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APPROXIMATION IN NON-ASPLUND SPACES

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ABSTRACT. Results on the uniform approximation of continuous functions by C^k-smooth functions on the boundary of certain convex subsets in Banach spaces which are non-Asplund are given.

1. Introduction. The uniform approximation of continuous functions by C^k-smooth maps on Banach spaces which admit C^k-smooth bump functions (C^k-smooth real-valued functions with bounded, non-empty support) has received much attention over the years, see, e.g., [**2**]. Corresponding results in non-Asplund spaces has been less common, although related work can be traced back to the seminal papers [**4**] and [**1**], while more recently, the behavior of smooth functions on non-Asplund spaces and their 'harmonic' behavior has been considered in [**2**, Theorem III.1.3 and Proposition III.1.7], and results in a similar vein are in [**3**].

A simple yet important observation is that if one is able to uniformly approximate arbitrary continuous functions on an open set G in a Banach space X via maps C^1-smooth on G, then by approximating a suitable continuous bump function on G with a C^1-smooth map on G subsequently composed with an appropriate smooth bump function on $\mathbf{R}$, one can construct a C^1-smooth bump function on X. This in turn implies that X is Asplund. Hence, for non-Asplund spaces X, it is not possible to uniformly approximate arbitrary continuous maps on open sets by functions C^1-smooth on X. This is in stark contrast to the situation for many Banach spaces which admit C^1-smooth bump functions such as reflexive spaces or, more generally, weakly compactly generated Asplund spaces.

It follows that, for non-Asplund spaces, approximation theorems are much more constrained. Nevertheless, we obtain some interesting

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positive results. In particular, if one restricts the smooth approximation to the boundary ∂C of convex sets C, we show that smooth approximation of continuous maps is possible for certain C. We note below, see Proposition 1, that for such approximation results, some stronger convexity assumptions on C are also necessary when X is non-Asplund, and in our main result these assumptions concern the extremal structure of ∂C. More specifically, the convex sets C we shall work with are subsets of separable spaces which are convex, closed, bounded and such that each $x \in \partial C$ is strongly exposed. For such a C we show, for example, that, given a continuous map $F : C \to \mathbf{R}$, $n \geq 1$ and $\varepsilon > 0$, there exists an open set $U = U_\varepsilon \supset \partial C$ and a C^n-smooth function $K : U \to \mathbf{R}$ which approximates F to within ε on ∂C. It has been noted [**6**] that even uniform approximation of continuous functions on $S_X = \partial B_X$, where B_X is the closed unit ball of X, by maps C^1-smooth on X would enable one to construct a smooth bump function on X, indicating that we cannot expect the maps K above to be smooth off of U in the case that X is non-Asplund.

Our results apply, for example, to $C = B_{l_1}$, with B_{l_1} being the closed unit ball of an equivalent locally uniformly rotund norm of l_1. Even in separable Asplund spaces X, we obtain new results when the smoothness of K is chosen to exceed the highest order of smoothness of the norm of X. For example, in l_p with p odd, $n > p$, and choosing $C = B_{l_p}$ (again, with a locally uniformly rotund renorming).

The fact that the approximation here is only on ∂C is not as restrictive as one might suppose if the function $F : X \to Y$ to be approximated is C^1-smooth and Y is Asplund while X is not, since then [**2**, Proposition II.1.7] gives us that $\overline{F(\partial C)} = F(C)$. A related result has recently been obtained in [**3**], where it is shown that if X does not have finite cotype, while Y does, and if $G \subset X$ is open and bounded with $F : X \to Y$ having uniformly continuous derivative in a neighborhood of $\overline{G}$, then $\overline{F(\partial \overline{G})} = F(\overline{G})$. Hence, in this last situation, we can also see that approximation on $\partial \overline{G}$ gives more information than might otherwise be expected.

The motivation, as well as many of the techniques in this note, originate in [**4**].

