\gamma,\mu}(W)$ just as in (16.3) with B being now a domain in $\mathbf{R}^n$. If $s > 1/2$, then we define the space $H^{s-1/2,\gamma,\mu}(\partial W)$ to consist of the traces on $\partial W \setminus (\mathbf{R}^q \times \{0\})$ of functions in $H^{s,\gamma,\mu}(W)$. It is a Hilbert space under the canonical quotient norm.

Assuming $s > \max m_i + 1/2$, we assign an operator

$$\begin{pmatrix} A \\ \oplus r_{\partial W} B_i \end{pmatrix} : H^{s,\gamma,\mu}(W) \longrightarrow \begin{array}{c} H^{s-m,\gamma,\mu-m}(W) \\ \oplus \\ \oplus H^{s-m_i-1/2,\gamma,\mu-m_i}(\partial W) \end{array} \tag{19.2}$$

to the boundary value problem (19.1) where $r_{\partial W}$ means restriction to (the smooth part of) the boundary of W. Denote the operator (19.2) by $\mathcal{A}$. It can be written as a weighted pseudodifferential operator over the half-line $\mathbf{R}_+$ with an operator-valued symbol $a(r,\zeta)$ taking its values in the space $\mathcal{L}(H_1,H_2)$ where

$$H_1 = H^s(\mathbf{R}^q \times B),$$
$$H_2 = H^{s-m}(\mathbf{R}^q \times B) \oplus (\oplus H^{s-m_i-1/2}(\mathbf{R}^q \times \partial B)),$$

cf. the remark after Proposition 12.3. In fact $\mathcal{A} = (\delta'(r))^m \mathrm{op}_{w,\gamma}(a(r,\zeta))$ where

$$a(r,\zeta) = \begin{pmatrix} \sum_{|\beta|+j\le m} a_{\beta,j}(y,r)\zeta^j \mathbf{D}_y^\beta \\ \oplus \sum_{|\beta|+j\le m_i} r_{\partial B} \circ b_{i,\beta,j}(y,r)\zeta^j \mathbf{D}_y^\beta \end{pmatrix}.$$

Consider the weight functions

$$\begin{aligned}\lambda_1(r,\varrho) &= ((1+(\delta'(r))^{-2}\Delta_y+\varrho^2)^{1/2}+\Lambda_B)^s,\\ \lambda_2(r,\varrho) &= ((1+(\delta'(r))^{-2}\Delta_y+\varrho^2)^{1/2}+\Lambda_B)^{s-m}\\ &\quad \oplus (\oplus\lambda_{s-m_i-1/2}(r,\varrho))\end{aligned}$$

where $\Lambda_B : H^\sigma(B) \to H^{\sigma-1}(B)$, $\sigma > -1/2$, is the order-reducing operator described in Example 9.5 and

$$\lambda_{s-m_i-1/2}(r,\varrho) = (1+(\delta'(r))^{-2}\Delta_y+\varrho^2+\Delta_{\partial B})^{[(s-m_i-(1/2))/2]},$$

cf. (16.5). If $\delta(r)$ meets the first condition of (12.3), then

$$\lambda_1(r,\varrho) \in \Lambda_w(H_1,\tilde{H}_1), \qquad \lambda_2(r,\varrho) \in \Lambda_w(H_2,\tilde{H}_2)$$

where

$$\begin{aligned}\tilde{H}_1 &= L^2(\mathbf{R}^q \times B),\\ \tilde{H}_2 &= L^2(\mathbf{R}^q \times B) \oplus (\oplus L^2(\mathbf{R}^q \times \partial B)).\end{aligned}$$

It is a standard matter to verify that $a(r,\varrho+i\gamma) \in \mathcal{S}_{w,sv}(\lambda_1,\lambda_2)$ for each $\lambda \in \mathbf{R}$.

Having disposed of this preliminary step, we now turn to the problem of local invertibility of the boundary value problem $\mathcal{A}$ near the edge $\mathbf{R}^q$, i.e., at $r=0$.

