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HYPERCYCLIC AND CHAOTIC CONVOLUTION OPERATORS ON CHÉBLI-TRIMÈCHE HYPERGROUPS

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ABSTRACT. In this paper a universality property for Chébli-Trimèche convolution operators is proved. The results obtained extend prior analysis of the Fourier and Hankel transforms. We also investigate hypercyclic and chaotic convolution operators on Chébli-Trimèche hypergroups in some distribution spaces.

1. Introduction. In this paper we investigate new properties for the generalized Fourier transformation, also called Chébli-Trimèche transform, $\mathcal{F}$ defined, when f is a suitable function defined on $(0,\infty)$, by

$$(\mathcal{F}(f)(\lambda) = \int_0^\infty \psi_\lambda(x) f(x) A(x)\, dx, \quad \lambda \geq 0,$$

where, for every $\lambda \geq 0$, ψ_λ represents the solution of the equation

$$\Delta\psi_\lambda(x) = (\lambda^2+\rho^2)\psi_\lambda(x), \quad x>0, \tag{1.1}$$

satisfying that

$$\psi_\lambda(0)=1 \quad \text{and} \quad \frac{d}{dx}\,\psi_\lambda(0)=0.$$

Here $\rho \geq 0$ and Δ denotes the differential operator

$$\Delta = -\frac{1}{A(x)}\,\frac{d}{dx}\bigg(A(x)\frac{d}{dx}\bigg), \tag{1.2}$$

where A is a real function on $[0,\infty)$ of the form $A(x)=x^{2\alpha+1}B(x)$, $\alpha > -1/2$, with B an even positive analytic function on $\mathbf{R}$ satisfying $B(0)=1$. We assume that A satisfies the following conditions

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(i) A is increasing and unbounded on $(0,\infty)$,

(ii) A'/A is a decreasing C^∞-function on $(0,\infty)$. Hence there exists $\lim_{x\to\infty} A'(x)/A(x) \geq 0$.

In the sequel the positive real number ρ appearing in (1.1) is defined by

$$\rho = \frac{1}{2}\lim_{x\to\infty}\frac{A'(x)}{A(x)},$$

when Δ is given by (1.2).

(iii) There exist η and $M>0$ and a smooth function $\mathcal{C}$ such that $\mathcal{C}^{(k)}$ is bounded on $(0,\infty)$, for every $k\in\mathbf{N}$, and for which, when $x\in(M,\infty)$,

$$\frac{A'(x)}{A(x)} = \begin{cases} 2\rho + e^{-\eta x}\mathcal{C}(x) & \text{if } \rho>0,\\ \dfrac{2\alpha+1}{x} + e^{-\eta x}\mathcal{C}(x) & \text{if } \rho=0. \end{cases}$$

(iv) There exists a positive real number δ such that $(B'(x)/B(x))' = e^{-\delta x}D(x)$, $x\in[0,\infty)$, with D being a continuous bounded function on $[0,\infty)$.

In particular, the generalized Fourier transform $\mathcal{F}$ reduces to the Hankel transform [**11**] when $A(x)=x^{2\alpha+1}$, $x\in[0,\infty)$ and $\alpha>-1/2$. Also the Jacobi transform [**8**] and [**14**] that can be interpreted in certain cases as the spherical transform on noncompact symmetric spaces of rank one, appears when $A(x)=(\sinh x)^{2\alpha+1}(\cosh x)^{2\beta+1}$, $x\in[0,\infty)$ with $\alpha\geq\beta\geq -1/2$ and $\alpha\neq -1/2$.

The inversion formula of the transform $\mathcal{F}$ is given by [**5**]

$$f(x) = \int_0^\infty \psi_\lambda(x)(\mathcal{F}f)(\lambda)\,\frac{d\lambda}{|c(\lambda)|^2}$$

where $|c(\lambda)|^{-2}$ is a continuous function on $[0,\infty)$. The function $c(\lambda)$ can be seen as a function of the Harish-Chandra type.

For the Chébli-Trimèche transform the following Plancherel formula [**21**] and [**2**, Theorem 2.2.13] holds

$$\int_0^\infty |f(x)|^2 A(x)\,dx = \int_0^\infty |\mathcal{F}(f)(\lambda)|^2\,\frac{d\lambda}{|c(\lambda)|^2}, \tag{1.3}$$

for every $f \in L^2((0,\infty), A(x)\,dx)$. As usual, for every $1 \leq p \leq \infty$, by $L^p((0,\infty), d\mu(x))$ we represent the Lebesgue p-space on $(0,\infty)$ with respect to the positive measure μ.

Since $|\psi_\lambda(x)| \leq 1$, $x, \lambda \in (0,\infty)$ [**5**, Corollary 9] and [**3**, Lemma 3.4] it is not hard to see that $\mathcal{F}$ maps $L^1((0,\infty), A(x)\,dx)$ into $L^\infty((0,\infty), d\lambda/|c(\lambda)|^2)$. Hence, by (1.3), Riesz-Thorin interpolation theorem implies that $\mathcal{F}$ can be extended as a bounded operator from $L^p((0,\infty), A(x)\,dx)$ into $L^{p'}((0,\infty), d\lambda/|c(\lambda)|^2)$, provided that $1 \leq p \leq 2$, where p' denotes the exponent conjugated to p.

Chébli [**5**] and Trimèche [**19**] have established Paley-Wiener theorems for the generalized Fourier transform. For every $a > 0$, the space $\mathcal{D}_a$ is constituted by all those even and C^∞-functions ϕ on $\mathbf{R}$ such that $\phi(x) = 0$, $|x| \geq a$. We consider on $\mathcal{D}_a$ the topology associated with the family $\{p_m\}_{m\in\mathbf{N}}$ of semi-norms, where for every $m \in \mathbf{N}$:

$$p_m(\phi) = \sup_{x\in\mathbf{R}} |\phi^{(n)}(x)|, \quad \phi \in \mathcal{D}_a.$$

By $\mathcal{D}$ we understand the strict inductive limit $\cup_{a>0}\mathcal{D}_\alpha$ and $\mathcal{D}'$ denotes the dual space of $\mathcal{D}$. On $\mathcal{D}'$ we consider the weak $*$ topology.

Bloom and Xu [**3**] studied the generalized Fourier transform on Schwartz type spaces. They introduced the space $S_p((0,\infty), A)$ for each $0 < p \leq 2$, as follows. A complex-valued function ϕ defined on $(0,\infty)$ is in $S_p((0,\infty), A)$ if and only if there exists an even function $\Phi \in C^\infty(\mathbf{R})$ such that $\phi = \Phi$ on $(0,\infty)$ and that

$$\mu^p_{k,l}(\phi) = \sup_{x\in(0,\infty)} (1+x^2)^l \psi_0(x)^{-2/p} \Big| \frac{d^k}{dx^k}\phi(x) \Big| < \infty,$$

for every $l, k \in \mathbf{N}$. The image by the Chébli-Trimèche transform of $S_p((0,\infty), A)$ is characterized in [**3**, Proposition 4.26].

A one-dimensional hypergroup on $(0,\infty)$, also called Chébli-Trimèche hypergroup (see [**22**, Chapter 6]) is associated to the generalized Fourier transform (see [**2**]). The generalized translation $u(x,y) = (\tau_x f)(y)$ of any real-valued function f on $(0,\infty)$ which is the restriction of an even C^∞-function on $\mathbf{R}$ is the solution of the following Cauchy problem:

$$\begin{aligned} (\Delta_x - \Delta_y)u(x,y) &= 0 \\ u(x,0) &= f(x), \quad x \geq 0 \\ u_y(x,0) &= 0, \quad x \geq 0. \end{aligned}$$

This generalized translation $\tau_x f$ of f can be written

$$(\tau_x f)(y) = \int_0^\infty f(z)D(x,y,z)A(z)\,dz, \quad x,y \in \,]0,\infty)\,,$$
$$(\tau_x f)(0) = f(x), \quad x \in \,]0,\infty)\,,$$

and we define $(\tau_x f)(y) = (\tau_{|x|}f)(|y|)$, $x,y \in \mathbf{R}$. Here, for every $x,y \in \,]0,\infty)$, $D(x,y,\cdot)$ is a positive function such that $D(x,y,z) = 0$, when $z \notin [|x-y|, x+y]$. Moreover, the eigenfunction ψ_λ of the operator Δ and the function D are related through the following product formula [**21**, Section II.3]

$$\int_0^\infty D(x,y,z)\psi_\lambda(z)A(z)\,dz = \psi_\lambda(x)\psi_\lambda(y), \quad x,y \in \,]0,\infty)\,. \tag{1.4}$$

The convolution operation on the Chébli-Trimèche hypergroup is defined as follows. If f and g are in $L^1((0,\infty), A(x)\,dx)$, then the convolution $f\#g$ of f and g is given through

$$(f\#g)(x) = \int_0^\infty f(y)(\tau_x g)(y)A(y)\,dy, \quad x \geq 0.$$

The #-convolution is also defined on bounded measures on $[0,\infty)$, [**15**].

