ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 35, Number 1, 2005

COMPOSITION OF ULTRADISTRIBUTIONS AND CORRESPONDING MICROFUNCTIONS WITH $\mathcal{E}^*$-FUNCTIONS

ATSUHIKO EIDA AND STEVAN PILIPOVIĆ

ABSTRACT. The pull back $f^\star u \in \mathcal{D}'^*(X)$ of $u \in \mathcal{D}'^*(Y)$ is determined, where $f = (f_1, \dots, f_m) : X \to Y$ is an $\mathcal{E}^*$-function, X and Y are open sets of $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. The $SS_*(f^\star u)$ is estimated. Also, the composition is done for the corresponding spaces of microfunctions $\mathcal{C}_*^*(X \times \mathbf{R}^n \setminus \{0\})$ and $\mathcal{C}_*^*(Y \times \mathbf{R}^m \setminus \{0\})$.

0. Introduction. Although the structural analysis of ultradistribution spaces is almost completed there is an important question not answered up to now. It concerns the composition of an ultradistribution and an ultradifferentiable function as well as of a microfunction in $\mathcal{C}_*^*$, cf. [**1, 2, 3, 6**], and an ultradifferentiable function. The composition of $u \in \mathcal{D}'^*(X)$ and a real analytic function f, under the condition that the singular spectrum of u does not intersect the set of normals of f, is given in [**4**] and [**9**]. We refer to [**4**] for the background in distribution theory. However, the known procedure could not be used for the composition in the class of ultradifferentiable functions.

Using an almost analytic extension, we analyze in this paper the composition with $f = (f_1, \dots, f_m) : X \to Y$, $f_i \in \mathcal{E}^*(X)$, $i = 1, \dots, m$, X and Y are open sets in $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. Also, we estimate the corresponding ultradistributional singular spectrum.

In Section 1 is recalled the definition of ultradistributional singular spectrum SS_*, cf. [**1, 4**] and [**11**]. The main assertion on the composition is given in Section 2. The corresponding assertion for microfunctions is given in Section 3.

1. Notations and notions. We denote by M_p, $p \in \mathbf{N}_0$, a sequence of positive numbers with $M_0 = 1$ and refer to [**5**] for the meaning of

The work of the second author is supported by MNTR of Serbia, Project No. 1835.
Received by the editors on June 21, 2002, and in revised form on July 28, 2002.

conditions (M.1), (M.2)$'$, (M.2), (M.3)$'$ and (M.3). Also we use the following one [**8**]:

$$\text{(M.1)}^* \qquad {M^*}_p^2 \leq M^*_{p-1}M^*_{p+1}, p \in \mathbf{N},$$

where $M^*_0 = 1$, $M^*_p = M_p/p!$, $p \in \mathbf{N}$.

We will always assume conditions (M.1)*, (M.2) and (M.3)$'$ hold. The associated and growth functions $M(\rho)$ and $M^*(\rho)$ related to M_p are defined by

$$M(\rho) = \sup_{p\in\mathbf{N}_0} \ln \frac{\rho^p}{M_p}, \quad M^*(\rho) = \sup_{p\in\mathbf{N}_0} \ln \frac{\rho^p}{M^*_p}, \quad \rho > 0.$$

Let Ω be an open set in $\mathbf{R}^n$. Then $K \subset\subset \Omega$ means that K, or its closure, is a compact subset of Ω. Recall, for $\varphi \in C^\infty(\Omega)$,

$$\|\varphi\|_{K,h,M_p} = \sup_{\substack{x\in K\\ \alpha\in\mathbf{N}_0^n}} \frac{|\varphi^{(\alpha)}(x)|}{h^{|\alpha|}M_{|\alpha|}}, \quad h > 0 \quad (K \subset\subset \Omega).$$

The basic spaces $\mathcal{E}^*(\Omega)$, $\mathcal{D}^*_K(\Omega)$, $\mathcal{D}^*(\Omega)$ and their strong duals are defined by the meaning of these semi-norms. The symbol $*$ is used for both (M_p) and $\{M_p\}$.

Eida [**1**] and Komatsu [**6**] have defined SS_*- and SS^*- singular support of a hyperfunction. We will recall the definition of SS_* for ultradistributions. Let $f \in \mathcal{D}'^*$. Then $(x,\omega) \in S^*\Omega = \Omega \times S^{n-1}$ is not in SS_*f if and only if there exist a neighborhood $U \subset \Omega$ of x and a conic neighborhood Γ of ω of the form

$$\Gamma = \{\xi \neq 0;\ |\ \xi/|\ \xi\ | - \omega\ | < \eta\}$$

such that for every $\phi \in \mathcal{D}^*(U)$ the following holds.

