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# EVERY ABSOLUTELY HENSTOCK-KURZWEIL INTEGRABLE FUNCTION IS MCSHANE INTEGRABLE: AN ALTERNATIVE PROOF

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ABSTRACT. We give an alternative proof of the well-known result that every absolutely Henstock-Kurzweil integrable function is McShane integrable.

**1. Introduction.** It is well known that the Lebesgue integral is a proper extension of the Riemann integral. In the late 1950s, Henstock [**4**] and Kurzweil [**6**] independently gave a slight, but ingenious, modification of the classical Riemann integral to obtain a Riemann-type definition of the Perron integral. This integral is now commonly known as the Henstock-Kurzweil integral [**9, 12**], the Kurzweil-Henstock integral [**8, 14**], the gauge integral [**13**] or the Henstock integral [**1, 3, 7**], and we shall use the term "Henstock-Kurzweil integral." Later, McShane [**10**] modified the Henstock-Kurzweil integral to yield a Riemann-type definition of the Lebesgue integral, which is also commonly referred to as the McShane integral [**1, 3, 7, 8, 12**–**14**]. It turns out that $f$ and $|f|$ are both Henstock-Kurzweil integrable on a compact subinterval $E$ of the real line if and only if $f$ is McShane integrable there. In 1980 Pfeffer in [**11**, p. 46] proposed a problem to prove, using only the definitions of Henstock-Kurzweil and McShane integrals, that absolutely Henstock-Kurzweil integrable functions are McShane integrable. Since then a fairly large number of proofs have been offered. See, for example, [**1, 3, 7, 8, 13, 14**]. However, their proofs either involve convergence theorems or the existing techniques rely heavily on the real-valued property of integrable functions. In this paper we give an alternative proof of the above result which is also valid for Banach-valued integrable functions satisfying the Saks-Henstock lemma. Moreover our method, unlike the existing known proofs, uses neither the measurability of the integrand nor convergence theorems.

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**2. Preliminaries.** Unless stated otherwise, the following conventions and notations will be used. The set of all real numbers is denoted by $\mathbf{R}$, and the ambient space of this paper is $\mathbf{R}^m$, where $m$ is a fixed positive integer. The norm in $\mathbf{R}^m$ is the maximum norm $|||\cdot|||$. For $x \in \mathbf{R}^m$ and $r > 0$, set $B(x,r) := \{y \in \mathbf{R}^m : |||y-x||| < r\}$. Let $E := \prod_{i=1}^m [a_i, b_i]$ be a fixed interval in $\mathbf{R}^m$. For a set $A \subseteq E$, we denote by $\chi_{_A}$, $\operatorname{diam}(A)$ and $\mu_m(A)$ the characteristic function, diameter and $m$-dimensional Lebesgue outer measure of $A$, respectively. Moreover, we denote its interior and closure of $A \subseteq E$ with respect to the subspace topology of $E$ by $\operatorname{int}(A)$ and $\overline{A}$, respectively. The distance between $Y \subseteq E$ and $Z \subseteq E$ will be denoted by $\operatorname{dist}(Y,Z)$. A set $A \subset E$ is called *negligible* whenever $\mu_m(A) = 0$. We say that two sets are nonoverlapping if their intersection is negligible. Let $X$ be a Banach space equipped with norm $\|\cdot\|$. A function is always $X$-valued. When no confusion is possible, we do not distinguish between a function defined on a set $Z$ and its restriction to a set $W \subset Z$.

An *interval* in $\mathbf{R}^m$ is the cartesian product of $m$ nondegenerate compact intervals in $\mathbf{R}$. $\mathcal{I}$ denotes the family of all nondegenerate subintervals of $E$. If $I \in \mathcal{I}$, we shall write $\mu_m(I)$ as $|I|$. For each $J \in \mathcal{I}$, the *regularity* of an $m$-dimensional interval $J \subseteq E$, denoted by $\operatorname{reg}(J)$, is the ratio of its shortest and longest sides. A function $F$ defined on $\mathcal{I}$ is said to be additive if $F(I \cup J) = F(I) + F(J)$ for each nonoverlapping interval $I, J \in \mathcal{I}$ with $I \cup J \in \mathcal{I}$. In particular, if we follow the proof of [**8**, Corollary 6.2.4], then we can verify that if $F$ is an additive interval function on $\mathcal{I}$ with $J \in \mathcal{I}$, and $\{K_1, K_2, \dots, K_r\}$ is a collection of nonoverlapping subintervals of $J$ with $\bigcup_{i=1}^r K_i = J$, then