The integral transform $\mathcal{F}$ is related to the generalized translation and to the #-convolution as the following formulas show [**3**, Theorem 2.4]

(i) $\mathcal{F}(\tau_x f)(\lambda) = \psi_\lambda(x)\mathcal{F}(f)(\lambda)$, $f \in L^1((0,\infty), A(x)\,dx)$ and $x \geq 0$,

(ii) $\mathcal{F}(f\#g)(\lambda) = \mathcal{F}(f)(\lambda)\mathcal{F}(g)(\lambda)$, $f,g \in L^1((0,\infty), A(x)\,dx)$.

Herzog [**12**] has proved a universality property of solutions of the heat equation. He introduced, for every $\beta > 0$, the space $\mathcal{E}_\beta$ constituted by all those real and continuous functions ϕ defined on $\mathbf{R}$ such that

$$\lim_{|x|\to\infty} e^{-\beta|x|}\phi(x) = 0.$$

For every $t > 0$ and $\phi \in \mathcal{E}_\beta$, $\beta > 0$, $T_t\phi$ is defined by

$$(T_t\phi)(x) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} \exp(-(x-s)^2/4t)\phi(s)\,ds, \quad x \in \mathbf{R}.$$

Note that, for every $t > 0$, T_t defines a usual convolution operator. Herzog proved in [**12**, Theorem 1.1] that, for every $\beta > 0$, the set

$$U_\beta = \{\phi \in \mathcal{E}_\beta : \overline{\{T_n \phi : n \in \mathbf{N}\}} = C(\mathbf{R}, \mathbf{R})\}$$

is a residual subset of $\mathcal{E}_\beta$, that is, $\mathcal{E}_\beta \setminus U_\beta$ is of first category in $\mathcal{E}_\beta$. Here $C(\mathbf{R}, \mathbf{R})$ denotes the space of real and continuous functions on $\mathbf{R}$ and it is endowed with the topology of the uniform convergence of compact subsets of $\mathbf{R}$.

In Section 2, we establish the corresponding universality property for certain #-convolution operators. Our results can be seen as an extension of those obtained in [**1**] where Hankel and Fourier convolution operators were considered.

Suppose now X is a locally convex space and T is a continuous linear operator from X into itself. We say that T is a hypercyclic operator when there exists $x \in X$, called a hypercyclic vector for T, such that $\{T^n x\}_{n \in \mathbf{N}}$ is a dense subset of X. Every hypercyclic operator on X is topologically transitive in the sense of dynamical systems; that is, for every pair of open and nonempty subsets U and V of X, there exists $n \in \mathbf{N}$ for which $T^n(U) \cap V \neq \varnothing$. As in [**4**] (see also [**6**]), we call a linear and continuous operator on X chaotic if it is topologically transitive and it has a dense set of periodic points.

Trimèche [**19**] considered the space $\mathbf{E}$ that consists of all those even and C^∞-functions on $\mathbf{R}$. We consider on $\mathbf{E}$ the topology generated by the family $\{p_{m,n}\}_{m,n \in \mathbf{N}}$ of semi-norms where, for every $m, n \in \mathbf{N}$,

$$p_{m,n}(f) = \sup_{|x| \leq n} |f^{(m)}(x)|, \quad f \in \mathbf{E}.$$

According to [**3**, Lemma 4.18] we can see that the topology of $\mathbf{E}$ is also generated by the systems $\{q_{m,n}\}_{m,n \in \mathbf{N}}$ and $\{r_{m,n}\}_{m,n \in \mathbf{N}}$ of semi-norms where, for each $m, n \in \mathbf{N}$,

$$q_{m,n}(f) = \sup_{|x| \leq n} |\Delta^m f(x)|, \quad f \in \mathbf{E},$$

and

$$r_{m,n}(f) = \left\{ \int_0^n |\Delta^m f(x)|^2 A(x)\, dx \right\}^{1/2}, \quad f \in \mathbf{E}.$$

The space $\mathbf{E}$ can be seen as a subspace of $\mathcal{D}'$, the dual space of $\mathcal{D}$, in the following sense. Each $f \in \mathbf{E}$ is identified with the element S_f of $\mathcal{D}'$ defined by

$$\langle S_f, \phi \rangle = \int_0^\infty f(x)\phi(x)A(x)\,dx, \quad \phi \in \mathcal{D}.$$

Thus $\mathbf{E}$ is a dense subspace of $\mathcal{D}'$. This can be proved by using #-approximate identities. The dual space of $\mathbf{E}$ is denoted by $\mathbf{E}'$ and it is constituted by distributions of compact support.

For every $x \in [0, \infty)$ the translation operator τ_x defines a continuous and linear mapping from $\mathbf{E}$ into itself [**19**, Proposition 8.3] and from $\mathcal{D}$ into itself [**19**, Corollary 8.2]. If $T \in \mathbf{E}'$, respectively $\mathcal{D}'$, and $f \in \mathbf{E}$, respectively $\mathcal{D}$, the convolution $T\#f$ of T and f is defined by

$$(T\#f)(x) = \langle T, \tau_x f \rangle, \quad x \in [0, \infty).$$

Trimèche proved in [**20**] that, for every $T \in \mathbf{E}'$, the linear mapping defined by $f \to T\#f$ is continuous from $\mathbf{E}$ into itself and from $\mathcal{D}$ into itself. Moreover, the space $\mathbf{E}'$ can be characterized as the space of the #-convolution operators on $\mathcal{D}$; that is, given $T \in \mathcal{D}'$, we have that $T\#\phi \in \mathcal{D}$, for each $\phi \in \mathcal{D}$ if and only if $T \in \mathbf{E}'$. The #-convolution $S\#T$ of $S \in \mathcal{D}'$ and $T \in \mathbf{E}'$ is the element of $\mathcal{D}'$ defined through

$$\langle S\#T, \phi \rangle = \langle S, T\#\phi \rangle, \quad \phi \in \mathcal{D}.$$

Inspired by the results in [**9**] and [**4**], in Section 3 of this paper we prove that if $T \in \mathbf{E}'$ is not a scalar multiple of the Dirac functional δ, then T defines a hypercyclic and chaotic #-convolution operator on $\mathbf{E}$ and on $\mathcal{D}'$.

Trimèche [**18**] and Fitouhi [**7**] investigated the convergence of generalized Taylor series associated to the operator Δ. We collect now some properties established in [**18**] and [**7**] that will be useful to us. The function $\lambda \to \psi_\lambda(x)$ for every $x \in \mathbf{R}$ is even and analytic and we can write ([**7**, p. 246])

$$\psi_\lambda(x) = \sum_{n=0}^{\infty} (-1)^n b_n(x)(\lambda^2 + \rho^2)^n, \quad \lambda \in \mathbf{C}, \tag{1.5}$$

where, for every $n \in \mathbf{N}$, b_n is an even smooth function on $\mathbf{R}$ defined by

$$b_n(x) = \int_0^x K(x,u) j_{n-1/2}(i\rho u) \frac{u^{2n}}{(2n)!}\, du, \quad x > 0, \tag{1.6}$$

where

$$j_\mu(z) = \begin{cases} 2^\mu \Gamma(\mu+1) z^{-\mu} J_\mu(z) & \text{if } z \neq 0, \\ 1 & \text{if } z = 0, \end{cases}$$

J_μ being the Bessel function of the first kind and index μ. Here for every $x > 0$, $K(x,\cdot)$ is a nonnegative integrable function with support in $[-x, x]$ (see [**19**]). We assume that $b_{-n} = 0$ when $n \in \mathbf{N} \setminus \{0\}$.

The main properties of the functions b_n, $n \in \mathbf{N}$, were established in [**18**] and [**7**, Section 2.2]. In particular, we have that

$$b_0 = 1, \quad b_n(0) = 0, \quad \Delta b_n = -b_{n-1}, \quad n \in \mathbf{N},\ n \geq 1. \tag{1.7}$$

Moreover, according to [**7**, Corollary 2.1], for every $n \in \mathbf{N}$,

$$0 \leq b_n(x) \leq \frac{x^{2n}}{(2n)!}, \quad x \in \mathbf{R}, \tag{1.8}$$

and

$$0 \leq b_n'(x) \leq \frac{x^{2n-1}}{(2n-1)!}, \quad x \in \mathbf{R}. \tag{1.9}$$

Functions b_n, $n \in \mathbf{N}$, play in the generalized Taylor series of Trimèche [**18**] the role of the power functions in the usual Taylor series.

Throughout this paper we always represent by C a positive constant, which is not necessarily the same in each occurrence.

2. A universality property for #-convolution operators. In this section, inspired by the paper of Herzog [**12**], we investigate a universality property for certain #-convolution operators.

As in [**1**], we consider the set $\mathcal{A}$ constituted by all functions h defined on $[0, \infty)$ that are positive, decreasing, continuous on $[0, \infty)$ and that satisfy the following inequality

$$h(x+y) \geq C h(x) h(y), \quad x, y \in [0, \infty), \tag{2.1}$$

where $C > 0$ is independent of $x, y \in [0, \infty)$.

If $h \in \mathcal{A}$, E_h represents the function space that consists of all those continuous functions f on $[0, \infty)$ such that $\lim_{x\to\infty} h(x)f(x) = 0$. On E_h we define the norm $\|\cdot\|_h$ through

$$\|f\|_h = \sup_{x\in[0,\infty)} h(x)|f(x)|.$$

Thus E_h is a separable Banach space.