In the (M_p) case – for every $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$|\ \widehat{\phi f}(\xi)\ | \leq C_\varepsilon e^{-M(\varepsilon|\xi|)}, \quad \xi \in \Gamma.$$

In the $\{M_p\}$ case – there exist $k > 0$ and $C > 0$ such that

$$|\ \widehat{\phi f}(\xi)\ | \leq C e^{-M(k|\xi|)}, \quad \xi \in \Gamma.$$

We denote by $\sum_* \widehat{\phi f}$ the set of all $\xi \in \mathbf{R}^n \setminus \{0\}$ having no conic neighborhood Γ in which the estimates given above hold.

The notion $SS_{\{M_p\}}$ equals Hörmander's notion WF_L. The definition of the singular spectrum SSf, where $f \in \mathcal{B}(\Omega)$, is given by Sato, see [**11**]. Let $f \in \mathcal{D}'^*(\Omega)$. Then $(x,\omega) \in S^*\Omega$ is not in SSf if (x,ω) is not in $SS\{f\}$, where $\{f\}$ denotes the corresponding hyperfunction, see Section 3. This notion equals Hörmander's $WF_A f$, the analytic wave front set of f, cf. [**4**, Definition 9.3.2 and Theorem 9.6.3].

It is said that $(x,\xi) \in \Omega \times (\mathbf{R}^n \setminus \{0\})$ is an element of the corresponding singular spectrum defined above if this holds for $(x, \xi/|\xi|)$.

As in [**8**], the almost analytic extension of $\phi \in \mathcal{D}^*(\mathbf{R}^n)$ is defined by

$$(1) \qquad \Phi(z) = \sum_{p \in \mathbf{N}_0^n} \frac{\varphi^{(p)}(x)}{p!} (\sqrt{-1}y)^p \kappa_p(y), \quad z = x + \sqrt{-1}y \in \mathbf{C}^n,$$

where

$$p! = p_1! \dots p_n!, \quad (\sqrt{-1}y)^p = (\sqrt{-1}y_1)^{p_1} \dots (\sqrt{-1}y_n)^{p_n},$$

$$\kappa_p(y) = \kappa_{p_1}(y_1) \dots \kappa_{p_n}(y_n)$$

and κ is a nonnegative function of $\mathcal{D}^*(\mathbf{R})$ such that $\operatorname{supp} \kappa \subset [-2,2]$, $\kappa \mid_{[-1,1]} = 1$ and

$$\kappa_p(t) = \kappa(4tm_p^*/h), \quad h > 0, \quad p \in \mathbf{N}_0,$$

where $m_p = M_{p+1}/M_p$, $m_p^* = m_p/p$, $p \in \mathbf{N}$, and h depends on φ.

Note, $\Phi(z)$ is a smooth function on $\mathbf{C}^n$ which extends $\varphi(x)$ and for every $0 \neq y_0 \in \mathbf{R}^n$

$$(2) \qquad \sup_{i=1,\dots,n} \left\{ e^{M^*(k/t)} \left| \frac{\partial}{\partial \bar{z}_i} \Phi(x + \sqrt{-1}ty_0) \right| \right\} \leq C,$$

$$| \Phi(x + \sqrt{-1}ty_0) | \leq C, \quad x \in \mathbf{R}^n, t \in [0,1)$$

hold in the $(M_p)-$case for every $k > 0$ and corresponding $C > 0$ and in the $\{M_p\}-$case for some $k > 0$ and some $C > 0$, cf. [**8**].

The following two theorems are needed in the sequel. We refer to [**5, 8**] and [**10**] for the first part of Theorem 1. Its last part was proved for tempered ultradistributions in [**9**]. The proof for ultradistributions can be adapted easily.

Recall, $\Gamma_1 \subset\subset \Gamma$ means that $\bar{\Gamma}_1 \subset \Gamma \cup \{0\}$; $\bar{\Gamma}$ denotes the closure of Γ and Γ^0 denotes the dual cone of Γ.