$$F(J) = \sum_{i=1}^{r} F(K_i).$$

A *partition* $P$ is a finite collection $\{(I_i, \xi_i)\}_{i=1}^p$, where $I_1, I_2, \dots, I_p$ are nonoverlapping subintervals of $E$. Given $Z \subseteq E$, a positive function $\delta$ on $Z$ is called a *gauge* on $Z$. We say that a partition $\{(I_i, \xi_i)\}_{i=1}^p$ is

(i) a partition *in* $Z$ if $\bigcup_{i=1}^p I_i \subseteq Z$,

(ii) a partition *of* $Z$ if $\bigcup_{i=1}^p I_i = Z$,

(iii) *anchored* in $Z$ if $\{\xi_1, \xi_2, \dots, \xi_p\} \subset Z$,

(iv) $\delta$-*fine* if $I_i \subset B(\xi_i, \delta(\xi_i))$ for each $i = 1, 2, \dots, p$,

(v) *Perron* if $\xi_i \in I_i$ for each $i = 1, 2, \ldots, p$,

(vi) *McShane* if $\xi_i$ need not belong to $I_i$ for all $i = 1, 2, \ldots, p$.

According to Cousin's lemma [**8**, Lemma 6.2.6], for any given gauge $\delta$ on $E$, $\delta$-fine Perron partitions of $E$ exist. Hence the following definition is meaningful.

**Definition 2.1.** A function $f : E \to X$ is said to be strongly Henstock-Kurzweil integrable (respectively strongly McShane integrable) on $E$ if there exists an additive interval function $F : \mathcal{I} \to X$ with the following property: for each $\epsilon > 0$ there exists a gauge $\delta$ on $E$ such that

$$\sum_{i=1}^{p} \|f(\xi_i)|I_i| - F(I_i)\| < \varepsilon$$

for each $\delta$-fine Perron partition (respectively $\delta$-fine McShane partition) $\{(I_i, \xi_i)\}_{i=1}^{p}$ in $E$. The function $F$ is called the indefinite strong Henstock-Kurzweil integral (respectively indefinite strong McShane integral) of $f$ on $E$.

**Remark 2.2**. When $X = \mathbf{R}$, the reader can verify that Definition 2.1 is equivalent to the classical definition of the Henstock-Kurzweil integral. A similar result also holds for the McShane integral.

For additional properties of the Henstock-Kurzweil integral, the reader may wish to consult, for example, [**8, 13**] or [**9**]. Unless stated otherwise, for the rest of this paper, every integral of real-valued function will be understood as a Henstock-Kurzweil integral.

**Theorem 2.3.** *If $f : E \to X$ is strongly Henstock-Kurzweil integrable on $E$, and $F$ is the indefinite strong Henstock-Kurzweil integral of $f$, then for $\mu_m$-almost all $x \in E$, given $\varepsilon > 0$ there exists $\delta_0(x) > 0$ such that*

$$\left\| f(x) - \frac{F(J)}{|J|} \right\| < \varepsilon$$

*whenever $x \in J \in \mathcal{I}$ with $\operatorname{diam}(J) < \delta_0(x)$ and $\operatorname{reg}(J) = 1$. In particular, $f$ is strongly measurable.*

*Proof.* The proof is similar to the case for real-valued Henstock-Kurzweil integrable functions. See, for example, [**5**, Note 1.5, Theorem 2.8].

## 3. Main results.

**Definition 3.1.** A function $f : E \to X$ is absolutely strongly Henstock-Kurzweil integrable on $E$ if $f$ is strongly Henstock-Kurzweil integrable on $E$, and $\|f\|$ is Henstock-Kurzweil integrable on $E$.

Our aim is to prove every absolutely strongly Henstock-Kurzweil integrable function $f$ on $E$ is also strongly McShane integrable there. Moreover, the indefinite strong integrals coincide.