2. Definitions and notation. All Banach spaces are assumed real and are denoted by X, Y, etc., and their continuous duals by X^*, Y^*, etc. The closed unit ball and sphere of X are written B_X and S_X, respectively, with similar notation for the dual space. A closed ball of radius $r > 0$ and center $p \in X$ is denoted $B_r(p)$. The interior of a set S is written $\mathbf{int}\, S$ and its closure as $\overline{S}$. If $C \subset X$ is a bounded, convex set, a point $x \in C$ is said to be an *exposed point* if there exists a functional $x^* \in S_{X^*}$ (the exposing functional) with $x^*(x) > x^*(y)$ for all $y \in C$, $y \neq x$. The point x is said to be *strongly exposed* if, in addition to being exposed, we have that for any sequence $\{x_n\} \subset C$, $x^*(x_n - x) \to 0$ implies $x_n \to x$, where x^* is the exposing functional for x. Recall that the norm of X is *locally uniformly rotund* (LUR) if, for $x, x_n \in S_X$ with $\|x + x_n\| \to 2$ we have $\|x - x_n\| \to 0$. The importance of this notion for us is that if the norm of X is LUR, then every $x \in S_X$ is strongly exposed. Any separable, or more generally any weakly compactly generated space, admits an equivalent LUR norm. In this note smoothness is meant in the Fréchet sense. As mentioned in the introduction, a *C^k-smooth bump function* on X is a C^k-smooth, real-valued function with bounded, nonempty support.

X is said to be an Asplund space if every continuous, convex function on X is Fréchet differentiable on a dense G_δ subset. For example, if X admits a C^1-smooth bump function, then X is Asplund, see, e.g., [**2**, Lemma II.5.4]. We refer the reader to the text [**5**] for more information on Asplund spaces.

3. Main results. As noted in the introduction, in non-Asplund spaces X some restrictions on the 'size' of the set over which one is approximating are necessary, since the uniform approximation of continuous functions over a set with nonempty interior by smooth functions enables one to construct a smooth bump function on X. Our approximation results shall take place on the boundaries of certain convex subsets C of separable spaces. In addition, some further convexity assumptions are necessary on C when X is non-Asplund, as the following proposition indicates.

Proposition 1. *Let X be a Banach space and U an open neighborhood of S_X. Suppose that any continuous function on S_X can be uniformly approximated on S_X by a map C^1-smooth on U. Further* *suppose that* $Y \subset X$ *is a translate of a hyperplane such that* $Y \cap S_X$ *is relatively open in* S_X. *Then* X *admits a* C^1*-smooth bump function.*

Proof. We can suppose that, for some $x_0 \in Y \cap S_X$, we have $B_{3/4}(x_0) \cap S_X \subset Y \cap S_X$. For $x \in Y \cap S_X$, put $d(x) = \mathbf{dist}\,(x, B_{1/2}(x_0))$. Using our hypothesis, let $b_1 : U \to [0,1]$ be a map C^1-smooth on a neighborhood U of $Y \cap S_X$ such that $|b_1(x) - d(x)| < 1/8$ on $Y \cap S_X$. Let $\xi \in C^1(\mathbf{R}, [0,1])$ be such that $\xi(t) = 1$ if $t \leq 1/8$ and $\xi(t) = 0$ if $t \geq 1/4$. For $x \in Y \cap S_X$, put $b_2(x) = \xi(b_1(x))$ and extend b_2 to all of Y, still calling it b_2, by setting $b_2 = 0$ on $Y \setminus B_{3/4}(x_0)$. Then b_2 is C^1-smooth on Y with support $(b_2) \subset B_{3/4}(x_0) \cap S_X$ and $b_2(x_0) = 1$. For $x \in Y - x_0$, put $b_3(x) = b_2(x + x_0)$.