Let $0 < p \le 2$ and $\phi \in S_p((0,\infty), A)$. We define by T_ϕ the #-convolution operator given by

$$T_\phi(f) = \phi\# f, \quad f \in E_h.$$

Also, for every $n \in \mathbf{N}$, the operator $T_{\phi,n}$ is defined by

$$T_{\phi,n} f = \phi^{\# n}\# f, \quad f \in E_h,$$

where $\phi^{\# n} = \overbrace{\phi\#\cdots\#\phi}^{n}$.

Our universality result for the #-convolution operator T_ϕ is the following.

Proposition 2.1. *Let $\phi \in S_1((0,\infty), A)$ and $h \in \mathcal{A}$. Assume that the following conditions are satisfied*

(i) $\phi/h \in L^1((0,\infty), A(x)\,dx)$,

(ii) $\mathcal{F}(\phi)(i\rho) = 1$ *and* $|\mathcal{F}(\phi)(x)| < 1$ *for almost all* $x \in (0,\infty)$,

(iii) $h(x)\cosh(x)$ *is bounded on* $[0,\infty)$,

(iv) *the function* $F(z) = \sum_{n=0}^{\infty} \delta_n(\phi) z^n$, $z \in \mathbf{C}$, *is holomorphic in a neighborhood of the closed unit disk* $\overline{D}(0,1)$ *where* $\delta_n(\phi) = \int_0^\infty b_n(x)\phi(x)A(x)\,dx$, $n \in \mathbf{N}$.

Then the set

$$U_\phi = \{f \in E_h : \overline{\{T_{\phi,n}(f) : n \in \mathbf{N}\}} = C([0,\infty))\}$$

is a residual set of E_h, where $C([0,\infty))$ denotes the space of continuous functions on $[0,\infty)$ endowed of the topology of uniform convergence in $[0,a]$ for every $a > 0$.

Proof. Note firstly that the operator T_ϕ is bounded from E_h into $C[0,\infty)$. Indeed, let $f \in E_h$. According to (2.1) and by taking into account that the measure of the generalized translation operator is a probability measure on $[0,\infty)$ for every $x, y \in [0,\infty)$ ([**5**, p. 453] and [**3**, p. 90]), we have that

$$|(\tau_x f)(y)| \le C \, \frac{\|f\|_h}{h(x)h(y)}, \quad x, y \in [0,\infty).$$

Then it follows

$$|T_\phi(f)(x)| \le C\bigg(\frac{1}{h(x)} \int_0^\infty \Big(\frac{|\phi(y)|}{h(y)} A(y)\Big)\, dy\bigg) \|f\|_h, \; x \in [0,\infty).$$

Hence, by (i), if $a > 0$, one has

$$\sup_{0\le x\le a} |T_\phi(f)(x)| \le C\|f\|_h.$$

Note that, to see that T_ϕ is bounded from E_h into $C([0,\infty))$, it is sufficient that (i) holds.

We now prove that $T_{\phi,n}$ is a bounded linear mapping from E_h into $C([0,\infty))$, for every $n \in \mathbf{N}$. For this purpose, we will see that $\phi^{\#n}/h \in L^1((0,\infty), A(x)\,dx)$, for every $n \in \mathbf{N}$. We only establish this property for $n = 2$. An inductive procedure allows us to show the property for every $n \in \mathbf{N}$.

Since the measure of the generalized translation operator is supported on $[|x-y|, x+y]$, for every $x, y \in [0,\infty)$ and the function h is decreasing on $[0,\infty)$, we can write

$$\int_0^\infty \frac{|\phi \# \phi(x)|}{h(x)} A(x)\, dx \\ \le \int_0^\infty \int_0^\infty \frac{|\phi(y)|}{h(x)}\, h(|x-y|) \tau_x\bigg(\frac{|\phi|}{h}\bigg)(y) A(y) A(x)\, dy\, dx.$$

Moreover it is not hard to see that $h(y) \ge C h(y-x) h(x)$, $0 \le x \le y < \infty$. Then by dividing the integral in the y-variable as follows

$$\int_0^\infty |\phi(y)| h(|x-y|) \tau_x\bigg(\frac{|\phi|}{h}\bigg)(y) A(y)\, dy \\ = \bigg(\int_0^x + \int_x^\infty\bigg) |\phi(y)| h(|x-y|) \tau_x\bigg(\frac{|\phi|}{h}\bigg)(y) A(y)\, dy, \quad x \in (0,\infty),$$

a straightforward manipulation leads to

$$\int_0^\infty \frac{|\phi\#\phi(x)|}{h(x)}\,A(x)\,dx \le C\left\|\frac{|\phi|}{h}\#\frac{|\phi|}{h}\right\|_{L^1((0,\infty),A(x)\,dx)}.$$

Hence [**21**, (II.14)] and [**3**, Theorem 2.4] imply that $(\phi\#\phi)/h \in L^1((0,\infty),A(x)\,dx)$.

To simplify we denote by E_h^0 the subspace of E_h that consists of all those $f\in E_h$ such that $f(x)=0$, $x\ge a$, for some $a>0$. It is not hard to see that E_h^0 is a dense subspace of E_h.

We are going to see that, for every $f\in E_h^0$,

$$\lim_{n\to\infty} T_{\phi,n}(f)(x)=0,$$

uniformly in $x\in(0,\infty)$.

Fix $f\in E_h^0$. By [**3**, Theorem 2.4] and [**21**, Theorem II.3], we can write

$$T_{\phi,n}(f)(x)=\int_0^\infty \mathcal{F}(f)(y)(\mathcal{F}(\phi)(y))^n\psi_y(x)\,\frac{dy}{|c(y)|^2},$$
$$x\in(0,\infty)\quad\text{and}\quad n\in\mathbf{N}.$$

Trimèche [**21**] proved that there exist two constants, $c_1,c_2>0$, such that

$$c_1|y|^{2\alpha+1}\le|c(y)|^{-2}\le c_2|y|^{2\alpha+1},$$

when y is large enough. We choose $l\in\mathbf{N}$ such that $l>\alpha+1$. Since $(1+y^2)^l\mathcal{F}(\phi)(y)\to 0$, as $y\to\infty$, there exists $y_0>0$ for which $(1+y^2)^l|\mathcal{F}(\phi)(y)|<1/2$, $y>y_0$. Hence, according to [**3**, Lemma 3.4] and since $f\in E_h^0$, we obtain

$$\begin{aligned}&\left|\int_{y_0}^\infty \mathcal{F}(f)(y)(\mathcal{F}(\phi)(y))^n\psi_y(x)\frac{dy}{|c(y)|^2}\right|\\ &\qquad\le C\int_{y_0}^\infty((1+y^2)^l|\mathcal{F}(\phi)(y)|)^n\frac{y^{2\alpha+1}}{(1+y^2)^{nl}}\,dy \qquad (2.2)\\ &\qquad\le C\left(\frac{1}{2}\right)^n\int_0^\infty\frac{y^{2\alpha+1}}{(1+y^2)^{nl}}\,dy,\quad x\in(0,\infty).\end{aligned}$$

On the other hand, from the fact that $|c(\lambda)|^{-2}$ is continuous on $[0,\infty)$ we can write

$$\left|\int_0^{y_0} \mathcal{F}(f)(y)(\mathcal{F}(\phi)(y))^n \psi_y(x)\,\frac{dy}{|c(y)|^2}\right| \le C\int_0^{y_0} |\mathcal{F}(\phi)(y)|^n\,dy,\quad x\in(0,\infty). \tag{2.3}$$

Then, by combining (2.2) and (2.3) and using the dominated convergence theorem, we conclude in view of (ii) that $\lim_{n\to\infty} T_{\phi,n}(f)(x)=0$, uniformly in $x\in(0,\infty)$.

Thus we prove that the set

$$\{f\in E_h : \lim_{n\to\infty} T_{\phi,n} f \text{ exists in } C[0,\infty)\}$$

is dense in E_h.

Hence, according to [**10**], our result is shown when we prove that the set U_ϕ, defined by

$$U_\phi=\{f\in E_h : \overline{\{T_{\phi,n}(f) : n\in\mathbf{N}\}} = C([0,\infty))\},$$

is not empty.

We now define the Banach space F_h as follows. We define first the space F. We say that a function f is in F when it can be written as

$$f(x)=\sum_{n=0}^{\infty} b_n(x)a_n,\quad x\in[0,\infty), \tag{2.4}$$

where $(a_n)_{n\in\mathbf{N}}\in c_0$. Here c_0 denotes the space of real sequences that converge to zero.

Note that by (1.8) the series in (2.4) is convergent, for every $x\in(0,\infty)$.

For $f\in F$ we put

$$\|f\|_h=\sup_{\substack{p\in\mathbf{N}\\ x\in(0,\infty)}} h(x)|\Delta^p f(x)|.$$

Note that, as f has the representation (2.4), then by (1.6), (1.7), (1.8) and (1.9), it follows

$$h(x)|\Delta^p f(x)| \leq Ch(x) \sum_{n=p}^{\infty} b_{n-p}(x) \leq Ch(x) \cosh(x), \quad x \in (0, \infty).$$

Hence $\sup_{p\in \mathbf{N}, x\in(0,\infty)} h(x)|\Delta^p f(x)| < \infty$. Then $\|\cdot\|_h$ defines a norm in F. We represent by F_h the space F when it is equipped with the norm $\|\cdot\|_h$.