**Theorem 3.2.** *If $f : E \to X$ is absolutely strongly Henstock-Kurzweil integrable on $E$, and $F$ is the indefinite strong Henstock-Kurzweil integral of $f$, then the inequality*

$$\|F(I)\| \leq \int_I \|f\|$$

*holds for each $I \in \mathcal{I}$.*

*Proof.* The proof is similar to the case for real-valued absolutely Henstock-Kurzweil integrable functions.

**Definition 3.3.** An additive interval function $F$ on $\mathcal{I}$ is said to be strongly absolutely continuous if given $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\sum_{i=1}^{q} \|F(I_i)\| < \varepsilon$$

whenever $\{I_1, I_2, \dots, I_q\}$ is a collection of nonoverlapping subintervals of $E$ with $\sum_{i=1}^{q} |I_i| < \eta$.

In order to proceed further, we need to prove the following result: if $F$ is the indefinite strong Henstock-Kurzweil integral of an absolutely strongly Henstock-Kurzweil integrable function on $E$, then $F$

is strongly absolutely continuous on $\mathcal{I}$. Unlike the existing classical proofs, our method uses neither the strong measurability of $f$ nor convergence theorems. In particular, our method does not depend on the truncations of $f$. We need a lemma.

**Lemma 3.4.** *If $g : E \to \mathbf{R}$ is a nonnegative Henstock-Kurzweil integrable function on $E$, and $X \subseteq E$ is closed, then $g\chi_X$ is Henstock-Kurzweil integrable on $E$.*

*Proof.* If $X = \varnothing$ or $X = E$, then the result is obvious, so we may assume that both sets $X$ and $E\backslash X$ are nonempty. Since $X$ is closed, $E\backslash X$ is relatively open in $E$. An application of [**2**, Lemma 2.43] shows that $E\backslash X$ can be written as countable union of nonoverlapping intervals $\{J_i\}_{i=1}^{\infty}$. Since $g$ is Henstock-Kurzweil integrable on $E$, it follows from [**8**, Theorem 6.4.2] that $g$ is also Henstock-Kurzweil integrable on each of the intervals $J_i$. Moreover, it follows from the additivity of the indefinite Henstock-Kurzweil integral of $g$ that

$$\begin{aligned}\sum_{i=1}^{\infty}\int_{J_i} g = \lim_{n\to\infty}\sum_{i=1}^{n}\int_{J_i} g = \lim_{n\to\infty}\int_E g\chi_{\cup_{i=1}^{n} J_i}\\ \leq \int_E g < \infty.\end{aligned} \tag{1}$$

Since $g$ is assumed to be Henstock-Kurzweil integrable on $E$ with $g\chi_X = g - g\chi_{E\backslash X}$, it remains to prove that $g\chi_{E\backslash X}$ is Henstock-Kurzweil integrable on $E$ with integral value $\sum_{i=1}^{\infty}\int_{J_i} g$.

Let $\varepsilon > 0$. In view of (1), we may fix a positive number $N$ satisfying

$$\sum_{i=N+1}^{\infty}\int_{J_i} g < \frac{\varepsilon}{2}. \tag{2}$$

Since $g$ is Henstock-Kurzweil integrable on $E$, there exists a gauge $\Delta$ on $E$ such that

$$\sum_{i=1}^{p}\left|g(x_i)|I_i'| - \int_{I_i'} g\right| < \frac{\varepsilon}{2}$$

whenever $\{(I_i', x_i)\}_{i=1}^p$ is a $\Delta$-fine Perron partition in $E$. Since $E\backslash X$ is relatively open in $E$, we may also assume that

$$E \cap B(x, \Delta(x)) \subset E\backslash X \tag{3}$$

whenever $x \in E\backslash X$, and

$$B(x, \Delta(x)) \cap \bigcup_{i=1}^{N} J_i = \varnothing \tag{4}$$

whenever $x \in E\backslash \cup_{i=1}^N J_i$.