Putting $H = Y - x_0$, we have $X = H \oplus \mathbf{R}$, and we norm the direct sum in the standard way by setting, for $x = (h, r) \in X$, $\|x\| = \|(h, r)\| = \sqrt{\|h\|^2 + r^2}$. We have the projections $\pi_1(h, r) = h$ and $\pi_2(h, r) = r$. Finally, define the C^1-smooth map $b : X \to [0,1]$ by $b(x) = b_3(\pi_1(x))\xi(\pi_2(x))$. Then $b(0) = b_3(0)\xi(0) = 1$, while if $\|x\| = \|(h, r)\| \geq \sqrt{2}$, then either $\|h\| \geq 1$ or $|r| \geq 1$, and so either $b_3(\pi_1(x)) = b_3(h) = 0$ or $\xi(\pi_2(x)) = \xi(r) = 0$, and hence support $(b) \subset \sqrt{2}B_X$. $\square$

Proposition 1 shows that, if C is a subset of X with X non-Asplund, then in general the uniform approximation of continuous maps by C^k-smooth maps on ∂C will be possible only if we place some stronger convexity assumptions on C. For our main result we shall in fact assume that each $x \in \partial C$ is a strongly exposed point.

Theorem 1. *Let* X *be a separable Banach space and* $C \subset X$ *a closed, convex and bounded subset such that each* $x \in \partial C$ *is strongly exposed. Then, if* Y *is an arbitrary Banach space,* $F : C \to Y$ *is continuous,* $n \geq 1$ *and* $\varepsilon > 0$, *there exists an open set* $U \supset \partial C$ *and a* C^n*-smooth map* $K : U \to Y$ *such that*

$$\|F(x) - K(x)\| < \varepsilon \quad \textit{for all } x \in \partial C.$$

Proof. Let F and C be as in the theorem statement, and let $\varepsilon \in (0,1)$ and $n \geq 1$ be given. For $x \in \partial C$, using the continuity of F, pick $\delta_x > 0$

such that $y \in B_{\delta_x}(x) \cap C$ implies

$$\|F(x) - F(y)\| < \varepsilon/2.$$

Since any $x \in \partial C$ is a strongly exposed point, for each such x there exists an $x^* \in S_{X^*}$ with $x^*(x) > x^*(y)$ for all $y \in C$, $y \neq x$, and moreover, for the given $\delta_x > 0$, there exists $\varepsilon_x \in (0, \delta_x)$ with

$$x^*(x - y) > \varepsilon_x \quad \text{for all} \quad y \in C \quad \text{with } \|x - y\| \geq \delta_x. \tag{3.1}$$

Similarly, for the same x and x^*, we can find an $\varepsilon'_x < \varepsilon_x$ such that

$$x^*(x - y) > \varepsilon'_x \quad \text{for all} \quad y \in C \quad \text{with } \|x - y\| \geq \varepsilon_x/2. \tag{3.2}$$

Define a relatively open cover of ∂C by the sets

$$C_x^1 = \mathbf{int}\,(B_{\varepsilon'_x/2}(x)) \cap \partial C \quad \text{for } x \in \partial C,$$

and note that, since C is separable, there is a countable subcover $\{C_{x_j}^1\}_{j=1}^\infty$ for ∂C. With notation as above, associated with the x_j are the strongly exposing functionals $x_j^* \in S_{X^*}$, and the numbers $\delta_{x_j}, \varepsilon_{x_j}$ and ε'_{x_j}.

Also define sets

$$C_{x_j}^2 = \mathbf{int}\,(B_{\varepsilon_{x_j}/2}(x_j)) \cap \partial C,$$

and

$$C_{x_j}^3 = \mathbf{int}\,(B_{\delta_{x_j}}(x_j)) \cap \partial C,$$

noting that the collections $\{C_{x_j}^2\}_{j=1}^\infty$ and $\{C_{x_j}^3\}_{j=1}^\infty$ are also covers for ∂C.

To simplify notation, we shall write the sequences

$$\{\delta_{x_j}\}, \{\varepsilon_{x_j}\}, \{\varepsilon'_{x_j}\}, \{C_{x_j}^1\}, \{C_{x_j}^2\}, \text{ etc.},$$

as simply $\{\delta_j\}, \{\varepsilon_j\}, \{\varepsilon'_j\}, \{C_j^1\}$, etc.