We shall make two standing assumptions on the functions $t=\delta(r)$ under consideration, namely (9.1) and (16.6). These guarantee that both $\lambda_1(r,\varrho)$ and $\lambda_2(r,\varrho)$ vary slowly as $r \to 0$.

From Proposition 12.4 it follows that the operator $\mathcal{A}$ can be thought of as acting from $H_w(\lambda_1;\gamma,\mu)$ to $H_w(\lambda_2;\gamma,\mu-m)$.

In this section we discuss conditions of local invertibility for $\mathcal{A}$ with coefficients "frozen" at any point $y^0 \in \mathbf{R}^q$. Similarly to Section 16, we write $\mathcal{A}_{y^0}$ for this operator. It still belongs to $\mathcal{OPS}_{w,sv}(\lambda_1,\lambda_2)$.

It was Feigin [**9**] who observed that the local invertibility of $\mathcal{A}$ at $r=0$ is controlled by the "compressed" symbol of $\mathcal{A}$ with respect to action in y and r. It is defined by

$$\sigma(\mathcal{A})(y,r;\eta,\varrho) = \begin{pmatrix} \sum_{|\beta|+j\le m} a_{\beta,j}(y,r)\eta^\beta\zeta^j \\ \oplus \sum_{|\beta|+j\le m_i} r_{\partial B}\circ b_{i,\beta,j}(y,r)\eta^\beta\zeta^j \end{pmatrix} \tag{19.3}$$

for $(y, r; \eta, \varrho) \in T^*(\mathbf{R}^q \times \mathbf{R}_+)$. Thus,

$$\sigma(\mathcal{A})(y, r; \eta, \varrho) : H^s(B) \longrightarrow \begin{array}{c} H^{s-m}(B) \\ \oplus \\ \oplus H^{s-m_i-1/2}(\partial B) \end{array}$$

is a C^∞ function on $T^*(\mathbf{R}^q \times \mathbf{R}_+)$ taking its values in the space of boundary value problems on B.

We also apply the symbol mapping σ to our weight functions $\lambda_j(r, \varrho)$ for $j = 1, 2$, by

$$\sigma(\lambda_1)(\eta, \varrho) = (\langle \eta, \varrho \rangle + \Lambda_B)^s,$$
$$\sigma(\lambda_2)(\eta, \varrho) = (\langle \eta, \varrho \rangle + \Lambda_B)^{s-m} \oplus \left(\oplus (\langle \eta, \varrho \rangle^2 + \Delta_{\partial B})^{\frac{s-m_i-(1/2)}{2}} \right).$$

Theorem 19.1. *Let $s \in \mathbf{Z}_+$ satisfy $s > \max m_i$ and $\gamma, \mu \in \mathbf{R}$. Then $\mathcal{A}_{y^0}$ acting as in* (19.2) *is locally invertible at $r = 0$ if and only if there exists $\varepsilon > 0$ such that $\sigma(\mathcal{A})(y^0, r; \eta, \varrho + i\gamma)$ is invertible for all $(r, \eta, \varrho) \in (0, \varepsilon) \times \mathbf{R}^{q+1}$ and*

(19.4)
$$\sup_{(0,\varepsilon)\times\mathbf{R}^{q+1}} \|\sigma(\lambda_1)(\eta, \varrho)\sigma(\mathcal{A})^{-1}(y^0, r; \eta, \varrho + i\gamma)\sigma(\lambda_2)^{-1}(\eta, \varrho)\| < \infty$$

where $\|\cdot\|$ means the operator norm in $\mathcal{L}(L^2(B) \oplus (\oplus L^2(\partial B)), L^2(B))$.