Theorem 1. *Let Γ be an open convex cone in $\mathbf{R}^n$ and U an analytic function in*

$$Z = \{z \in \mathbf{C}^n; \quad \operatorname{Re} z \in \Omega, \ \operatorname{Im} z \in \Gamma, \quad | \operatorname{Im} z |< d\}$$

for some $d > 0$.

Assume that for every compact set $K \subset\subset \Omega$ and every open convex cone $\Gamma_1 \subset\subset \Gamma$

$$| U(x + \sqrt{-1}y) | \leq C_a e^{M^*(a/|y|)}, \quad x + \sqrt{-1}y \in K + \sqrt{-1}\Gamma_1,$$

holds, in the $(M_p)-$case for some $a > 0$ and $C_a > 0$, and in the $\{M_p\}-$case for every $a > 0$ there exists $C_a > 0$. Then

$$U(x + \sqrt{-1}y) \overset{\mathcal{D}'^*(\Omega)}{\longrightarrow} U(x + \sqrt{-1}0), \quad y \to 0, \quad y \in \Gamma.$$

Moreover, with $y_0 \in \Gamma$, $|y_0| < d$, we have

$$\begin{aligned}
&\langle U(x + \sqrt{-1}0), \varphi(x)\rangle \\
&\quad = \int_{\mathbf{R}^n} U(x + \sqrt{-1}y_0)\Phi(x + \sqrt{-1}y_0)\, dx \\
&\qquad + 2\sqrt{-1} \sum_{i=1}^{n} y_{0,i} \int_0^1 \int_{\mathbf{R}^n} \frac{\partial}{\partial \bar{z}_i} \Phi(x+ \sqrt{-1}ty_0)U(x+ \sqrt{-1}y_0 t)\, dt\, dx.
\end{aligned}$$

We refer to [**6**] and [**10**] for the next theorem.

Theorem 2. *Let Γ be an open convex cone in $\mathbf{R}^n$, $u \in \mathcal{D}'^*(\Omega)$, $\Omega \subset \mathbf{R}^n$ and $SS_* u \subset \Omega \times \Gamma^0$. Let $\Omega_1 \subset\subset \Omega$ and let Γ_1 be an open convex cone such that $\Gamma_1 \subset\subset \Gamma$. Then there is a function U analytic in*

$$\{x + \sqrt{-1}y; \quad x \in \Omega_1, \quad y \in \Gamma_1, \quad | y |< d\}$$

such that for some $k > 0$ and $C > 0$ in the $(M_p)-$case, for every $k > 0$ there is $C > 0$ in the $\{M_p\}-$case,

$$(3) \qquad | U(x + \sqrt{-1}y) | \leq Ce^{M^*(k/|y|)}, \quad x \in \Omega_1, \quad y \in \Gamma_1, \quad | y |< d,$$

and

$$U(\cdot + \sqrt{-1}0) - u |_{\Omega_1} \in \mathcal{E}^*(\Omega_1).$$

Remark 1. If an analytic function $U(x+\sqrt{-1}y)$ satisfies (3), then the same type of estimate holds for its derivatives, i.e., for every $j \in \mathbf{N}_0^n$

$$| U^{(j)}(x + \sqrt{-1}y) | \leq Ce^{M^*(k/|y|)}, \quad x \in \Omega_1, \quad y \in \Gamma_1, \quad | y |< d,$$

for some $k > 0$ and $C > 0$ in the $(M_p)-$case, for every $k > 0$ there is a $C > 0$ in the $\{M_p\}-$case.

This follows by the Cauchy formula and the inequality

$$M^*(t_1 + t_2) \leq M^*(t_1) + M^*(t_2), \quad t_1, t_2 \geq 0.$$

2. Composition. Let X and Y be open sets in $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively, and $f = (f_1, \dots , f_m) : X \to Y$ such that $f_i \in \mathcal{E}^*(X)$, $i = 1, \dots , m$.

We are going to define the pullback $f^\star u$ of $u \in \mathcal{D}'^*$ with a suitable property of its singular spectrum. Denote

$$\begin{aligned} N_f &= \{(f(x), \eta); \quad {}^t f'(x)\eta = 0\}, \\ \mathcal{D}'^*_\gamma(Y) &= \{u \in \mathcal{D}'^*(Y);\ SS_* u \subset \gamma^0\}, \\ f^\star\gamma &= \{(x,{}^t f'(x)\eta); \quad (f(x), \eta) \in \gamma\} \end{aligned}$$

where γ is a closed conic subset of $Y \times (\mathbf{R}^m \setminus \{0\})$ conic in the second variable and γ^0 is the dual conic set (in the second variable).