We now prove that if c_0 is endowed with its usual topology, that is, the topology associated to the norm $\|\cdot\|_\infty$ defined by

$$\|(a_n)_{n\in\mathbf{N}}\|_\infty = \sup_{n\in\mathbf{N}} |a_n|, \quad (a_n)_{n\in\mathbf{N}} \in c_0,$$

then the mapping L defined by

$$L((a_n)_{n\in\mathbf{N}})(x) = \sum_{n=0}^{\infty} a_n b_n(x), \quad x \in (0, \infty),$$

is a homeomorphism from c_0 onto F_h.

Note first that, according to (1.7), L is a one-to-one mapping from c_0 onto F.

Indeed, let $(a_n)_{n\in\mathbf{N}} \in c_0$. We have that

$$\begin{aligned}\|L((a_n)_{n\in\mathbf{N}}\|_h &= \sup_{\substack{p\in\mathbf{N}\\ x\in(0,\infty)}} h(x)\left|\sum_{n=p}^{\infty} b_{n-p}(x) a_n\right| \\ &\leq \|(a_n)_{n\in\mathbf{N}}\|_\infty \sup_{x\in(0,\infty)} h(x)\cosh(x).\end{aligned}$$

Thus, from (iii), we have seen that L is a continuous mapping from c_0 into F_h.

Let now $f \in F$. Assume that $f = L((a_n)_{n\in\mathbf{N}})$ where $(a_n)_{n\in\mathbf{N}} \in c_0$. Then by [**18**, p. 1016], since $a_n = (-1)^n \Delta^n f(0)$, $n \in \mathbf{N}$, we have

$$\|(a_n)_{n\in\mathbf{N}}\|_\infty = \sup_{n\in\mathbf{N}} |a_n| = \sup_{p\in\mathbf{N}} |\Delta^p f(0)| \leq \frac{1}{h(0)} \|f\|_h.$$

Hence L^{-1} is continuous from F_h into c_0.

Since c_0 is a separable Banach space, F_h is also a separable Banach space.

The space F is contained in E_h. Indeed, let $f \in F$. Assume that

$$f(x) = \sum_{n=0}^{\infty} a_n b_n(x), \quad x \in [0, \infty),$$

where $(a_n)_{n\in\mathbf{N}} \in c_0$. Fix $\varepsilon > 0$. There exists $n_0 \in \mathbf{N}$ such that $|a_n| < \varepsilon$, $n > n_0$. Hence, from (1.8) and (iii), one has

$$\left| h(x) \sum_{n=n_0+1}^{\infty} a_n b_n(x) \right| \le \varepsilon h(x) \sum_{n=0}^{\infty} b_n(x) \le C\varepsilon, \quad x \in (0, \infty).$$

Moreover, from (iii), since $\lim_{x\to\infty} h(x)x^{2n} = 0$, for every $n \in \mathbf{N}$, there exists $x_0 > 0$ such that, for all $x \ge x_0$,

$$\left| h(x) \sum_{n=0}^{n_0} a_n b_n(x) \right| \le \varepsilon.$$

Hence $\lim_{x\to\infty} f(x) = 0$.

Moreover, the space F is dense in $C([0, \infty))$. To see this it is sufficient to prove that, for every $k \in \mathbf{N}$, $p_k(z) = z^{2k}$, $z \in [0, \infty)$, is in the closure of the linear space generated by $\{b_n\}_{n\in\mathbf{N}}$ in $C([0, a])$ for each $a > 0$. Let $k > \mathbf{N}$ and $a > 0$. We define a C^∞-function q_k on $\mathbf{R}$ such that $q_k(x) = 0$, $|x| > a + 1$ and $q_k(x) = p_k(x)$, $|x| < a$. It is clear that $q_k \in S_p((0, \infty), A)$, with $0 < p \le 2$. Hence, according to [**3**, Theorem 4.27], $\mathcal{F}(q_k) \in L^1((0, \infty), dy/|c(y)|^2)$ and we can write

$$q_k(x) = \int_0^\infty \psi_y(x) \mathcal{F}(q_k)(y) \, \frac{dy}{|c(y)|^2}, \quad x \in [0, \infty).$$

Let $\varepsilon > 0$. By [**3**, Lemma 3.4] there exists $y_0 > 0$ for which

$$\left| \int_{y_0}^\infty \psi_y(x) \mathcal{F}(q_k)(y) \, \frac{dy}{|c(y)|^2} \right| \le \int_{y_0}^\infty |\mathcal{F}(q_k)(y)| \, \frac{dy}{|c(y)|^2} | < \varepsilon, \quad x \in [0, \infty).$$

Moreover, since the function $\psi_y(x)\mathcal{F}(q_k)(y)/|c(y)|^2$ is uniformly continuous in $\{(x,y) : x \in [0,a], y \in [0,y_0]\}$, we can put

$$\int_0^{y_0} \psi_y(x)\mathcal{F}(q_k)(y)\,\frac{dy}{|c(y)|^2} = \lim_{n\to\infty} \frac{y_0}{n} \sum_{j=1}^{n} \psi_{y_0 j/n}(x)\mathcal{F}(q_k)\bigg(\frac{y_0 j}{n}\bigg)\bigg|c\bigg(\frac{y_0 j}{n}\bigg)\bigg|^{-2},$$

uniformly in $x \in [0,a]$.

Hence, there exists $n_0 \in \mathbf{N}$ such that

$$\bigg|p_k(x) - \frac{y_0}{n} \sum_{j=1}^{n} \psi_{y_0 j/n}(x)\mathcal{F}(q_k)\bigg(\frac{y_0 j}{n}\bigg)\bigg|c\bigg(\frac{y_0 j}{n}\bigg)\bigg|^{-2}\bigg| < \varepsilon, \quad x \in [0,a],$$

provided that $n \geq n_0$.

Thus we prove that p_k is in the closure of the linear space generated by $\{\psi_y\}_{y>0}$ in $C([0,a])$.

On the other hand, by (1.5) we have that, for every $y > 0$,

$$\psi_y(x) = \sum_{n=0}^{\infty} (-1)^n b_n(x)(y^2 + \rho^2)^n, \quad x \in \mathbf{R},$$

where the series is uniformly convergent in $[0,a]$. Hence, for every $y > 0$, ψ_y is in the closure of the linear space generated by $\{b_n\}_{n\in\mathbf{N}}$ in $C([0,a])$.

Then we conclude that p_k belongs to the closure of the linear space generated by $\{b_n\}_{n\in\mathbf{N}}$ in $C([0,a])$.

Our next purpose is to see that, for every $f \in F$,

$$T_\phi(f) = K_\phi(f),$$

where

$$K_\phi(f)(x) = \sum_{n=0}^{\infty} (-1)^n \delta_n(\phi)\Delta^n f(x), \quad x \in (0,\infty).$$

Here $\delta_n = \int_0^\infty b_n(y)\phi(y)A(y)\,dy$, $n \in \mathbf{N}$. Note that by (1.8) and [**3**], the integral defining δ_n is absolutely convergent for every $n \in \mathbf{N}$. Moreover, by assumption $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

Suppose firstly that

$$f(x) = \sum_{n=0}^{p} b_n(x)a_n, \quad x \in (0,\infty),$$

with $p \in \mathbf{N}$ and $a_n \in \mathbf{R}$, $n = 0, 1, \dots, p$. It is clear that $f \in F$.

We have that, by [**18**, Theorem 4],

$$\begin{aligned} T_\phi(f)(x) &= (\phi \# f)(x) \\ &= \int_0^\infty \phi(y)(\tau_x f)(y)A(y)\,dy \\ &= \int_0^\infty \phi(y)\bigg(\sum_{k=0}^{p}(-1)^k b_k(x)\Delta^k f(y)\bigg)A(y)\,dy \\ &= \sum_{k=0}^{p} b_k(x)\bigg(\sum_{n=k}^{p} a_n \delta_{n-k}(\phi)\bigg), \quad x \in (0,\infty). \end{aligned}$$

Moreover, we derive

$$\begin{aligned} K_\phi(f)(x) &= \sum_{n=0}^{\infty}(-1)^n \delta_n(\phi)\Delta^n f(x) \\ &= \sum_{n=0}^{p} \delta_n(\phi)\bigg(\sum_{k=n}^{p} b_{k-n}(x)a_k\bigg) \\ &= \sum_{k=0}^{p} b_k(x)\bigg(\sum_{n=k}^{p} \delta_{n-k}(\phi)a_n\bigg), \quad x \in (0,\infty). \end{aligned}$$

Hence $K_\phi(f) = T_\phi(f)$.

