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# A QUASILINEARIZATION APPROACH FOR TWO POINT NONLINEAR BOUNDARY VALUE PROBLEMS ON TIME SCALES

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ABSTRACT. In this paper the quasilinearization method is used in an approach to the unique solution of the separated boundary value problem on time scales from below and above by monotone convergent sequences of upper and lower solutions. The rate of convergence is also determined.

**1. Introduction.** In this paper we consider the separated boundary value problems (SBVPs)

$$-(p(t)x^{\Delta})^{\Delta} + q(t)x^{\sigma} = f(t, x^{\sigma}) + g(t, x^{\sigma}), \quad t \in [a, b]^{\kappa^2} \tag{1}$$

$$x(a) = A, \quad x(b) = B \tag{2}$$

and

$$-(p(t)x^{\Delta})^{\nabla} + q(t)x = f(t, x) + g(t, x), \quad t \in [a, b] \tag{3}$$

$$x(\rho(a)) = A, \quad x(\sigma(b)) = B. \tag{4}$$

In Section 2 we give some preliminary results with respect to the calculus on time scales which can also be found in the books by Bohner and Peterson [**7**] and Kaymakçalan, Lakshmikantham, and Sivasundaram [**12**]. In Section 3 we introduce the theory of the method of lower and upper solutions for the SBVP (1)–(2). Under certain assumptions on $f$ and $g$ we prove existence theorems for solutions of the SBVP (1)–(2) on a time scale $\mathbf{T}$. Then, in Section 4, the idea of the quasilinearization method is used for the SBVP (1)–(2) on $\mathbf{T}$ for which $f$ and $g$ are $k$-hyperconvex and $k$-hyperconcave functions, respectively. This method has been studied by Cabada and Nieto [**8**], Lakshmikantham and Vatsala [**13**], Mohapatra, Vajravelu and Yin [**14**]

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for ordinary differential equations. The method of upper and lower solutions for the SBVPs has recently been developed by Akın [**3**] on $\mathbf{T}$. Then the quasilinearization method is applied by Eloe [**9**] for the SBVP (1)–(2) when $p = 1$, $q = g = 0$ and $f$ is 1-hyperconvex on a compact time scale. We also prove that the order of convergence of sequences of lower and upper solutions is $k$. In Section 5 we again use the idea of quasilinerization method for the SBVP (3)–(4) where $f$ and $g$ are respectively $k$-hyperconvex and $k$-hyperconcave functions on $\mathbf{T}$. This method is also used by Merdivenci Atıcı, Eloe and Kaymakçalan [**5**] for the SBVP (3) with periodic boundary conditions when $p = 1$, $f$ and $g$ are 1-hyperconvex and 1-hyperconcave functions on $\mathbf{T}$. We also determine the order of convergence of sequences of lower and upper solutions similarly as in Section 4. In Section 6, we emphasize that the sequences of lower and upper solutions are not unique in the main result of this paper.

**2. Calculus on time scales.** The theory of time scales was initiated by Stefan Hilger in his Ph.D. dissertation [**11**] in 1988 in order to unify continuous and discrete analysis. Some recent paper concerning dynamic equations on time scales include Agarwal, Bohner and O'Reagan [**2**] and Akın-Bohner and Bohner [**4**]. A *time scale* $\mathbf{T}$ is an arbitrary nonempty closed subset of real numbers $\mathbf{R}$. For our purposes, we let $\mathbf{T}$ be a time scale, $[a, b]$ be the closed and bounded interval in $\mathbf{T}$, i.e., $[a, b] := \{t \in \mathbf{T} : a \le t \le b\}$ and $a, b \in \mathbf{T}$.

Obviously a time scale $\mathbf{T}$ may or may not be connected. Therefore we have the concept of *forward* and *backwards jump operators* as follows. Define $\sigma, \rho : \mathbf{T} \mapsto \mathbf{T}$ by

$$\sigma(t) = \inf\{s \in \mathbf{T} : s > t\} \quad \text{and} \quad \rho(t) = \{s \in \mathbf{T} : s < t\}.$$