Proof. Applying the Fourier transform in $y \in \mathbf{R}^q$ shows that the estimate (16.8) just amounts to the estimate

$$\sup_{(0,\varepsilon)\times\mathbf{R}} \|\lambda_1(r, \varrho)\sigma^{-1}_{\mathcal{A}_{y^0}}(r, \varrho + i\gamma)\lambda_2^{-1}(r, \varrho)\|_{\mathcal{L}(\tilde{H}_2, \tilde{H}_1)} < \infty$$

adapting (13.2) to our problem. Moreover, our assumptions on δ guarantee that the hypotheses of Theorem 13.1 are fulfilled. Consequently, the desired conclusion follows from Theorem 13.1. □

For elliptic boundary value problems $\mathcal{A}$, the weight functions $\lambda_1(r, \varrho)$ and $\lambda_2(r, \varrho)$ can be removed from the condition (19.4).

Corollary 19.2. *Suppose $\sigma(\mathcal{A})(y^0, r; \eta, \varrho+i\gamma)$ is an elliptic boundary value problem on B with parameter $(n, \varrho) \in \mathbf{R}^{q+1}$, uniformly in $r > 0$ small enough. Then the operator $\mathcal{A}_{y^0}$ is locally invertible at $r = 0$ if and only if there exists $\varepsilon > 0$ such that $\sigma(\mathcal{A})(y^0, r; \eta, \varrho+i\gamma)$ is invertible for all $(r, \eta, \varrho) \in (0, \varepsilon) \times \mathbf{R}^{q+1}$, and*

$$\sup_{r\in(0,\varepsilon)} \|\sigma(\mathcal{A})^{-1}(y^0, r; \eta, \varrho + i\gamma)\| < \infty, \tag{19.5}$$

$\|\cdot\|$ meaning the operator norm in $\mathcal{L}(H^{s-m}(B) \oplus (\oplus H^{s-m_i-1/2}(\partial B)), H^s(B))$.

Proof. As is shown in the proof of Corollary 16.5, estimate (19.4) implies estimate (19.5). On the other hand, the latter implies the former for (η, ϱ) on any ball in $\mathbf{R}^{q+1}$ while for $|\eta, \varrho|$ large enough the estimate (19.4) is a consequence of the parameter-dependent ellipticity. $\square$

As the symbols of differential boundary value problems are polynomials in ζ, Theorem 13.2 applies to the operator $\mathcal{A}_{y^0}$. This results in an exponential estimate for solutions of the problem with coefficients frozen along the edge.

Theorem 19.3. *Under the hypotheses of Theorem* 19.1*, let moreover $\sigma(\mathcal{A})(y^0, r; \eta, \zeta)$ be invertible for all $r \in (0, \varepsilon)$ and $(n, \zeta) \in \mathbf{R}^q \times (\mathbf{R} + i(a, b))$ and the estimate* (19.4) *hold uniformly with respect to γ on compact intervals in (a, b). Then any function $u \in H^{s,a+0,\mu}(W)$ satisfying $\mathcal{A}_{y^0} u = f$ with $f \in H^{s-m,b-0,\mu}(W) \oplus (\oplus H^{s-m_i-1/2,b-0,\mu-m_i}(\partial W))$ near $r = 0$, is actually of class $H^{s,b-0,\mu}(W)$ near $r = 0$.*

As but two examples we consider the classical Dirichlet and Neumann problems for the Schrödinger operator.

Example 19.4. Suppose that the function $f(r)$ specifying the degeneracy of W along the edge meets (15.9). As mentioned, this implies $f(0) = 0$. The Dirichlet problem in W is

$$\begin{cases} \Delta u = f & \text{in } W, \\ u = u_0 & \text{on } \partial W \setminus (\mathbf{R}^q \times \{0\}), \end{cases}$$

where f and u_0 are given functions on W and the smooth part of ∂W, respectively. The operator $\mathcal{A}$ of this boundary value problem can be thought of as acting from $H^{s,\gamma,\mu}(W)$ to $H^{s-2,\gamma,\mu-2}(W) \oplus H^{s-1/2,\gamma,\mu}(\partial W)$ for each integer $s > 0$ and $\gamma, \mu \in \mathbf{R}$. By Proposition 15.2, the compressed symbol of $\mathcal{A}$ is given by