Theorem 3. *The pullback $f^\star u$ can be defined in one and only one way for all $u \in \mathcal{D}'^*(Y)$ with $N_f \cap SS_* u = \varnothing$ so that $f^\star u = u \circ f$ when u is a continuous function and for any closed conic subset γ of $Y \times (\mathbf{R}^n \setminus \{0\})$ with $\gamma \cap N_f = \varnothing$, if $u \in \mathcal{D}'^*_\gamma(Y)$, then $SS_*(f^\star u) \subset f^\star(SS_* u)$.*

Proof. Let Γ be an open convex cone in $\mathbf{R}^m$ such that $(Y \times \Gamma^0) \cap N_f = \varnothing$ and $\gamma = Y \times \Gamma$. Using a partition of unity we may suppose that $u \in \mathcal{D}'^*_\gamma(Y) \cap \mathcal{E}'^*(Y)$. Let $\Omega_1 \subset\subset Y$ and $\operatorname{supp} u \subset \Omega_1$. We apply Theorem 2. The corresponding U is analytic in $\Omega_1 + \sqrt{-1}\Gamma_1$, satisfies (3) in this domain and

$$U(\cdot + \sqrt{-1}0) - u \in \mathcal{E}^*(\Omega_1).$$

We define $f^\star u$ as an element of $\mathcal{E}'^*(X)$ as follows. Denote by X_0 an open relatively compact subset of X such that $f^{-1}(\operatorname{supp} u) \subset X_0$. For every $x_0 \in X_0$ choose $h_0 \in \mathbf{R}^n$ such that $f'(x_0)h_0 \in \Gamma_1$. Note, $f'(x_0)(mh_0) \in \Gamma_1$ for every $m > 0$.

Let $\theta \in \mathcal{D}^*(X)$, $\theta = 1$ on $\tilde{O}_{x_0}$, where $\tilde{O}_{x_0}$ is an open neighborhood of x_0 contained in X_0. We denote by

$$\xi = F_\theta = (F_{1\theta}, \dots , F_{m\theta})$$

an almost analytic extension of $f\theta = (f_1\theta, \dots f_m\theta)$ defined by

$$\xi_i = F_{i\theta}(z) = \sum_{p \in \mathbf{N}_0^n} \frac{(f_i\theta)^{(p)}(x)}{p!}(\sqrt{-1}y)^p \kappa_p(y),$$
$$z = x + \sqrt{-1}y, \quad x \in X, \quad y \in \mathbf{R}^n, \quad i = 1, \dots , m.$$

This is a smooth extension of $f\theta$. By Taylor's formula one can prove that there exist an open set O_{x_0}, $O_{x_0} \subset \tilde{O}_{x_0}$, $\varepsilon_0 > 0$ and $h_0 \in \mathbf{R}^n \setminus \{0\}$ such that

$$(4) \qquad F_\theta(x + \sqrt{-1}\varepsilon h_0) \in \Omega_1 + \sqrt{-1}\Gamma_1, \quad x \in O_{x_0}, \quad \varepsilon < \varepsilon_0.$$

In fact, we have

$$F_\theta(x + \sqrt{-1}\varepsilon h_0) = F_\theta(x_0) + (F_\theta)'(x_0)\sqrt{-1}\varepsilon h_0 + \mathcal{O}(|x - x_0|).$$
$$(F_\theta(x_0) = f(x_0), \quad (F_\theta)'(x_0) = f'(x_0).)$$

We denote by Φ the almost analytic extension of φ defined by (1). By Theorem 1, with suitable $Y \in \Gamma_1$ and sufficiently small ε, we have

$$\begin{aligned}
&\langle U(F_\theta(x+\sqrt{-1}\varepsilon h_0)), \varphi(x)\rangle \\
&\quad = \int_{\mathbf{R}^n} U(F_\theta(x+\sqrt{-1}\varepsilon h_0)+\sqrt{-1}Y)\Phi(x+\sqrt{-1}Y)\,dx \\
&\quad + 2\sqrt{-1}\sum_{i=1}^{n} Y_i \int_0^1 \int_{\mathbf{R}^n} \Big(\Big(\frac{\partial}{\partial \bar z_i}\Phi\Big)(x+\sqrt{-1}tY) \\
&\quad \times U(F_\theta(x+\sqrt{-1}\varepsilon h_0)+\sqrt{-1}\,tY) \\
&\quad + \Phi(x+\sqrt{-1}tY)\frac{\partial}{\partial \bar z_i}(U(F_\theta(z+\sqrt{-1}\varepsilon h_0)+\sqrt{-1}Y))|_{z=x}\Big)\,dt\,dx.
\end{aligned}$$