Let $\theta \in C^\infty(\mathbf{R}, \mathbf{R}^+)$ be the function

$$\theta(t) = \begin{cases} e^{-1/t^2} & \text{for } t > 0 \\ 0 & \text{for } t \leq 0, \end{cases}$$

and define maps $f_j : X \to [0,1]$ by

$$f_j(x) = \frac{\theta(\varepsilon_j - x_j^*(x_j - x))}{\theta(\varepsilon_j - x_j^*(x_j - x)) + \theta(x_j^*(x_j - x) - \varepsilon_j/2)}.$$

Therefore, using (3.1),

$$x \in \partial C \setminus C_j^3 \Longrightarrow \|x_j - x\| \geq \delta_j \Longrightarrow f_j(x) = 0.$$

Also,

$$x \in \partial C \cap C_j^2 \Longrightarrow \|x_j - x\| < \varepsilon_j/2 \Longrightarrow f_j(x) = 1.$$

For $j \geq 0$ we define maps $g_j : X \to [0,1]$ by setting $g_0 = 0$ and, for $j \geq 1$,

$$g_j(x) = \frac{\theta(\varepsilon_j' - x_j^*(x_j - x))}{\theta(\varepsilon_j' - x_j^*(x_j - x)) + \theta(x_j^*(x_j - x) - \varepsilon_j'/2)}.$$

Observe that, for $x \in C_j^1 \cap \partial C$, we have $g_j(x) = 1$ and $x \in \partial C \setminus C_j^2$ implies $g_j(x) = 0$ using (3.2).

We define sets by

$$\begin{aligned} D_1 &= C_1^2 \\ D_j &= C_j^2 \setminus (\cup_{i<j} C_i^2). \end{aligned}$$

Observe that the collection $\{D_j\}_{j=1}^{\infty}$ covers ∂C and, if $x \in D_j$, then $x \in C_j^2$ while $x \in \partial C \setminus C_i^2$ for $i = 1, \ldots, j-1$.

We finally define maps $\psi_j : X \to \mathbf{R}^+$ by

$$\psi_j(x) = (\|F(x_j)\| + 1) \exp\left\{ -\beta_j \left[(1 - f_j(x)) + \sum_{i=1}^{j-1} g_i(x) \right] \right\},$$

where $\beta_j \in \mathbf{R}^+$ shall be specified later.

Some properties of the ψ_j are collected in the following lemma.

Lemma 1. *For the functions ψ_j defined above, we have*

(i) $\psi_j(x) \leq (\|F(x_j)\| + 1)e^{-\beta_j}$ *for all* $x \in \partial C \setminus C_j^3$.

(ii) *For each* $x \in \partial C$, *there exists a* j_0 *such that*

$$\psi_{j_0}(x) - (\|F(x_{j_0})\| + 1) = 0.$$

(iii) *For each* $x \in \partial C$, *there exists a* j' *and an* $\eta = \eta(x) > 0$ *such that*

$$\psi_j(y) \le (\|F(x_j)\| + 1)e^{-\beta_j},$$

for all y *with* $\|y - x\| < \eta$ *and* $j > j'$.

Proof. (i) For $x \in \partial C$ we have, since $g_j \ge 0$,

$$\psi_j(x) \le (\|F(x_j)\| + 1)\exp\{-\beta_j(1 - f_j(x))\}.$$

Also, as noted above, $f_j(x) = 0$ for $x \in \partial C \setminus C_j^3$, and the result follows.