Now, we let $Q = \{(I_i, \xi_i)\}_{i=1}^q$ be any $\Delta$-fine Perron partition of $E$. For such a partition, we let

$$T = \{i : (I_i, \xi_i) \in Q \text{ with } \xi_i \in E\backslash X\}.$$

In view of (3) and (4), we have

$$\bigcup_{i=1}^{N} J_i \subseteq \bigcup_{i \in T} I_i.$$

Since $g$ is assumed to be nonnegative and integrable on $E$, we have

$$\int_E g\chi_{\cup_{i=1}^N J_i} \le \int_E g\chi_{\cup_{i\in T} I_i}. \tag{5}$$

Consequently, it follows from our choice of $\Delta$, (5) and (2) that

$$\begin{aligned}
\Big|\sum_{i=1}^{q} g(\xi_i)\chi_{E\backslash X}(\xi_i)|I_i| - \sum_{i=1}^{\infty}\int_{J_i} g\Big| \\
&\le \Big|\sum_{i\in T}\Big\{g(\xi_i)|I_i| - \int_{I_i} g\Big\}\Big| + \sum_{i=1}^{\infty}\int_{J_i} g - \sum_{i\in T}\int_{I_i} g \\
&< \frac{\varepsilon}{2} + \sum_{i=1}^{\infty}\int_{J_i} g - \int_E g\chi_{\bigcup_{i=1}^N J_i} < \varepsilon,
\end{aligned}$$

completing the proof of the lemma.

**Theorem 3.5.** *If $f : E \to X$ is absolutely strongly Henstock-Kurzweil integrable on $E$, and $F$ is the indefinite strong Henstock-Kurzweil integral of $f$, then $F$ is strongly absolutely continuous on $\mathcal{I}$.*

*Proof.* Since $f$ is assumed to be absolutely strongly Henstock-Kurzweil integrable on $E$, for any given $\varepsilon > 0$ there exists a gauge $\delta_0$ on $E$ such that for any $\delta_0$-fine Perron partition $\{(I_i', x_i)\}_{i=1}^s$ in $E$, we have

$$\sum_{i=1}^{s} \|f(x_i)|I_i'| - F(I_i')\| < \frac{\varepsilon}{3} \tag{6}$$

and

$$\sum_{i=1}^{s} \left| \|f(x_i)\||I_i'| - \int_{I_i'} \|f\| \right| < \frac{\varepsilon}{21}. \tag{7}$$

For each positive integer $n$, set

$$Y_n = \left\{ x \in E : \|f(x)\| \leq n \text{ and } \delta_0(x) \geq \frac{1}{n} \right\},$$

and $X_n := \overline{Y_n}$. We claim that there exists a positive number $N$ such that

$$\int_E \|f\| - \int_E \|f\chi_{X_N}\| < \frac{\varepsilon}{3}. \tag{8}$$

For each positive integer $n$, let $g_n := \|f\chi_{X_n}\|$. Since $g_n(x) = \|f(x)\chi_{X_n}(x)\| = \|f(x)\|\chi_{X_n}(x)$ for each $x \in E$, it follows from Lemma 3.4 that $g_n$ is Henstock-Kurzweil integrable on $E$. Hence given $\varepsilon > 0$ there exists a gauge $\delta_n$ on $E$ such that

$$\sum_{i=1}^{r} \left| g_n(y_i)|K_i| - \int_{K_i} g_n \right| < \frac{\varepsilon}{(21)(2^n)} \tag{9}$$

for each $\delta_n$-fine Perron partition $\{(K_i, y_i)\}_{i=1}^r$ in $E$.

For each $\xi \in E$, we may assume that $\{\delta_n(\xi)\}_{n=0}^{\infty}$ is a decreasing sequence of positive numbers.

Since $\{g_n\}_{n=1}^{\infty}$ is an increasing sequence of nonnegative Henstock-Kurzweil integrable functions with $g_n(x) \leq g(x) := \|f(x)\|$ for all positive integers $n$ and $x \in E$, we have

$$\int_E g_n \leq C := \sup_{n\geq 1}\left\{\int_E g_n\right\} \leq \int_E g < \infty$$

from which we deduce that there exists a positive integer $N_0$ such that

$$C - \int_E g_n < \frac{\varepsilon}{21} \tag{10}$$

for all $n \geq N_0$.