We define $f^\star u_{x_0}$ as an element of $\mathcal{D}'^*(O_{x_0})$ letting $\varepsilon \to 0$ in (5):
(6)

$$\begin{aligned}
\langle f^\star u_{x_0}, \varphi\rangle = &\int_{\mathbf{R}^n} \Phi(x+\sqrt{-1}Y)U(F_\theta(x)+\sqrt{-1}Y)\,dx \\
&+ 2\sqrt{-1}\sum_{i=1}^{m} Y_i \int_0^1 \int_{\mathbf{R}^n} \Big(\Big(\frac{\partial}{\partial \bar z_i}\Phi\Big)(x+\sqrt{-1}tY) \\
&\times U(F_\theta(x)+\sqrt{-1}tY) \\
&+ \Phi(x+\sqrt{-1}tY)\frac{\partial}{\partial \bar z_i}(U(F_\theta(z)+\sqrt{-1}Y)\Big)|_{z=x}\Big)\,dt\,dx.
\end{aligned}$$

Clearly, if $O_{x_0} \cap O_{x_1} \neq \varnothing$, then $f^\star u_{x_0|O_{x_0}\cap O_{x_1}} = f^\star u_{x_1|O_{x_0}\cap O_{x_1}}$. Thus the family $\{f^\star u_{x_0}; x_0 \in X_0\}$ defines an element of $\mathcal{E}'^*(X)$ which we denote by $f^\star u$.

More generally, let $u \in \mathcal{E}'^*(f(X_0))$, $f(X_0) \subset \Omega_1 \subset\subset Y$, such that $SS_* u \cap N_f = \varnothing$. There exist open convex cones Γ_j, $j = 1, \ldots, s$, such that

$$SS_* u \subset \bigcup_{i=1}^{s} Y \times \Gamma_j^0 \quad \text{and} \quad \bigcup_{j=1}^{s} (Y \times \Gamma_j^0) \cap N_f = \varnothing.$$

Theorem 2 implies that there are holomorphic functions U_j in $\Omega_1 + \sqrt{-1}\Gamma_{1j}$, $j = 1, \ldots, s$, which satisfy (3) and

$$u_j - U_j(\cdot + \sqrt{-1}\Gamma_{1j}0) \in \mathcal{E}^*(\Omega_1), \quad SS_* u_j \subset \Omega_1 \times \Gamma_{1j}^0,$$

and

$$u = \sum_{j=1}^{s} u_j + r,$$

where $r \in \mathcal{E}^*(\Omega_1)$. We define

$$f^\star u = \sum_{j=1}^{s} f^\star u_j + r \circ f.$$

One can simply prove that this definition does not depend on the decomposition of u.

Let us estimate $SS_* f^\star u$.

Let $\psi \in \mathcal{E}^*(O_{x_0})$ and $\psi = 1$ in a neighborhood of x_0. Then, by (6), we have

$$\begin{aligned}
\widehat{\psi f^\star u}(\xi) &= \langle (f^\star u_{x_0}) e^{-\sqrt{-1}\langle \cdot, \xi\rangle}, \psi\rangle \\
&= \int_{\mathbf{R}^n} \Psi(x+\sqrt{-1}Y) U(F_\theta(x)+\sqrt{-1}Y) \\
&\quad \times e^{-\sqrt{-1}\langle x+\sqrt{-1}Y,\xi\rangle}\, dx + 2\sqrt{-1}\sum_{i=1}^{m} Y_i \\
&\quad \times \int_0^1 \int_{\mathbf{R}^n} \bigg(\Big(\frac{\partial}{\partial \bar z_i}\Psi\Big)(x+\sqrt{-1}tY) e^{-\sqrt{-1}\langle x+\sqrt{-1}Y,\xi\rangle} \\
&\quad \times U(F_\theta(x)+\sqrt{-1}tY) + \Psi(x+\sqrt{-1}tY) \\
&\quad \times e^{-\sqrt{-1}\langle x+\sqrt{-1}Y,\xi\rangle} \frac{\partial}{\partial \bar z_i}(U(F_\theta(z)+\sqrt{-1}Y)|_{z=x} \bigg)\, dt\, dx,
\end{aligned}$$

where Ψ and F_θ are the almost analytic extensions of ψ and $f\theta$, respectively.