$$\sigma(\mathcal{A})(y, r; \eta, \zeta) = \begin{pmatrix} |\eta|^2 + \zeta^2 + (\phi(r))^{-2}\Delta_B \\ r_{\partial B} \end{pmatrix}$$

modulo operators of small local norm as $r \to 0$, $\phi(r)$ being the function controlling the oscillation of W near the edge. Denote by $k^2 > 0$ the first eigenvalue of the Laplacian Δ_B acting on functions $u \in H^s(B)$ which satisfy $r_{\partial B} u = 0$. Then Corollary 19.2 enables us to conclude that the operator $\mathcal{A}$ is locally invertible near the edge $r = 0$ if

$$-\frac{k}{\varphi_0} < \gamma < \frac{k}{\varphi_0},$$

where

$$\phi_0 = \liminf_{r \to 0} \phi(r) > 0.$$

Moreover, Theorem 19.3 shows that if $-k/\phi_0 < \gamma' < \gamma'' < k/\phi_0$, then any solution $u \in H^{s,\gamma',\mu}(W)$ of the homogeneous problem $\mathcal{A}u = 0$ actually belongs to $H^{\infty,\gamma'',\mu}(W)$.

Note that the Neumann problem for the Laplacian in the canonical domain is not locally invertible at $r = 0$ in any space $H^{s,\gamma,\mu}(W)$, for 0 is an eigenvalue of the Laplacian Δ_B acting on functions u which satisfy $(\partial/\partial n)u = 0$ on the boundary of B.

Example 19.5. Let us consider the boundary value problem

$$\begin{cases} (\Delta + p(r))u = f & \text{in } W, \\ \partial u/\partial n = u_1 & \text{on } \partial W \setminus (\mathbf{R}^q \times \{0\}), \end{cases}$$

where f and u_1 are given functions on W and the smooth part of ∂W, respectively. We assume that the potential $p(r)$ is of the form $p(r) = (\delta'(r))^2 a(r)$, with $a(r)$ meeting the conditions of Example 16.7. In particular, we require

$$\liminf_{r \to 0} a(r) = k^2 > 0.$$

The operator $\mathcal{A}$ of this boundary value problem can be regarded as acting from $H^{s,\gamma,\mu}(W)$ to $H^{s-2,\gamma,\mu-2}(W) \oplus H^{s-3/2,\gamma,\mu-1}(\partial W)$ for each integer $s > 1$ and $\gamma, \mu \in \mathbf{R}$. Once again we invoke Proposition 15.2 to see that the compressed symbol of $\mathcal{A}$ is given by

$$\sigma(\mathcal{A})(y,r;\eta,\zeta) = \begin{pmatrix} |\eta|^2 + \zeta^2 + (\phi(r))^{-2}\Delta_B + a(r) \\ r_{\partial B} \circ (\partial/\partial n) \end{pmatrix}$$

up to an operator of small local norm as $r \to 0$. If $|\gamma| < k$, then the operator-valued function

$$\sigma(\mathcal{A})(y,r;\eta,\varrho + i\gamma) : H^s(B) \longrightarrow \begin{matrix} H^{s-2}(B) \\ \oplus \\ H^{s-3/2}(\partial B) \end{matrix}$$

is invertible for all $(\eta, \varrho) \in \mathbf{R}^{q+1}$ uniformly in $r \in \mathbf{R}_+$, and the inverse fulfills (19.5). Thus, Corollary 19.2 applies to show that the Neumann problem for the Schrödinger operator on W, when posed in any weighted space $H^{s,\gamma,\mu}(W)$ with $\gamma \in (-k, k)$, is locally invertible near the edge $r = 0$.

20. Fredholm property of a boundary value problem in a domain with cuspidal wedges. We now turn to boundary value problems in a domain $\mathcal{D} \subset \mathbf{R}^N$, $N = q + n + 1$, with a cuspidal edge S of dimension q on the boundary, as is described in Section 18.