For any given $\xi \in E$, we observe that $\{g_n(\xi)\}_{n=1}^{\infty}$ is a nondecreasing sequence of positive numbers with $\lim_{n\to\infty} g_n(\xi) = g(\xi)$. Hence there exists a positive integer $p(\xi) \geq N_0$ such that

$$g(\xi) - g_n(\xi) < \frac{\varepsilon}{21|E|} \tag{11}$$

for all positive integers $n \geq p(\xi)$. Define a gauge $\Delta$ on $E$ by

$$\Delta(\xi) = \delta_{p(\xi)}(\xi). \tag{12}$$

In view of Cousin's lemma [**8**, Lemma 6.2.6], we may fix a $\Delta$-fine Perron partition $Q = \{(I_i, \xi_i)\}_{i=1}^{q}$ of $E$, and put $N = \max\{p(\xi_i) : (I_i, \xi_i) \in Q\}$. Then for $n \geq N$, it follows from (7), (11), (10) and our choice of $\Delta$ that

$$\begin{aligned}
&\int_E \|f\| - \int_E \|f\chi_{X_n}\| \\
&= \sum_{i=1}^{q}\left\{\int_{I_i} g - \int_{I_i} g_n\right\} \\
&\leq \sum_{i=1}^{q}\left|g(\xi_i)|I_i| - \int_{I_i} g\right| + \sum_{i=1}^{q}\left|g_n(\xi_i)|I_i| - \int_{I_i} g_n\right| + \sum_{i=1}^{q}|g(\xi_i) - g_n(\xi_i)||I_i| \\
&< \frac{\varepsilon}{21} + \sum_{i=1}^{q}\left|g_n(\xi_i)|I_i| - \int_{I_i} g_n\right| + \frac{\varepsilon}{21}
\end{aligned}$$

$$
\begin{aligned}
&< \frac{\varepsilon}{21} + \sum_{i=1}^{q} \{g_n(\xi_i) - g_{p(\xi_i)}(\xi_i)\}|I_i| + \sum_{i=1}^{q} \left\{ \int_{I_i} g_n - \int_{I_i} g_{p(\xi_i)} \right\} \\
&\quad + \sum_{i=1}^{q} \left| g_{p(\xi_i)}(\xi_i)|I_i| - \int_{I_i} g_{p(\xi_i)} \right| + \frac{\varepsilon}{21} \\
&\leq \frac{\varepsilon}{21} + \frac{2\varepsilon}{21} + \int_E \{g_n - g_{N_0}\} + \sum_{n=1}^{\infty} \sum_{p(\xi_i)=n} \left| g_{p(\xi_i)}(\xi_i)|I_i| - \int_{I_i} g_{p(\xi_i)} \right| + \frac{\varepsilon}{21} \\
&< \frac{\varepsilon}{3},
\end{aligned}
$$

proving that (8) holds.

Now we set $\eta := \varepsilon/3N$. Given any finite collection $\{J_i\}_{i=1}^{p}$ of nonoverlapping subinterval of $E$ with $\sum_{i=1}^{p} |J_i| < \eta$, which we may assume that $\operatorname{diam}(J_i) < 1/N$ for each $i = 1, 2, \ldots, p$, we let

$$S_1 = \{i \in \{1, 2, \ldots, p\} : X_N \cap \operatorname{int}(J_i) \neq \varnothing\}$$

and

$$S_2 = \{i \in \{1, 2, \ldots, p\} : X_N \cap \operatorname{int}(J_i) = \varnothing\}.$$

If $i \in S_1$, it follows from the density of $Y_N$ in $X_N$ that we may choose and fix $x_i \in Y_N \cap \operatorname{int}(J_i)$. Then $\{(J_i, x_i)\}_{i=1}^{p}$ is a $1/N$-fine, and hence $\delta_0$-fine, Perron partition anchored in $Y_N$. Thus it follows from (6), our construction of $Y_N$, our choice of $\eta$, Theorem 3.2 and (8) that

$$
\begin{aligned}
\sum_{i=1}^{p} \|F(J_i)\| &\leq \sum_{i \in S_1} \|f(x_i)|J_i| - F(J_i)\| + \sum_{i \in S_1} \|f(x_i)\||J_i| + \sum_{i \in S_2} \|F(J_i)\| \\
&< \frac{\varepsilon}{3} + N\frac{\varepsilon}{3N} + \int_E [\|f\| - \|f\chi_{X_N}\|] \\
&< \frac{\varepsilon}{3} + N\frac{\varepsilon}{3N} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}
$$

proving that $F$ is strongly absolutely continuous on $\mathcal{I}$. The proof is complete.