Let $\langle Y, \xi\rangle < 0$. Using the estimates given in Remark 1 and in (2), it follows

$$|\, \widehat{\psi f^\star u}(\xi)\, | \le C_\psi \Big(e^{\langle Y,\xi\rangle} + \int_0^1 e^{t\langle Y,\xi\rangle} e^{-M^*(k/t)}\Big)\, dt. \tag{7}$$

Note,

$$-\inf_{t>0} \left\{ M^*\left(\frac{k}{t}\right) - |\, t\langle Y,\xi\rangle\, | \right\} = \sup_{t>0} \left\{ t\langle Y,\xi\rangle - M^*\left(\frac{k}{t}\right) \right\} = -M(k_1|\xi|).$$

This implies that the righthand side of (8) is $\mathcal{O}(e^{-kM(\xi)})$ in a conic neighborhood of any point in the half space $\langle Y, \xi\rangle < 0$.

The dual cone for ${}^t f'(x_0)\Gamma_1^0$ is $\{h; f'(x_0)h \in \bar{\Gamma}_1\}$. Thus,

$$ {}^t f'(x_0)\Gamma_1^0 = \{\xi\,; \langle h, \xi\rangle \geq 0 \text{ if } f'(x_0)h \in \bar{\Gamma}_1\}. $$

Letting $\Gamma_1 \to \Gamma$, we obtain $\sum_*(\psi f^\star u) \subset f^\star \Gamma^0$.

Remark 2. We can introduce spaces $\mathcal{D}'^*_\Gamma(X)$, where X is open in $\mathbf{R}^n$ and Γ is a closed cone in $\mathbf{R}^n \setminus \{0\}$. Then, as in [**4**, Lemma 8.2.1, Definition 8.2.2 and Theorem 8.2.3], we can consider the corresponding statements with

$$ \text{“}\sup_V e^{M(\varepsilon\xi)} \dots, \text{ for every } \varepsilon > 0\text{”} \quad (\text{in the } (M_p) \text{ case}) $$

or

$$ \text{“}\sup_V e^{M(k\xi)} \dots, \text{ for some } k > 0\text{”} \quad (\text{in the } \{M_p\} \text{ case}). $$

Then one can prove that the analogous assertions as in Lemma 8.2.1 and Theorem 8.2.3 hold. Having this, one can prove that the mapping

$$ f^\star : \mathcal{D}'^*_\Gamma(Y) \to \mathcal{D}'^*_{f^*\Gamma}(X) $$

is continuous. We note that this can be done also in the case considered in [**4**], when f is real analytic.

3. Microlocalization. Denote by $\mathcal{B}$ and $\mathcal{C}$ shaves of Sato's hyperfunctions and microfunctions on $\mathbf{R}^n$, cf. [**10**] and [**5**]). By $Sp : \pi^{-1}\mathcal{B} \to \mathcal{C}$ is denoted the canonical surjective spectrum map. Then the singular spectrum $\mathrm{SS}(u)$ of $u \in \mathcal{B}$ is $\mathrm{SS}(u) = \operatorname{supp}(Sp(u))$. The injections $\mathcal{D}'^* \to \mathcal{B}$, respectively, $\mathcal{D}^* \to \mathcal{B}$, induce sheaf homomorphisms

$$ \pi^{-1}\mathcal{D}'^* \longrightarrow \mathcal{C}, \quad \text{respectively,} \quad \pi^{-1}\mathcal{D}^* \longrightarrow \mathcal{C}. $$

Subsheaves of $\mathcal{C}$, $\mathcal{C}^*$, respectively, $\mathcal{C}_*$, are defined as images of above respective morphisms.

We define the sheaf $\mathcal{C}_*^*$, by the exact sequence

$$0 \longrightarrow \mathcal{C}_* \longrightarrow \mathcal{C}^* \longrightarrow \mathcal{C}_*^* \longrightarrow 0$$

and refer to [**3**] for similar kind of microfunction spaces. Note, $\mathcal{C}^*$ and $\mathcal{C}_*$ are soft sheaves, so

$$(\mathcal{C}^*/\mathcal{C}_*)(\omega) = \mathcal{C}^*(\omega)/\mathcal{C}_*(\omega), \quad \text{for any open set } \omega.$$

Define $Sp_* : \pi^{-1}\mathcal{D}'^* \to \mathcal{C}_*^*$.