If $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, $\rho(t) < t$, then $t \in \mathbf{T}$ is called *right-dense* (rd), *right-scattered*, *left-dense*, *left-scattered*, respectively. We also define the *graininess function* $\mu : \mathbf{T} \mapsto [0, \infty)$ as $\mu(t) = \sigma(t) - t$. The sets $\mathbf{T}^\kappa$, $\mathbf{T}_\kappa$ which are derived from $\mathbf{T}$ are as follows: If $\mathbf{T}$ has a left-scattered maximum $t_1$, then $\mathbf{T}^\kappa = \mathbf{T} - \{t_1\}$, otherwise $\mathbf{T}^\kappa = \mathbf{T}$. If $\mathbf{T}$ has a right-scattered minimum $t_2$, then $\mathbf{T}_\kappa = \mathbf{T} - \{t_2\}$, otherwise $\mathbf{T}_\kappa = \mathbf{T}$. If $f : \mathbf{T} \mapsto \mathbf{R}$ is a function, we define the functions $f^\sigma : \mathbf{T}^\kappa \mapsto \mathbf{R}$ by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbf{T}^\kappa$, $f^\rho : \mathbf{T}_\kappa \mapsto \mathbf{R}$ by $f^\rho(t) = f(\rho(t))$ for all $t \in \mathbf{T}_\kappa$ and $\sigma^0(t) = \rho^0(t) = t$. If $f : \mathbf{T} \mapsto \mathbf{R}$ is a function and $t \in \mathbf{T}^\kappa$, then the *delta-derivative* of $f$ at a point $t$ is defined to be the number $f^{\Delta}(t)$, provided it exists, with the property that for each $\varepsilon > 0$ there is a neighborhood of $U_1$ of $t$ such that

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|,$$

for all $s \in U_1$. If $t \in \mathbf{T}_{\kappa}$, then we define the *nabla derivative* of $f$ at a point $t$ to be the number $f^{\nabla}(t)$, provided it exists, with the property that for each $\varepsilon > 0$ there is a neighborhood of $U_2$ of $t$ such that

$$|[f(\rho(t)) - f(s)] - f^{\nabla}(t)[\rho(t) - s]| \leq \varepsilon|\rho(t) - s|,$$

for all $s \in U_2$.

*Remark* 2.1. If $\mathbf{T} = \mathbf{R}$, then $f^{\Delta}(t) = f^{\nabla}(t) = f'(t)$, and if $\mathbf{T} = \mathbf{Z}$, then $f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t)$ and $f^{\nabla}(t) = \nabla f(t) = f(t) - f(t-1)$.

In the next two theorems we give some important properties of delta-derivative and nabla-derivative.

**Theorem 2.1.** *Assume $f : \mathbf{T} \mapsto \mathbf{R}$ is a function and let $t \in \mathbf{T}^{\kappa}$. Then we have the following*:

(i) *If $f$ is delta-differentiable at $t$, then $f$ is continuous at $t$.*

(ii) *If $f$ is continuous at $t$ and $t$ is right-scattered, then $f$ is delta-differentiable at $t$ with*

$$f^{\Delta}(t) = \frac{f^{\sigma}(t) - f(t)}{\mu(t)}.$$

(iii) *If $f$ is delta-differentiable and $t$ is right-dense, then*

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) *If $f$ is delta-differentiable at $t$, then*

$$f^{\sigma}(t) = f(t) + \mu(t) f^{\Delta}(t).$$

The product and quotient rules are given by

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f^{\sigma}(t)g^{\Delta}(t)$$

and

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g^{\sigma}(t)}$$

where $f$ and $g$ are two delta-differentiable functions such that $gg^{\sigma} \neq 0$.

**Theorem 2.2.** *Assume $f : \mathbf{T} \mapsto \mathbf{R}$ is a function and let $t \in \mathbf{T}_{\kappa}$. Then we have the following*:

(i) *If $f$ is nabla-differentiable at $t$, then $f$ is continuous at $t$.*

(ii) *If $f$ is continuous at $t$ and $t$ is left-scattered, then $f$ is nabla-differentiable at $t$ with*

$$f^{\nabla}(t) = \frac{f^{\rho}(t) - f(t)}{\rho(t) - t}.$$

(iii) *If $f$ is nabla-differentiable and $t$ is left-dense, then*

$$f^{\nabla}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) *If $f$ is nabla-differentiable at $t$, then*

$$f^{\rho}(t) = f(t) + (\rho(t) - t)f^{\nabla}(t).$$

The product and quotient rules are given by

$$(fg)^{\nabla}(t) = f^{\nabla}(t)g(t) + f^{\rho}(t)g^{\nabla}(t)$$

and

$$\left(\frac{f}{g}\right)^{\nabla}(t) = \frac{f^{\nabla}(t)g(t) - f(t)g^{\nabla}(t)}{g(t)g^{\rho}(t)}$$

where $f$ and $g$ are two nabla-differentiable functions such that $gg^{\rho} \neq 0$. Notice that, in general,

$$f^{\Delta}(\rho(t_0)) \neq f^{\nabla}(t_0) \quad \text{and} \quad f^{\nabla}(\sigma(t_0)) \neq f^{\Delta}(t_0).$$

However if $\sigma(\rho(t_0)) = t_0$ for any $t_0 \in \mathbf{T}_\kappa$, then $f^\Delta(\rho(t_0)) = f^\nabla(t_0)$ and while if $\rho(\sigma(t_0)) = t_0$ for any $t_0 \in \mathbf{T}^\kappa$, then $f^\nabla(\sigma(t_0)) = f^\Delta(t_0)$.