We now introduce the set $\mathcal{H}$ defined as follows

$$\mathcal{H} = \{f \in F : f(x) = \sum_{n=0}^{p} a_n b_n(x) \quad \text{with } a_n \in \mathbf{R},\ n = 0, 1, \dots, p \text{ and } p \in \mathbf{N}\}.$$

Thus $\mathcal{H}$ is a dense subspace of $\mathcal{F}_h$. Indeed, let $f \in F$ with the representation (2.4) where $(a_n)_{n\in\mathbf{N}} \in c_0$. Then we can write

$$\sup_{\substack{p\in\mathbf{N}\\ x\in(0,\infty)}} h(x)\bigg|\Delta^p\bigg(\sum_{n=k}^{\infty} b_n(x)a_n\bigg)\bigg| \le \sup_{n\ge k}|a_n| \sup_{x\in(0,\infty)} h(x)\cosh(x), \quad k \in \mathbf{N}.$$

Hence by (iii) since $(a_n)_{n\in\mathbf{N}} \in c_0$ we deduce that

$$\lim_{k\to\infty} \sum_{n=0}^{k} b_n(x)a_n = f(x),$$

in the sense of convergence in F_h.

Then, to see that $K_\phi(f) = T_\phi(f)$, $f \in F$, it is sufficient to show that K_ϕ is a continuous linear mapping from F_h into $C[0,\infty)$. Let $a > 0$. We can write for all $x \in [0,a]$:

$$\begin{aligned}|K_\phi(f)(x)| &\le \sum_{n=0}^{\infty} |\delta_n(\phi)|\,|\Delta^n f(x)| \\ &\le \bigg(\sum_{n=0}^{\infty} |\delta_n(\phi)| \frac{1}{h(a)}\bigg)\Big(\sup_{\substack{p\in\mathbf{N}\\ z\in(0,\infty)}} h(z)|\Delta^p f(z)|\Big).\end{aligned}$$

Hence K_ϕ defines a continuous operator from F_h into $C([0,\infty))$.

It is clear that the operator Δ is bounded from F_h into F_h. Then by assumption (iv), T_ϕ is bounded from F_h into F_h.

We now show that

$$T_\phi(T_\phi(f)) = T_{\phi,2}(f), \quad f \in F.$$

Indeed, let $k \in \mathbf{N}$. According to (1.5), we have

$$b_k(x) = \frac{(-1)^k}{2^k k!}\bigg(\frac{1}{\lambda}\frac{d}{d\lambda}\bigg)^k \psi_\lambda(x)_{|\lambda=i\rho}, \quad x \in (0,\infty).$$

Hence, by (1.4) we get for $x, y \in (0,\infty)$

$$\begin{aligned}(\tau_x b_k)(y) &= \frac{(-1)^k}{2^k k!}\bigg(\frac{1}{\lambda}\frac{d}{d\lambda}\bigg)^k \int_0^\infty \psi_\lambda(z)D(x,y,z)A(z)\,dz_{|\lambda=i\rho} \\ &= \frac{(-1)^k}{2^k k!}\bigg(\frac{1}{\lambda}\frac{d}{d\lambda}\bigg)^k (\psi_\lambda(x)\psi_\lambda(y))_{|\lambda=i\rho} \\ &= \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j} b_j(x)b_{k-j}(y)j!(k-j)! \\ &= \sum_{j=0}^{k} b_j(x)b_{k-j}(y), \quad k \in \mathbf{N}.\end{aligned} \tag{2.5}$$

This result is also true for $x = y = 0$ and for x or y equal to zero.

Then, by [**3**, Theorem 2.4], it follows for every $x \in (0, \infty)$

$$\begin{aligned} T_\phi(b_k)(x) &= \int_0^\infty b_k(y)(\tau_x\phi)(y)A(y)\,dy \\ &= \int_0^\infty (\tau_x b_k)(y)\phi(y)A(y)\,dy \\ &= \sum_{j=0}^{k} b_j(x)\int_0^\infty b_{k-j}(y)\phi(y)A(y)\,dy, \quad x \in (0,\infty). \end{aligned}$$

Hence, for every $x \in (0, \infty)$,

$$\begin{aligned} &T_\phi(T_\phi(b_k))(x) \\ &= \sum_{j=0}^{k}\sum_{l=0}^{j} b_l(x)\bigg(\int_0^\infty b_{j-l}(y)\phi(y)A(y)\,dy\bigg)\bigg(\int_0^\infty b_{k-j}(y)\phi(y)A(y)\,dy\bigg). \end{aligned}$$

On the other hand, we obtain in a similar way

$$\begin{aligned} T_{\phi\#\phi}(b_k)(x) &= \sum_{j=0}^{k} b_j(x)\int_0^\infty b_{k-j}(y)(\phi\#\phi)(y)A(y)\,dy \\ &= \sum_{j=0}^{k} b_j(x)\frac{(-1)^{k-j}}{2^{k-j}(k-j)!}\bigg(\frac{1}{\lambda}\frac{d}{d\lambda}\bigg)^{k-j}\mathcal{F}(\phi\#\phi)(\lambda)_{|\lambda=i\rho} \\ &= \sum_{j=0}^{k} b_j(x)\sum_{l=0}^{k-j}\frac{(-1)^{l}}{2^{l}l!}\bigg(\frac{1}{\lambda}\frac{d}{d\lambda}\bigg)^{l}\mathcal{F}(\phi)(y)_{|\lambda=i\rho} \\ &\quad\times \frac{(-1)^{k-j-l}}{2^{k-j-l}(k-j-l)!}\bigg(\frac{1}{\lambda}\frac{d}{d\lambda}\bigg)^{k-j-l}\mathcal{F}(\phi)(\lambda)|_{\lambda=i\rho} \\ &= \sum_{j=0}^{k} b_j(x)\sum_{l=0}^{k-j}\bigg(\int_0^\infty b_l(y)\phi(y)A(y)\,dy\bigg) \\ &\quad\times\bigg(\int_0^\infty b_{k-j-l}(y)\phi(y)A(y)\,dy\bigg), \quad x \in (0,\infty). \end{aligned}$$

By interchanging the order of summation we conclude that

$$T_\phi(T_\phi(b_k)) = T_{\phi,2}(b_k).$$

Hence

$$T_\phi(T_\phi(f)) = T_{\phi,2}(f), \quad f \in \mathcal{H}. \tag{2.6}$$

On the other hand, $\phi\#\phi$ satisfies the conditions (i), (ii) and (iv) listed in this proposition. Indeed, property (i) was established above for $\phi\#\phi$.

Also by [**3**, Lemma 2.4], we have that

$$\mathcal{F}(\phi\#\phi)(i\rho) = \mathcal{F}(\phi)(i\rho)\mathcal{F}(\phi)(i\rho) = 1,$$

and

$$|\mathcal{F}(\phi\#\phi)(x)| = |\mathcal{F}(\phi)(x)||\mathcal{F}(\phi)(x)| < 1, \text{ for almost all } x \in (0,\infty).$$

Thus $\phi\#\phi$ fulfills (ii).

Finally we see that $\phi\#\phi$ verifies (iv). Let $n \in \mathbf{N}$. As we proved above, according to the definition of $\phi\#\phi$ and [**18**, Theorem 4], we now find

$$\begin{aligned}\delta_n(\phi\#\phi) &= \int_0^\infty b_n(x)(\phi\#\phi)(x)A(x)\,dx \\ &= \sum_{j=0}^n \bigg(\int_0^\infty b_j(y)\phi(y)A(y)\,dy\bigg)\bigg(\int_0^\infty b_{n-j}(y)\phi(y)A(y)\,dy\bigg) \\ &= \sum_{j=0}^n \delta_j(\phi)\delta_{n-j}(\phi),\end{aligned}$$

where $\delta_j(\phi)$ and $\delta_j(\phi\#\phi)$ have the obvious meaning, for every $j \in \mathbf{N}$.

Also we can write for z in a neighborhood of the closed unit disk $\overline{D}(0,1)$

$$\begin{aligned}\bigg(\sum_{k=0}^\infty \delta_n(\phi)z^n\bigg)\bigg(\sum_{j=0}^\infty \delta_j(\phi)z^n\bigg) &= \sum_{n=0}^\infty \bigg(\sum_{k=0}^n \delta_k(\phi)\delta_{n-k}(\phi)\bigg)z^n \\ &= \sum_{n=0}^\infty \delta_n(\phi\#\phi)z^n.\end{aligned}$$

Hence (iv) is satisfied for $\phi\#\phi$.

Since $\phi\#\phi$ satisfies the same properties as ϕ, we infer that the operator $T_{\phi,2}$ is bounded from F_h into itself. Hence, since $\mathcal{H}$ is a dense subspace of F_h, (2.6) implies that $T_{\phi,2}f = T_\phi(T_\phi(f))$, for every $f \in F$.

An inductive argument allows us to conclude that

$$T_{\phi,n}(f) = \overbrace{T_\phi \dots T_\phi}^{n}(f), \quad f \in F \quad \text{and} \quad n \in \mathbf{N}.$$

On the other hand, according to (1.7), the operator Δ is onto. Also the set $\mathcal{H}$ is contained in $\cup_{n\in\mathbf{N}}\mathrm{Ker}\,(\Delta^n)$. Hence $\cup_{n\in\mathbf{N}}\mathrm{Ker}\,(\Delta^n)$ is a dense subset of F_h. Since $\delta_0(\phi) = \mathcal{F}(\phi)(i\rho) = 1$ and the spectrum $\sigma(\Delta)$ of Δ is contained in the closed unit disk $\overline{D}(0,1)$, [**16**, Corollary 1] allows us to deduce that T_ϕ is hypercyclic. That is, the set

$$\{f \in F : \overline{\{T_{\phi,n}f : n \in \mathbf{N}\}}^{F_h} = F\}$$

is not empty. By taking into account that the topology of F_h is stronger than the one induced on F by $C([0,\infty))$ and that F is a dense subspace of $C([0,\infty))$, we conclude that the set U_ϕ is not empty.