Blowing up $\mathcal{D}$ along the edge S by using the cylindrical coordinates (r, y, θ) near S does not remove the singularity. What we obtain in this way is still a domain with edges on the boundary. However, this new domain is of product type close to the singularities on the boundary. In fact, it bears the structure of a fibered bundle over the edge S whose typical fiber is the semi-cylinder $\mathbf{R}_+ \times B$ over B. This "resolution of singularities" simplifies the analysis of pseudodifferential operators near S in $\mathcal{D}$.

We consider a boundary value problem

$$\begin{cases} Au = f & \text{in } \mathcal{D}, \\ B_i u = u_i & \text{on } \partial\mathcal{D} \setminus S, \end{cases} \tag{20.1}$$

where A is a differential operator in $\mathcal{D}$ and (B_i) a system of differential operators defined near $\partial\mathcal{D} \setminus S$ in $\mathcal{D}$. The coefficients of both A and (B_i) are assumed to be C^∞ functions up to the smooth part of $\partial\mathcal{D}$.

The pull-backs of A and (B_i) under cylindrical coordinates (15.2) close to S are of the form

$$\pi^\sharp A = (\delta'(r))^m \sum_{|\beta|+j\le m} a_{\beta,j}(y,r)\mathbf{D}_y^\beta \mathbf{D}_r^j,$$
$$\pi^\sharp B_i = (\delta'(r))^{m_i} \sum_{|\beta|+j\le m_i} b_{i,\beta,j}(y,r)\mathbf{D}_y^\beta \mathbf{D}_r^j,$$

where $a_{\beta,j}$ is a C^∞ function of $(y,r) \in B(0,\varepsilon)\times(0,\varepsilon)$ whose values are differential operators of order $m-|\beta|-j$ on B, and $b_{i,\beta,j}$ is a C^∞ function of $(y,r)\in B(0,\varepsilon)\times(0,\varepsilon)$ whose values are differential operators of order $m_i-|\beta|-j$ in a neighborhood of ∂B. We require the coefficients $a_{j,\alpha}$ and $b_{i,j,\alpha}$ to satisfy (16.2) uniformly in y on compact subsets of $B(0,\mathcal{E})$ and $r\in(0,\varepsilon]$, $\varepsilon<\mathcal{E}$.

For $s\ge 0$ and $\gamma,\mu\in\mathbf{R}$, the weighted Sobolev spaces $H^{s,\gamma,\mu}(\mathcal{D})$ on $\mathcal{D}$ are introduced by gluing together the local Sobolev spaces $H^s_{\mathrm{loc}}(\mathcal{D})$ with the weighted Sobolev spaces $H^{s,\gamma,\mu}(W)$ of (16.3).

If $s>1/2$, then $H^{s-1/2,\gamma,\mu}(\partial\mathcal{D})$ stands for the space formed by the traces on $\partial\mathcal{D}\setminus S$ of functions in $H^{s,\gamma,\mu}(\mathcal{D})$. It is a Hilbert space under the canonical quotient norm. When regarded as a normal space, $H^{s-1/2,\gamma,\mu}(\partial\mathcal{D})$ coincides with the weighted Sobolev space on the surface $\partial\mathcal{D}$ defined in Section 17.

Assuming $s>\max m_i+1/2$, we assign an operator

$$(20.2)\quad \mathcal{A} = \begin{pmatrix} A \\ \oplus r_{\partial\mathcal{D}} B_i\end{pmatrix} : H^{s,\gamma,\mu}(\mathcal{D}) \longrightarrow \begin{matrix} H^{s-m,\gamma,\mu-m}(\mathcal{D}) \\ \oplus \\ \oplus H^{s-m_i-1/2,\gamma,\mu-m_i}(\partial\mathcal{D})\end{matrix}$$

to the boundary value problem (20.1), $r_{\partial\mathcal{D}}$ being restricted to (the smooth part of) the boundary of $\mathcal{D}$.