(ii) Fix any $x \in \partial C$. From the construction of the cover $\{D_j\}$, we have that there exists j_0 with $x \in C_{j_0}^2$ while $x \in \partial C \setminus C_i^2$ for $i = 1, \dots, j_0 - 1$. It now follows from the construction of the f_j and the g_j as noted above that

$$\begin{aligned}
&|\psi_{j_0}(x) - (\|F(x_{j_0})\| + 1)| \\
&\quad = (\|F(x_{j_0})\| + 1)\left|\exp\left\{ - \beta_{j_0}\left[(1 - f_{j_0}(x)) + \sum_{i=1}^{j_0-1} g_i(x)\right]\right\} - 1\right| \\
&\quad = (\|F(x_{j_0})\| + 1)|e^0 - 1| \\
&\quad = 0.
\end{aligned}$$

(iii) Fix $x \in \partial C$ and fix j' such that $x \in C_{j'}^1$. Then since $C_{j'}^1$ is (relatively) open, there exists an $\eta = \eta(x) > 0$ such that $\partial C \cap \mathbf{int}\, B_\eta(x) \subset C_{j'}^1$. We choose η smaller, if necessary, so that for all $y \in \mathbf{int}\, B_\eta(x)$ we have $\|x_{j'} - y\| < \varepsilon_j'/2$.

Using the definition of g_j, we then have

$$g_{j'}(y) = 1 \quad \text{for } y \in \mathbf{int}\, B_\eta(x).$$

Then, for $j > j'$ and $y \in \mathbf{int}\, B_\eta(x)$, we have, using $1 - f_j \ge 0$ and $g_i \ge 0$,

$$\begin{aligned}
\psi_j(y) &\le (\|F(x_j)\| + 1)\exp\left\{ - \beta_j \sum_{i=1}^{j-1} g_i(y)\right\} \\
&\le (\|F(x_j)\| + 1)\exp\{-\beta_j g_{j'}(y)\} \\
&= (\|F(x_j)\| + 1)e^{-\beta_j}. \qquad \square
\end{aligned}$$

Referring to the proof of Lemma 1(iii), we define the open set $U = \cup_{x \in \partial C} (\textbf{int}\, B_{\eta(x)}(x)) \supset \partial C$.

Now, returning to the proof of the theorem, we define for $x \in U$,

$$\psi(x) = \sum_{j=1}^{\infty} \psi_j(x) \quad \text{and} \quad K(x) = \frac{\sum_{j=1}^{\infty} F(x_j)\psi_j(x)}{\psi(x)}.$$

Choose the $\beta_j > 0$ large enough that $(\|F(x_j)\| + 1)^2 e^{-\beta_j} \leq 2^{-(j+4)}$. Then Lemma 1(iii) implies that $\psi(x)$ and $\sum_{j=1}^{\infty} F(x_j)\psi_j(x)$ are continuous on the open set $U \supset \partial C$. Also, for any $x \in X$ and $j \geq 1$,

$$\begin{aligned}\psi(x) \geq \psi_j(x) &= (\|F(x_j)\| + 1) \exp\left\{ -\beta_j \left[(1 - f_j(x)) + \sum_{i=1}^{j-1} g_i(x) \right] \right\} \\ &\geq \exp\{-j\beta_j\} \\ &> 0.\end{aligned}$$

It follows that K is continuous on U.

Let us next investigate the first derivative of $\psi(x)$. Fix $x \in U$ and also $x_0 \in \partial C$ with $x \in \textbf{int}\, B_{\eta(x_0)}(x_0)$ according to the definition of U. We choose a neighborhood N_x of x with $N_x \subset \textbf{int}\, B_{\eta(x_0)}(x_0)$. An easy calculation shows that all the derivatives $f_j^{(k)}$ and $g_j^{(k)}$ for $k = 1, \ldots, n$ are bounded on X by some constants $A_j > 0$. We have for all $y \in X$ and $\|h\| \leq 1$,

$$\psi_j'(y)(h) \leq \psi_j(y)\beta_j \left[A_j + \sum_{i=1}^{j-1} A_i \right]. \tag{3.3}$$

Pick $j' = j'(x_0)$ so that the conditions of Lemma 1 (iii) are met at x_0 and choose the β_j larger if necessary so that the righthand side of (3.3) is less than 2^{-j} which, by Lemma 1 (iii) will hold for all $y \in N_x$ and $j > j'$. It follows that the partial sums $\sum_{j=1}^{m} \psi_j'(x)(h)$ and $\sum_{j=1}^{m} F(x_j)\psi_j'(x)(h)$ converge uniformly on N_x for all $\|h\| \leq 1$. Hence, K is C^1-smooth on U. Similar calculations show that, given $n \geq 1$, the β_j can be chosen such that $K \in C^n(U, Y)$.