Thus the proof is finished. □

3. Hypercyclic and chaotic convolution operators on the spaces E and $\mathcal{D}'$. Godefroy and Shapiro characterized the continuous linear mapping on $H(\mathbf{C}^n)$, the space of holomorphic functions on $\mathbf{C}^n$ that commutes with usual translations [**9**, Proposition 5.2]. As a consequence of that result, they extended classical works of Birkhoff and MacLane about the hypercyclicity of translation and differentiation on $H(\mathbf{C})$, and they proved that every partial differential operator on $\mathbf{R}^n$ which is not a scalar multiple of the identity is hypercyclic and chaotic [**9**, Theorems 5.1 and 6.2]. Recently, Bonet [**4**] showed that usual convolution operators on spaces of ultradifferentiable functions of Beurling and Roumieu type are hypercyclic and chaotic when they are not scalar multiples of the identity. Our purpose in this section is to obtain a version of Bonet's result for #-convolution operators on **E** and $\mathcal{D}'$.

We first introduce a space of functions that will play the same role in our study as the space of entire functions in the theory developed by Godefroy and Shapiro [**9**].

We denote by $\mathcal{G}$ the space that consists of all the complex sequences $(a_n)_{n\in\mathbf{N}}$ such that the series $\sum_{n=0}^{\infty}|a_n|(|x|^n/(2n)!)$ converges, for every $x \in \mathbf{R}$. The function space $\mathbf{H}$ is defined as follows. An even function f is defined on $\mathbf{R}$ is in $\mathbf{H}$ if and only if there exists $(a_n)_{n\in\mathbf{N}} \in \mathcal{G}$ such that $f(x) = \sum_{n=0}^{\infty} a_n b_n(x)$, $x \in \mathbf{R}$. Note that, according to (1.8), if $(a_n)_{n\in\mathbf{N}} \in \mathcal{G}$, then $\sum_{n=0}^{\infty}|a_n| b_n(x)$ converges uniformly in $x \in [0, a]$ for every $a > 0$.

Proposition 3.1. *If $f \in \mathbf{H}$, then $f \in \mathbf{E}$. Moreover, if $(a_n)_{n\in\mathbf{N}} \in \mathcal{G}$ then the series $\sum_{n=0}^{\infty} a_n b_n$ converges in $\mathbf{E}$.*

Proof. According to (1.6) we can write

$$b_n = \mathcal{X}\biggl(\frac{u^{2n}}{(2n)!}\, j_{n-1/2}\,(ipu)\biggr), \quad n \in \mathbf{N},$$

where $\mathcal{X}$ represents the generalized Riemann-Liouville transform defined by [**19**]

$$\mathcal{X}(f)(x) = \int_0^x K(x,y) f(y)\, dy, \quad x \in [0, \infty).$$

Here the function K is understood as in the introduction. Since $\mathcal{X}$ is an automorphism on $\mathbf{E}$, for every $m \in \mathbf{N}$ and $a > 0$ there exist $s, l \in \mathbf{N}$ and $C, w > 1$ such that

(3.1)

$$\begin{aligned}\sup_{|x|\le a}\left|\frac{d^m}{dx^m} b_n(x)\right| &\le C \max_{j=0,1,\dots,s}\sup_{|y|\le w}\left|\frac{d^j}{dy^j}\biggl(\frac{y^{2n}}{(2n)!} j_{n-1/2}(i\rho y)\biggr)\right| \\ &\le C \max_{j=0,1,\dots,l}\sup_{|y|\le w}\left|\biggl(\frac{1}{y}\frac{d}{dy}\biggr)^j\biggl(\frac{y^{2n}}{(2n)!} j_{n-1/2}(i\rho y)\biggr)\right| \\ &\le C\frac{w^{2n}}{(2(n-l))!}, \quad n \in \mathbf{N}, \quad n > l.\end{aligned}$$

In the last inequality we have used [**23**, Section 5.1, (7)] and that $j_\mu(iu) \le \cosh u$, $u \in \mathbf{R}$ [**7**, p. 246]. Let $(a_n)_{n\in\mathbf{N}}$ be in $\mathcal{G}$. From (3.1) we deduce that, for every $m \in \mathbf{N}$, the series $\sum_{n=0}^{\infty} a_n(d^m/dx^m) b_n(x)$ is uniformly convergent in $[-a, a]$ for each $a > 0$. Hence the function f defined by

$$f(x) = \sum_{n=0}^{\infty} a_n b_n(x), \quad x \in \mathbf{R},$$

is in $\mathbf{E}$. The above argument shows also that the series $\sum_{n=0}^{\infty} a_n b_n$ converges in E. □

Proposition 3.2. $\mathbf{H}$ *is a dense subspace of* $\mathbf{E}$.

Proof. Let $T \in \mathbf{E}'$. The generalized Fourier transform $\mathcal{F}T$ is defined by

$$(\mathcal{F}T)(\lambda) = \langle T(x), \psi_\lambda(x) \rangle, \quad \lambda \in \mathbf{C}$$

(see [**19**]). Suppose that $T_{|\mathbf{H}} = 0$. Then, since $\psi_\lambda \in \mathbf{H}$, for every $\lambda \in \mathbf{C}$ (see (1.5)), $\mathcal{F}T = 0$. Hence $T = 0$. Hahn-Banach's theorem allows us to conclude that $\mathbf{H}$ is dense in $\mathbf{E}$. □

We consider on the space $\mathbf{H}$ the topology induced on it by the space $\mathbf{E}$. By (1.7) it is not hard to see that the operator Δ defines a linear and continuous mapping from $\mathbf{H}$ into itself. The behavior of the translation operator τ_x, $x \in [0, \infty)$ on $\mathbf{H}$ is presented in the following.

Proposition 3.3. *Let* $x \in [0, \infty)$. *The translation operator* τ_x *is a linear and continuous mapping from* $\mathbf{H}$ *into itself. Moreover, if* $f = \sum_{n=0}^{\infty} a_n b_n$ *where* $(a_n)_{n \in \mathbf{N}} \in \mathcal{G}$, *then*

$$\tau_x f = \sum_{n=0}^{\infty} (-1)^n b_n(x) \Delta^n f,$$

where the last series converges on $\mathbf{E}$.

Proof. Assume that $(a_n)_{n \in \mathbf{N}} \in \mathcal{G}$ and write

$$f(y) = \sum_{n=0}^{\infty} a_n b_n(y), \quad y \in \mathbf{R}.$$

Let $x \in [0,\infty)$. Since the last series converges uniformly in every compact subset of $\mathbf{R}$, we have by using (2.5),

$$(3.2)\qquad \begin{aligned}(\tau_x f)(y) &= \sum_{n=0}^{\infty} a_n (\tau_x b_n)(y)\\ &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} b_k(x) b_{n-k}(y)\\ &= \sum_{n=0}^{\infty} b_n(y) \sum_{j=0}^{\infty} a_{n+j} b_j(x), \quad y \in [0,\infty).\end{aligned}$$

Note that by (1.8), for every $y \in [0,\infty)$,

$$\begin{aligned}\sum_{k=0}^{\infty} \left| \sum_{n=0}^{\infty} b_n(x) a_{n+k} \right| \frac{|y|^k}{(2k)!} &\leq \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} b_n(x) |a_{n+k}| \frac{|y|^k}{(2k)!}\\ &\leq \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} |a_{n+k}| \frac{|x|^{2n}}{(2n)!} \frac{|y|^k}{(2k)!}\\ &\leq \sum_{n=0}^{\infty} |a_n| \sum_{k=0}^{n} \binom{2n}{2k} |x|^{2(n-k)} |y|^k\\ &\leq \sum_{n=0}^{\infty} \frac{|a_n|}{(2n)!} (x+\sqrt{y})^{2n} < \infty.\end{aligned}$$

Hence $\tau_x f \in H$.

Finally, from [**19**, Proposition 8.3], we infer that τ_x defines a continuous linear mapping from $\mathbf{H}$ into itself.