Let $u \in \mathcal{D}'^*$. Then we have $SS_*(u) = \operatorname{supp}(Sp_*(u))$.

Let X, Y and f be as in Section 2 and $\mathcal{C}_*^*(X \times \mathbf{R}^n \setminus \{0\})$, $\mathcal{C}_*^*(Y \times \mathbf{R}^m \setminus \{0\})$ be spaces of sections. The pull back

$$f^\star : \mathcal{C}_*^*(Y \times \mathbf{R}^m \setminus \{0\}) \longrightarrow \mathcal{C}_*^*(X \times \mathbf{R}^n \setminus \{0\}), \quad h \longmapsto f^\star(h),$$

is defined as follows.

Let $h \in \mathcal{C}_*^*(Y \times \mathbf{R}^m \setminus \{0\})$ and $\operatorname{supp} h \subset \gamma$, cf. Section 2 for γ. Denote by h the corresponding element in $h \in \mathcal{C}^*(Y \times \mathbf{R}^m \setminus \{0\})$. Denote by h_0 an element in $\pi^{-1}\mathcal{D}'^*(Y \times \mathbf{R}^m \setminus \{0\})$ with the property $Sp(h_0) = h$. Denote by h_0, the corresponding element in $\mathcal{D}'^*(Y)$; it has the property $SS_* h_0 \subset \gamma$. Then, put $f^\star h_0$ for the corresponding element in $\mathcal{D}'^*(X)$. We know, $SS_*(f^\star h_0) \subset f^\star \gamma$. Finally, we put $f^\star h_0$ for its image in $\pi^{-1}\mathcal{D}'^*(X \times \mathbf{R}^n \setminus \{0\})$ and $Sp_*(f^\star h_0) = f^\star h \in \mathcal{C}_*^*(X \times \mathbf{R}^n \setminus \{0\})$. Clearly, $\operatorname{supp} f^\star h \subset f^\star \gamma$.

REFERENCES

1. A. Eida, *On the microlocal decomposition of ultradistributions*, Master's Thesis, University of Tokyo, 1989. (in Japanese)

2. A. Eida and S. Pilipović, *On the microlocal decomposition of some classes of ultradistributions*, Math. Proc. Cambridge Philos. Soc. **125** (1999), 455–461.

3. ———, *Suppleness of some quotient of microfunctions*, Rocky Mountain J. Math. **30** (2000), 165–177.

4. L. Hörmander, *The analysis of linear partial differential operators* I, Springer, Berlin, 1983.

5. H. Komatsu, *Ultradistributions*, I–III, J. Fac. Sci. Univ. Tokyo **20** (1973), 25–105; **24** (1977), 607–628; **29** (1982), 653–717.

6. ———, *Microlocal analysis in Gevrey classes and in convex domains*, Lecture Notes in Math., vol. 1726, Springer-Verlag, New York, 1989, pp. 426–493.

7. D. Kovačević and S. Pilipović, *Structural properties of the space of tempered ultradistributions*, in *Complex analysis and generalized functions*, Proc. of Conf. "Complex Analysis and Applications '91 with Symposium on Generalized Functions" (Varna 1991), 1993.

8. H.J. Petzsche, *Generalized functions and the boundary values of holomorphic functions*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **31** (1984), 391–431.

9. S. Pilipović, *Microlocal analysis of ultradistributions*, Proc. Amer. Math. Soc. **126** (1998), 105–113.

10. J.W. de Roever, *Hyperfunctional singular support of ultradistributions*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **31** (1985), 585–631.

11. M. Sato, T. Kawai and M. Kashiwara, *Microfunctions and Pseudo-differential equations*, Lecture Notes in Math, vol. 287, Springer, New York, 1973, pp. 265–529.

School of Computer Science, Tokyo University of Technology, 1401-1, Katakura, Hachioji, Tokyo 192-0982, Japan
E-mail address: `eida@cs.teu.ac.jp`

Institute of Mathematics, University of Novi Sad, Trg D. Obradovića 4, 21000 Novi Sad, Serbia and Montenegro
E-mail address: `pilipovic@im.ns.ac.yu`