In view of Remark 2.2, the next theorem generalizes the well-known classical theorem that every absolutely Henstock-Kurzweil integrable function is McShane integrable.

**Theorem 3.6.** *If $f : E \to X$ is absolutely strongly Henstock-Kurzweil integrable on $E$, then it is strongly McShane integrable there.*

*Proof.* Let $F$ denote the indefinite strong Henstock-Kurzweil integral of $f$ on $E$. Given $\varepsilon > 0$, choose a gauge $\delta_k$ on $E$ such that

$$\sum_{i=1}^{q} \|f(x_i)|J_i| - F(J_i)\| < \frac{\varepsilon}{2^{k+3}} \tag{13}$$

for each $\delta_k$-fine Perron partition $\{(J_i, x_i)\}_{i=1}^{q}$ in $E$.

By Theorem 2.3, there exists a negligible set $Z \subset E$ such that for each $\xi \in E\backslash Z$, there exists $\nu(\xi) > 0$ such that

$$\left\| f(\xi) - \frac{F(I)}{|I|} \right\| < \min\left\{ \frac{\varepsilon}{8}, \frac{\varepsilon}{8|E|} \right\} \tag{14}$$

whenever $\xi \in I \in \mathcal{I}$ with $\operatorname{diam}(I) < \nu(\xi)$ and $\operatorname{reg}(I) = 1$. We may also assume that $f(x) = 0$ for each $x \in Z$.

For each positive integer $k$, set

$$W_k = \left\{ x \in E\backslash Z : \|f(x)\| \le k \text{ and } \nu(x) \ge \frac{1}{k} \right\}$$

and $X_k := \overline{W_k}$. Choose an open set $G_k \supset X_k$ so that $\mu_m(G_k\backslash X_k) < \eta_k$, where

$$0 < \eta_k < \frac{\varepsilon}{(k + 2\varepsilon)2^{k+3}}$$

corresponds to

$$\frac{\varepsilon}{2^{k+3}}$$

in the definition of strong absolute continuity of $F$. Choose also an open set $O \supset Z$ so that $\mu_m(O) < \eta_1$.

For each positive integer $k$, set $V_k := X_k\backslash X_{k-1}$ with $X_0 = \varnothing$. We may also assume that $V_k$ is nonempty for all $k$. Now, we define a gauge $\Delta$ on $E$ by

$$\Delta(\xi) = \begin{cases} \min\{\nu(\xi), 1/k, \operatorname{dist}(\{\xi\}, (E\backslash G_k) \cup X_{k-1}), \delta_k(\xi)\} \\ \qquad\qquad\qquad \text{if } \xi \in V_k\backslash Z \text{ for some positive integer } k, \\ \operatorname{dist}(\{\xi\}, E\backslash O) \quad \text{if } \xi \in Z. \end{cases}$$

Let $P = \{(I_i, \xi_i)\}_{i=1}^p$ be any $\Delta$-fine McShane partition in $E$. If $P_k := \{(I, \xi) \in P : \xi \in V_k \backslash Z\}$ is nonempty, then we have

$$
\begin{aligned}
&\sum \big\{ \|f(\xi)|I| - F(I)\| : (I,\xi) \in P_k \big\} \\
&\qquad \le \sum \big\{ \|f(\xi)|I| - F(I)\| : (I,\xi) \in P_k \quad \text{with} \quad \xi \in I \big\} \\
&\qquad\quad + \sum \big\{ \|f(\xi)|I| - F(I)\| : (I,\xi) \in P_k \quad \text{with} \quad \xi \notin I \\
&\qquad\qquad\quad \text{and } (V_k \backslash Z) \cap \operatorname{int}(I) = \varnothing \big\} \\
&\qquad\quad + \sum \big\{ \|f(\xi)|I| - F(I)\| : (I,\xi) \in P_k \quad \text{with} \quad \xi \notin I \\
&\qquad\qquad\quad \text{and } (V_k \backslash Z) \cap \operatorname{int}(I) \neq \varnothing \big\} \\
&\qquad = \alpha_k + \beta_k + \gamma_k \qquad \text{(say)}.
\end{aligned}
\tag{15}
$$