According to (3.2) we can write that

$$(3.3)\qquad (\tau_x f)(y) = \sum_{n=0}^{\infty} (-1)^n b_n(x) \Delta^n f(y), \quad y \in [0,\infty).$$

Let $x \in]0, \infty)$. We have that, for every m, $l_1, l_2 \in \mathbf{N}$, $l_1 < l_2$.

$$\begin{aligned}
&\left|\Delta_y^m \sum_{n=l_1}^{l_2} (-1)^n b_n(x) \Delta^n f(y)\right| \\
&\quad = \left|\Delta_y^m \sum_{n=l_1}^{l_2} b_n(x) \sum_{j=0}^{\infty} a_{n+j} b_j(y)\right| \\
&\quad \leq \sum_{n=l_1}^{l_2} b_n(x) \sum_{j=m}^{\infty} |a_{n+j}| b_{j-m}(y) \\
&\quad \leq \sum_{j=l_1+m}^{l_2+m} \frac{|a_j|}{(2(j-m))!} \sum_{n=l_1}^{j-m} \frac{x^{2n}}{(2n)!} \frac{y^{2(j-m-n)}}{(2(j-m-n))!} (2(j-m))! \\
&\qquad + \sum_{j=l_2+m}^{\infty} \frac{|a_j|}{(2(j-m))!} \sum_{n=l_1}^{l_2} \frac{x^{2n}}{(2n)!} \frac{y^{2(j-m-n)}}{(2(j-m-n))!} (2(j-m))! \\
&\quad \leq \sum_{j=l_1+m}^{l_2+m} \frac{|a_j|}{(2(j-m))!} \sum_{n=l_1}^{j-m} \binom{2(j-m)}{2(j-m-n)} x^{2n} y^{2(j-m-n)} \\
&\qquad + \sum_{j=l_2+m}^{\infty} \frac{|a_j|}{(2(j-m))!} \sum_{n=l_1}^{l_2} \binom{2(j-m)}{2(j-m-n)} x^{2n} y^{2(j-m-n)} \\
&\quad \leq \sum_{j=l_1+m}^{\infty} \frac{|a_j|}{(2(j-m))!} (x+y)^{2(j-m)}, \qquad y \in [0, \infty).
\end{aligned}$$

Hence the series in (3.3) converges in $\mathbf{E}$. For $x = 0$ the result is also true. $\square$

We will say that a complex sequence $(c_n)_{n\in\mathbf{N}}$ is in $\mathcal{P}$ when there exist C, $r > 0$ and $l \in \mathbf{N}$ such that

$$|c_n| \leq C \frac{r^{2n}}{(2n-l))!}, \quad n \in \mathbf{N}.$$

We define the operator $T_{(c_n)_{n\in\mathbf{N}}}$ on $\mathbf{H}$ as follows

$$T_{(c_n)_{n\in\mathbf{N}}} f = \sum_{n=0}^{\infty} c_n \Delta^n f, \quad f \in \mathbf{H}.$$

By proceeding as in the proof of Proposition 3.3, we can show the next property.

Proposition 3.4. *Let $(c_n)_{n\in\mathbf{N}} \in \mathcal{P}$. Then the operator $T_{(c_n)_{n\in\mathbf{N}}}$ is linear and continuous from $\mathbf{H}$ into itself. Moreover the series $\sum_{n=0}^{\infty} c_n \Delta^n f$ converges in $\mathbf{E}$ for every $f \in \mathbf{H}$.*

By using the Hahn-Banach and Riesz representation theorems we can establish the following representation of the elements of $\mathbf{E}'$ that will be useful in the sequel.

Proposition 3.5. *Let T be a linear operator from $\mathbf{E}$ into $\mathbf{C}$. Then $T \in \mathbf{E}'$ if and only if there exist $k \in \mathbf{N}$ and $F_0, F_1, \dots, F_k$ in $L^2((0,\infty), A(x)\,dx)$ with compact support in $[0,\infty)$ such that*

$$\langle T, f\rangle = \sum_{j=0}^{k} \int_0^\infty F_j(x)\Delta^j f(x) A(x)\,dx, \quad f \in E. \tag{3.4}$$

We now characterize the linear and continuous operator on $\mathbf{H}$ that commutes with the translation operator τ_x, $x \in [0,\infty)$. Our next result corresponds in our theory to [**9**, Proposition 5.2].

Proposition 3.6. *Assume that L is a linear and continuous mapping from $\mathbf{E}$ into itself. The following properties are equivalent.*

(a) *L commutes with the translation operator τ_x, $x \in [0,\infty)$, that is, $L\tau_x = \tau_x L$, $x \in [0,\infty)$,*

(b) *There exists $T \in \mathbf{E}'$ such that $Lf = T\#f$, $f \in \mathbf{E}$.*

(c) *There exist $k \in \mathbf{N}$ and $F_0, F_1, \dots, F_k$ functions in $L^2((0,\infty), A(x)\,dx)$ with compact support in $[0,\infty)$ such that*

$$L(f) = \sum_{j=0}^{k} \int_0^\infty F_j(y)\tau_x(\Delta^j f)(y) A(y)\,d(y), \quad f \in \mathbf{E} \text{ and } x \in [0,\infty).$$

(d) *There exists $(c_n)_{n\in\mathbf{N}} \in \mathcal{P}$ such that $L_{|\mathbf{H}} = T_{(c_n)_{n\in\mathbf{N}}}$.*

(e) L *commutes with the operator* Δ, *that is*, $L\Delta = \Delta L$.

Proof. Since the operator Δ and τ_x, $x \in [0,\infty)$, commute, the equivalence (b) $\Leftrightarrow$ (c) can be deduced from Proposition 3.5.

(a) $\Rightarrow$ (c). We define the functional

$$\langle T, f\rangle = (Lf)(0), \quad f \in \mathbf{E}.$$

It is clear that $T \in \mathbf{E}'$. Then by Proposition 3.5, there exist $k \in \mathbf{N}$ and $F_0, F_1, \dots, F_k$ functions in $L^2((0,\infty), A(x)\,dx)$ with compact support in $[0,\infty)$, such that

$$\langle T, f\rangle = \sum_{j=0}^{k} \int_0^\infty F_j(y)\Delta^j f(y) A(y)\,dy, \quad f \in \mathbf{E}.$$

Hence by (a) we can write

$$\begin{aligned}(Lf)(x) &= \tau_x(Lf)(0)\\ &= L(\tau_x f)(0)\\ &= \sum_{j=0}^{k} \int_0^\infty F_j(y)\tau_x(\Delta^j f)(y)A(y)\,dy, \quad f \in \mathbf{E}.\end{aligned}$$

(c) $\Rightarrow$ (d). Assume that, for every $f \in \mathbf{E}$,

$$(Lf)(x) = \sum_{j=0}^{k} \int_0^\infty F_j(y)\tau_x(\Delta^j f)(y)A(y)\,dy, \quad x \in [0,\infty),$$

where $F_0, F_1, \dots, F_k$ are functions in $L^2((0,\infty), A(x)\,dx)$ with compact support in $[0,\infty)$ for a certain $k \in \mathbf{N}$.

Let $f \in \mathbf{H}$. According to Proposition 3.3, we get

$$\begin{aligned}(Lf)(x) &= \sum_{j=0}^{k}\sum_{n=0}^{\infty}(-1)^n \Delta^{n+j} f(x) \int_0^\infty F_j(y) b_n(y) A(y)\,dy\\ &= \sum_{n=0}^{\infty} \Delta^n f(x) \sum_{l=0}^{k}(-1)^{n-l} \int_0^\infty F_l(y) b_{n-l}(y) A(y)\,dy,\\ &\qquad x \in [0,\infty).\end{aligned}$$

Let $a > 0$ be such that the supports of $F_0, \dots, F_k$ are contained in $[0, a]$. Then by (1.8) we have

$$\begin{aligned}\left|\sum_{l=0}^{k}\int_0^\infty F_l(y)b_{n-l}(y)A(y)\,dy\right| &\leq \sum_{l=0}^{k}\int_0^a F_l(y)\,\frac{y^{2(n-l)}}{(2(n-l))!}A(y)\,dy,\\ &\leq C\sum_{l=0}^{k}\frac{a^{2(n-l)}}{(2(n-l))!}\\ &\leq C\,\frac{a^{2n}}{(2(n-k))!},\quad n\in\mathbf{N}.\end{aligned}$$

Hence, by writing

$$c_n = \sum_{l=0}^{k}(-1)^{n-l}\int_0^\infty F_l(y)b_{n-l}(y)A(y)\,dy,\quad n\in\mathbf{N},$$

the sequence $(c_n)_{n\in\mathbf{N}}$ is in $\mathcal{P}$ and $L_{|\mathbf{H}} = T_{(c_n)_{n\in\mathbf{N}}}$.

(d) $\Rightarrow$ (a). Suppose that $L_{|\mathbf{H}} = T_{(c_n)_{n\in\mathbf{N}}}$ for a certain $(c_n)_{n\in\mathbf{N}} \in \mathcal{P}$. Then the series $\sum_{n=0}^\infty c_n\Delta^n f$ converges uniformly in every compact subset of $\mathbf{R}$ for every $f \in \mathbf{H}$. Hence we can write for $x, y \in\,]0,\infty)$:

$$\begin{aligned}\tau_x(Lf)(y) &= \int_{|x-y|}^{x+y} D(x,y,z)\sum_{n=0}^\infty c_n\Delta^n f(z)A(z)\,dz\\ &= \sum_{n=0}^\infty c_n\int_{|x-y|}^{x+y} D(x,y,z)\Delta^n f(z)A(z)\,dz\\ &= \sum_{n=0}^\infty c_n\tau_x(\Delta^n f)(y),\quad f\in\mathbf{H}.\end{aligned}$$

This relation is also true for $x = y = 0$ and for x or y equal to zero.