For any point $p\in S$, we can write $\mathcal{A}$ in the cylindrical coordinates near p in $\mathcal{D}$ and define the "compressed" symbol of $\mathcal{A}$ by formula (19.3). Analysis similar to that in Section 17 actually shows that the invertibility of $\sigma(\mathcal{A})(p,r;\eta,\varrho)$ is independent of the particular choice of local coordinates near p to evaluate $\sigma(\mathcal{A})$.

Theorem 20.1. *Let s be an integer satisfying $s>\max m_i$ and $\gamma,\mu\in\mathbf{R}$. Then the operator* (20.2) *is Fredholm if*:

1) $\mathcal{A}$ *is an elliptic boundary value problem away from the edge* S *on the boundary of* $\mathcal{D}$ *and*

2) *for each* $p \in S$ *there exists* $\varepsilon > 0$ *such that* $\sigma(\mathcal{A})(p, r; \eta, \zeta)$ *is an invertible operator on* B *for any* $(r, \eta, \zeta) \in (0, \mathcal{E}) \times \mathbf{R}^q \times (\mathbf{R} + i\gamma)$ *and the inverse satisfies* (19.4).

Proof. The proof is based on the standard localization procedure and Theorem 19.1, cf. the proof of Theorem 17.1. □

If $\sigma(\mathcal{A})(y, r; \eta, \varrho+i\gamma)$ is an elliptic boundary value problem on B with parameter $(\eta, \varrho) \in \mathbf{R}^{q+1}$, uniformly in $r > 0$ small enough, then condition 2) of Theorem 20.1 just amounts to saying that $\sigma(\mathcal{A})(p, r; \eta, \varrho+i\gamma)$ is an invertible operator on B for all $(p, r) \in S \times (0, \mathcal{E})$ and $(\eta, \varrho) \in \mathbf{R}^{q+1}$, and the inverse satisfies

$$\sup_{r \in (0,\varepsilon)} \|\sigma(\mathcal{A})^{-1}(p, r; \eta, \varrho + i\gamma)\| < \infty, \tag{20.3}$$

where $\|\cdot\|$ means the norm in $\mathcal{L}(H^{s-m}(B) \oplus (\oplus H^{s-m_i-1/2}(\partial B)), H^s(B))$. If, moreover, the condition (20.3) holds uniformly in γ on compact intervals in (a, b), then the conclusion of Theorem 19.3 is valid with W replaced by $\mathcal{D}$ and $\mathcal{A}_{y^0}$ replaced by $\mathcal{A}$.

Example 20.2. Consider the Dirichlet problem

$$\begin{cases} \Delta u = f & \text{in } \mathcal{D}, \\ u = u_0 & \text{on } \partial\mathcal{D} \setminus S, \end{cases}$$

where f and u_0 are given functions on $\mathcal{D}$ and $\partial\mathcal{D} \setminus S$, respectively. The operator $\mathcal{A}$ of this boundary value problem can be regarded as acting from $H^{s,\gamma,\mu}(\mathcal{D})$ to $H^{s-2,\gamma,\mu-2}(\mathcal{D}) \oplus H^{s-1/2,\gamma,\mu}(\partial\mathcal{D})$ for any integer $s > 0$ and $\gamma, \mu \in \mathbf{R}$. Combining Theorem 20.1 with Example 19.4 we deduce that the Dirichlet problem is Fredholm if

$$-\frac{k}{\varphi_0} < \gamma < \frac{k}{\varphi_0},$$

where $k^2 > 0$ is the first eigenvalue of the Laplacian Δ_B which acts on functions $u \in H^s(B)$ satisfying $r_{\partial B} u = 0$ and $\phi_0 = \liminf_{r \to 0} \phi(r)$.

Furthermore, any solution $u \in H^{s,-(k/\phi_0)+0,\mu}(M)$ of the homogeneous problem $\mathcal{A}u = 0$ is actually of the class $H^{\infty,(k/\phi_0)-0,\mu}(M)$.