By our choice of $\Delta$, $\Delta(\xi) \le \delta_k(\xi)$ for each $\xi \in V_k \backslash Z$, so the inequality

$$\alpha_k < \frac{\varepsilon}{2^{k+3}} \tag{16}$$

follows from (13). We shall next show that $\beta_k < 2\varepsilon/2^{k+3}$. Given that $(I, \xi) \in P_k$ and $x \in W_k \cap B(\xi, \Delta(\xi))$, we choose a 1-regular interval $K_{\xi,x} \subseteq B(\xi, \Delta(\xi))$ such that $\{\xi, x\} \subset K_{\xi,x}$. As $\Delta(\xi) \le \min\{\nu(\xi), 1/k\} \le \min\{\nu(\xi), \nu(x)\}$, it follows from (14) that

$$\|f(\xi) - f(x)\| \le \left\| f(\xi) - \frac{F(K_{\xi,x})}{|K_{\xi,x}|} \right\| + \left\| f(x) - \frac{F(K_{\xi,x})}{|K_{\xi,x}|} \right\| < \min\left\{ \frac{\varepsilon}{4}, \frac{\varepsilon}{4|E|} \right\}. \tag{17}$$

Hence it follows from (17), our choice of $G_k$, $\eta_k$ and the strong absolute continuity of $F$ that

$$\beta_k < \left( k + \frac{\varepsilon}{4} \right) \eta_k + \frac{\varepsilon}{2^{k+3}} < \frac{2\varepsilon}{2^{k+3}}. \tag{18}$$

For $\gamma_k$, we observe that for each $(I, \xi) \in P_k$ with $(V_k \backslash Z) \cap \operatorname{int}(I) \neq \varnothing$, we invoke the density of $W_k$ in $V_k \backslash Z$ to select and fix $x_{\xi,I} \in$

$(W_k \backslash W_{k-1}) \cap \operatorname{int}(I)$. Then it follows from (17) and (13) that

$$
\begin{aligned}
\gamma_k \leq & \sum \big\{ \|f(\xi) - f(x_{\xi,I})\| |I| : (I, \xi) \in P_k \quad \text{with} \quad \xi \notin I \\
& \qquad \text{and } (V_k \backslash Z) \cap \operatorname{int}(I) \neq \varnothing \big\} \\
& + \sum \big\{ \|f(x_{\xi,I})|I| - F(I)\| : (I, \xi) \in P_k \quad \text{with} \quad \xi \notin I \\
& \qquad \text{and } (V_k \backslash Z) \cap \operatorname{int}(I) \neq \varnothing \big\} \\
< & \frac{\varepsilon}{4|E|} \sum \big\{ |I| : (I, \xi) \in P_k \quad \text{with} \quad \xi \notin I \\
& \qquad \text{and } (V_k \backslash Z) \cap \operatorname{int}(I) \neq \varnothing \big\} + \frac{\varepsilon}{2^{k+3}}.
\end{aligned} \tag{19}
$$

Consequently, given any $\Delta$-fine McShane partition $P = \{(I_i, \xi_i)\}_{i=1}^p$ in $E$, it follows from (15), (16), (18), (19) and our choice of $O \supset Z$ that

$$
\begin{aligned}
\sum_{i=1}^p \|f(\xi_i)|I_i| - F(I_i)\| & \leq \sum_{k=1}^\infty \sum_{(I_i, \xi_i) \in P_k} \|f(\xi_i)|I_i| - F(I_i)\| \\
& \quad + \sum_{\xi_i \in Z} \|f(\xi_i)|I_i| - F(I_i)\| \\
& < \sum_{k=1}^\infty [\alpha_k + \beta_k + \gamma_k] + \frac{\varepsilon}{16} < \varepsilon,
\end{aligned}
$$

from which the strong McShane integrability of $f$ follows. The proof is complete. $\square$

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