Since the operators Δ and τ_x, $x \in [0,\infty)$, commute on $\mathbf{E}$, we conclude that $\tau_x L = L\tau_x$, $x \in [0,\infty)$ on $\mathbf{H}$. The proof of (a) can be completed by using Proposition 3.2 and [**19**, Proposition 8.3].

(e) $\Rightarrow$ (a). If (e) holds, then Proposition 3.3 implies that L and τ_x, $x \in [0,\infty)$ commute on $\mathbf{H}$. By [**19**, Proposition 8.3], we can conclude that L and τ_x, $x \in [0,\infty)$ commute on $\mathbf{E}$.

(d) $\Rightarrow$ (e). Let $(c_n)_{n\in\mathbf{N}} \in \mathcal{P}$. By Proposition 3.4 the series $\sum_{n=0}^{\infty} c_n \Delta^n f$ converges in $\mathbf{E}$ for every $f \in \mathbf{H}$.

Then, since Δ is a linear and continuous linear operator on $\mathbf{H}$, we get

$$\begin{aligned}\Delta T_{(c_n)_{n\in\mathbf{N}}} f &= \Delta\bigg(\sum_{n=0}^{\infty} c_n \Delta^n f\bigg)\\ &= \sum_{n=0}^{\infty} c_n \Delta^{n+1} f\\ &= T_{(c_n)_{n\in\mathbf{N}}} \Delta f, \quad f \in \mathbf{H}.\end{aligned}$$

Thus we prove that $\Delta L = L\Delta$ on $\mathbf{H}$. The proof finishes by using Proposition 3.2. □

The main results of this section are the following ones.

Proposition 3.7. *Let L be a continuous linear mapping from $\mathbf{E}$ into itself. Suppose that L is not a scalar multiple of the identity. If L commutes with the translation operator τ_x, $x \in [0,\infty)$, then L is hypercyclic and chaotic on $\mathbf{E}$ and there exists a dense L-invariant linear submanifold $\mathcal{M}$ of $\mathbf{E}$ such that each nonzero member of $\mathcal{M}$ is a hypercyclic vector of L.*

Proof. Assume first that V is a subset of $\mathbf{C}$ having adherence points. Then the space $S_V = \operatorname{span}\{\psi_\lambda : \lambda \in V\}$ is dense in $\mathbf{E}$. Indeed, let $T \in \mathbf{E}'$ be such that $\langle T, \psi_\lambda\rangle = 0$, $\lambda \in V$. According to Proposition 3.5, there exist $k \in \mathbf{N}$ and functions $F_0, F_1, \dots, F_k$ with support in $[0,\infty)$ such that

$$\langle T, f\rangle = \sum_{j=0}^{k} \int_0^{\infty} F_j(x)\Delta^j f(x) A(x)\,dx, \quad f \in \mathbf{E}. \tag{3.5}$$

In particular, by (1.1), for every $\lambda \in \mathbf{C}$, we have

$$\langle T, \psi_\lambda\rangle = \sum_{j=0}^{k} (\lambda^2+\rho^2)^j \int_0^{\infty} F_j(x)\psi_\lambda(x)A(x)\,dx. \tag{3.6}$$

The function $F(\lambda) = \langle T, \psi_\lambda \rangle$, $\lambda \in \mathbf{C}$, is an entire function. Moreover, $F(\lambda) = 0$, $\lambda \in V$. Hence, F is identically zero on $\mathbf{C}$.

From (3.6) we deduce that, for every $m \in \mathbf{N}$,

$$\left\langle T, \left(\frac{1}{\lambda} \frac{d}{d\lambda} \right)^m \psi_\lambda \right\rangle = 0, \quad \lambda \in \mathbf{C}.$$

Then by (1.5),

$$\langle T, b_m \rangle = \left\langle T, \left(\frac{(-1)^m}{2^m m!} \left(\frac{1}{\lambda} \frac{d}{d\lambda} \right)^m \psi\lambda \right)_{|\lambda = i\rho} \right\rangle = 0,$$

for every $m \in \mathbf{N}$. Since, by Propositions 3.1 and 3.2, span $\{b_m : m \in \mathbf{N}\}$ is a dense subspace of $\mathbf{E}$, we conclude that $T = 0$.

Hahn-Banach theorem allows us to show that span $\{\psi_\lambda : \lambda \in V\}$ is a dense subspace of $\mathbf{E}$.

Suppose that L is a continuous linear mapping from $\mathbf{E}$ into itself that commutes with the translation operator τ_x, $x \in [0, \infty)$. According to Proposition 3.6, there exists a sequence $(c_n)_{n \in \mathbf{N}}$ such that

$$Lf = \sum_{n=0}^{\infty} c_n \Delta^n f, \quad f \in \mathbf{H}.$$

Hence for every $\lambda \in \mathbf{C}$,

$$L\psi_\lambda = \sum_{n=0}^{\infty} c_n \Delta^n \psi_\lambda = \psi_\lambda \Phi(\lambda), \tag{3.7}$$

where $\Phi(\lambda) = \sum_{n=0}^{\infty} c_n (\lambda^2 + \rho^2)^n$, $\lambda \in \mathbf{C}$. Since $(c_n)_{n \in \mathbf{N}} \in \mathcal{P}$, Φ is entire. Φ is not identically zero and thus the set $W = \{\lambda \in \mathbf{C} : \Phi(\lambda) \neq 0\}$ is open and nonempty in $\mathbf{C}$. Hence S_W is a dense subspace of E. Since (3.7) implies that S_W is contained in the range of L, we conclude that the range of L is dense in $\mathbf{E}$.

By proceeding now as in the proof of [**9**, Theorem 5.1], we can prove as a consequence of the Fréchet space version of [**9**, Corollary 1.5], that L has a hypercyclic vector $f \in \mathbf{E}$. Moreover, the submanifold $\mathcal{M}$ of $\mathbf{E}$ defined by

$$\mathcal{M} = \{p(L)f : p \text{ is a polynomial}\}$$

is a dense, L-invariant, linear submanifold of $\mathbf{E}$ whose nonzero members are hypercyclic vectors for the operator L. To see this, we can argue as in [**9**, Sections 2 and 3].

To prove that L is chaotic on $\mathbf{E}$ it is sufficient to establish that the set of periodic points of L is dense in $\mathbf{E}$.

Let $(c_n)_{n\in\mathbf{N}}$ and Φ be as above. Φ is entire. Moreover, Φ is not constant because L is not a scalar multiple of the identity. Hence there exists $m \in \mathbf{N}$ such that

$$\Phi(\overline{D(0,m)}) \cap \partial D(0,1),$$

contains an open and nonempty subset of $\partial D(0,1)$ where $D(0,m)$ is the disk of center 0 and radius m.

Then the set $G = \{z \in \overline{D(0,m)} : \Phi(z)^l = 1 \text{ for some } l \in \mathbf{N}\}$ is infinity and hence G has adherence points in $\overline{D(0,m)}$. Hence the space span $\{\psi_\lambda : \lambda \in G\}$ is dense in $\mathbf{E}$. Moreover, if $\lambda \in G$ then, for some $l \in \mathbf{N}$,

$$L^l(\psi_\lambda) = \Phi(\lambda)^l \psi_\lambda = \psi_\lambda.$$

Thus we prove that each element of S_G is a periodic point of L and the proof is complete. $\square$

We now analyze the #-convolution operators on $\mathcal{D}'$.

Proposition 3.8. *Let $T \in \mathbf{E}'$. The convolution operator L_T on $\mathcal{D}'$ defined by*

$$L_T(S) = S\#T, \quad S \in \mathcal{D}',$$

is hypercyclic and chaotic provided that T is not a scalar multiple of the Dirac functional δ.

Proof. The space $\mathbf{E}$ is dense in $\mathcal{D}'$. Moreover, if $f \in \mathbf{E}$ then $L_T(f)$ coincides with the distribution generated by the function $T\#f \in \mathbf{E}$. Indeed, let $f \in \mathbf{E}$. The #-convolution $S_f\#T$ is defined on $\mathcal{D}$ by

$$\begin{aligned}\langle S_f\#T, \phi\rangle &= \langle S_f, T\#\phi\rangle \\ &= \int_0^\infty f(x)\langle T, \tau_x\phi\rangle A(x)\,dx, \quad \phi \in \mathcal{D}.\end{aligned}$$

Then, according to [**21**, p. 102], by proceeding as in [**23**, Theorem 5.6] using Riemann sums we get

$$\begin{aligned}\langle S_f \# T, \phi\rangle &= \Big\langle T_y, \int_0^\infty f(x)(\tau_x\phi)(y)A(x)\,dx\Big\rangle\\ &= \Big\langle T_y, \int_0^\infty (\tau_y f)(x)\phi(x)A(x)\,dx\Big\rangle\\ &= \int_0^\infty \phi(x)\langle T_y, (\tau_x f)(y)\rangle A(x)\,dx\\ &= \langle S_{T\#f}, \phi\rangle, \quad \phi\in\mathcal{D}.\end{aligned}$$

Now our result is a consequence of the comparison principle [**17**, p. 111], see also [**4**, Lemma 3] and Proposition 3.7. □

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