We leave it to the reader to examine in a similar way the Neumann problem for the Schrödinger operator in $\mathcal{D}$, cf. Example 19.5.

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Appendix

A. Hardy-Littlewood inequality on a half-line.

Lemma A.1. *Let $\Phi(t)$ be a C^2 function on a ray (T,∞) with values in a Banach space H such that*

$$\|\Phi^{(j)}\|_{L^\infty((T,\infty),H)} = \sup_{t\in(T,\infty)} \|\Phi^{(j)}(t)\|_H$$

is finite for $j = 0, 1, 2$. Then

$$\|\Phi'\|_{L^\infty((T,\infty),H)} \le \sqrt{8}\sqrt{\|\Phi\|_{L^\infty((T,\infty),H)}\|\Phi''\|_{L^\infty((T,\infty),H)}}.$$

Proof. We have

$$(e^{-t}\Phi'(t))' = e^{-t}\Phi''(t) - e^{-t}\Phi'(t),$$

and so integration by parts gives

$$\begin{aligned} -e^{-t}\Phi'(t) &= \int_t^\infty e^{-\theta}\Phi''(\theta)\,d\theta - \int_t^\infty e^{-\theta}\Phi'(\theta)\,d\theta \\ &= \int_t^\infty e^{-\theta}\Phi''(\theta)\,d\theta + e^{-t}\Phi(t) - \int_t^\infty e^{-\theta}\Phi(\theta)\,d\theta, \end{aligned}$$

for $t \in (T,\infty)$. Hence it follows that

$$-\Phi'(t) = \int_t^\infty e^{t-\theta}\Phi''(\theta)\,d\theta + \Phi(t) - \int_t^\infty e^{t-\theta}\Phi(\theta)\,d\theta.$$

Let us introduce the function

$$k(t) = \begin{cases} e^t & \text{if } t \leq 0, \\ 0 & \text{if } t > 0; \end{cases}$$

then the above equality for Φ' can be rewritten in the form

$$\Phi'(t) = (k * \Phi)(t) - \Phi(t) - (k * \Phi'')(t),$$

for $t \in (T, \infty)$. Since $\|k\|_{L^1(\mathbf{R})} = 1$, the well-known estimate for convolutions yields

$$\|\Phi'\|_{L^\infty((T,\infty),H)} \leq 2\|\Phi\|_{L^\infty((T,\infty),H)} + \|\Phi''\|_{L^\infty((T,\infty),H)}.$$

We now apply this estimate to the family of functions $\Phi(T+\lambda(t-T))$ on (T, ∞) parametrized by $\lambda \in \mathbf{R}_+$. This gives

$$\lambda\|\Phi'\|_{L^\infty((T,\infty),H)} \leq 2\|\Phi\|_{L^\infty((T,\infty),H)} + \lambda^2\|\Phi''\|_{L^\infty((T,\infty),H)},$$

or

$$\|\Phi'\|_{L^\infty((T,\infty),H)} \leq \frac{2}{\lambda}\|\Phi\|_{L^\infty((T,\infty),H)} + \lambda\|\Phi''\|_{L^\infty((T,\infty),H)},$$

for any $\lambda > 0$.

Taking the minimum over $\lambda > 0$ on the righthand side, we arrive at the desired estimate. $\square$

ENDNOTES

1. Note that $\overset{\circ}{H}{}^s(\overline{\mathbf{R}}^n_+)$ cannot be thought of as a subspace of $H^s(\overline{\mathbf{R}}^n_+)$ because the natural mapping of the former to the latter is not injective unless $s > -1/2$. Indeed the surface layer on $\partial\mathbf{R}^n_\pm$ belongs to $H^s(\mathbf{R}^n)$ if $s < -1/2$.

2. In the Appendix we give an independent proof of the Hardy-Littlewood inequality on the half-line.

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