Finally we show that, on ∂C, $\|K(x) - F(x)\| < \varepsilon$. Fix $x \in \partial C$. From Lemma 1 (ii) we have that there exists a j_0 with

$$|\psi_{j_0}(x) - (\|F(x_{j_0})\| + 1)| = 0.$$

Then also $x \in C_{j_0}^2 \subset C_{j_0}^3$ from the proof of Lemma 1 (ii), implying

$$\|F(x_{j_0}) - F(x)\| < \varepsilon/2.$$

From the previous two inequalities, we obtain

$$\psi(x) \geq \psi_{j_0}(x) \geq 1 \quad \text{and} \quad \psi(x) \geq \psi_{j_0}(x) > \|F(x)\|.$$

Set $J = \{j : x \in C_j^3\}$. Then from Lemma 1 (i) and again choosing the β_j sufficiently large(r), we obtain

$$\sum_{j \notin J} \psi_j(x) < \frac{\varepsilon}{4} \quad \text{and} \quad \sum_{j \notin J} \|F(x_j)\| \psi_j(x) < \frac{\varepsilon}{4}.$$

Therefore, using these estimates and the previous inequalities, we have

$$\begin{aligned} \|F(x) - K(x)\| &= \frac{1}{\psi(x)} \left\| \sum_{j=1}^{\infty} F(x)\psi_j(x) - \sum_{j=1}^{\infty} F(x_j)\psi_j(x) \right\| \\ &\leq \frac{1}{\psi(x)} \left\{ \sum_{j \in J} \|F(x) - F(x_j)\| \psi_j(x) \right\} \\ &\quad + \frac{1}{\psi(x)} \sum_{j \notin J} \|F(x)\| \psi_j(x) + \frac{1}{\psi(x)} \sum_{j \notin J} \|F(x_j)\| \psi_j(x) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon. \qquad \square \end{aligned}$$

Corollary 1. *Let X be a separable, non-Asplund Banach space and $G \subset X$ convex, open and bounded such that each $x \in \overline{G}$ is strongly exposed.*

Then, if Y is an Asplund space, $F : \overline{G} \to Y$ is continuous and C^1-smooth on G, and $n \geq 1$, $\varepsilon > 0$, there exists an open set $U \supset \partial\overline{G}$ and a map $K \in C^n(U, Y)$ such that, for any $x_0 \in \overline{G}$, there exists $x \in \partial\overline{G}$ with

$$\|F(x_0) - K(x)\| < \varepsilon.$$

Proof. Under the hypothesis of the corollary, by [**2**, Proposition III.1.7], we have that $F(\partial G)$ is dense in $F(\overline{G})$. Combining this with Theorem 1 gives the result. $\square$

Corollary 2. *Let X be a separable, non-Asplund Banach space with nonfinite cotype and $G \subset X$ convex, open and bounded such that each $x \in \overline{G}$ is strongly exposed.*

Then, if Y has finite cotype and $F : X \to Y$ is C^1-smooth on a neighborhood of $\overline{G}$ with uniformly continuous derivative on G and $n \geq 1$, $\varepsilon > 0$, there exists an open set $U \supset \partial\overline{G}$ and a map $K \in C^n(U, Y)$ such that, for any $x_0 \in \overline{G}$, there exists $x \in \partial\overline{G}$ with

$$\|F(x_0) - K(x)\| < \varepsilon.$$

Proof. Under the hypothesis of the corollary, by [**3**, Theorem 1], we have that $F(\partial\overline{G})$ is dense in $F(\overline{G})$. Combining this with Theorem 1 gives the result. $\square$

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