**Definition 2.1.** Let $f : \mathbf{T} \mapsto \mathbf{R}$ be a function. We say that $f$ is *rd-continuous* provided $f$ is continuous at each right-dense point in $\mathbf{T}$ and $\lim_{s\to t^-} f(s)$ exists as a finite number for all left-dense points in $\mathbf{T}$. The set of rd-continuous functions $f : \mathbf{T} \mapsto \mathbf{R}$ will be denoted in this paper by $\mathrm{C_{rd}} = \mathrm{C_{rd}}(\mathbf{T})$.

A function $F : \mathbf{T}^\kappa \mapsto \mathbf{R}$ is called a *delta-antiderivative* of $f : \mathbf{T} \mapsto \mathbf{R}$ provided $F^\Delta(t) = f(t)$ holds for all $t \in \mathbf{T}^\kappa$. In this case we define the integral of $f$ by

$$\int_a^t f(s)\Delta s = F(t) - F(a)$$

for all $a, t \in \mathbf{T}$. A function $G : \mathbf{T}_\kappa \mapsto \mathbf{R}$ is called a *nabla-antiderivative* of $f : \mathbf{T} \mapsto \mathbf{R}$ provided $G^\nabla(t) = f(t)$ holds for all $t \in \mathbf{T}_\kappa$. In this case we define the integral of $f$ by

$$\int_a^t f(s)\nabla s = G(t) - G(a)$$

for all $a, t \in \mathbf{T}$.

*Remark* 2.2. If $\mathbf{T} = \mathbf{R}$, then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)\nabla t = \int_a^b f(t)dt,$$

and if $\mathbf{T} = \mathbf{Z}$, then

$$\int_a^b f(t)\Delta t = \sum_{k=a}^{b-1} f(k)$$

and

$$\int_a^b f(t)\nabla t = \sum_{k=a+1}^{b} f(k).$$

We also need the following fundamental result which is proved in the article by Merdivenci Atıcı and Guseinov [**6**]. We refer the readers to [**6**] for further results for the nabla-derivative.

**Theorem 2.3.** *If $f : \mathbf{T} \mapsto \mathbf{R}$ is delta-differentiable on $\mathbf{T}^\kappa$ and if $f^\Delta$ is continuous on $\mathbf{T}^\kappa$, then $f$ is nabla-differentiable on $\mathbf{T}_\kappa$ and*

$$f^\nabla(t) = f^\Delta(\rho(t))$$

*for all $t \in \mathbf{T}_\kappa$.*

**3. The method of upper and lower solutions.** We consider the SBVP (1)–(2) with real-valued functions $f, g \in \mathrm{C}([a,b]^{\kappa^2} \times \mathbf{R})$ and $p, q \in \mathrm{C}_{\mathrm{rd}}([a,b]^{\kappa^2})$ such that $p(t) > 0$ and $q(t) \geq 0$ on $[a,b]^{\kappa^2}$. We define

$$\mathbf{D}_1 := \{x \in \mathbf{B} : \ x^\Delta \text{ is continuous and } px^\Delta \text{ is delta-differentiable on } [a,b]^\kappa \text{ and} (px^\Delta)^\Delta \text{ is rd-continuous on } [a,b]^{\kappa^2}\},$$

where the Banach space $\mathbf{B} = C([a,b])$ is equipped with the norm $||\cdot||$ defined by

$$||x|| = \max_{t \in [a,b]} |x(t)|.$$

A function $x$ is called a solution of the equation $-(p(t)y^\Delta)^\Delta + q(t)y^\sigma = 0$ on $[a,b]^{\kappa^2}$ if $x \in \mathbf{D}_1$ and the equation $-(px^\Delta)^\Delta(t) + q(t)x^\sigma(t) = 0$ holds for all $t \in [a,b]^{\kappa^2}$. Next we define for any $u, v \in \mathbf{D}_1$ the sector $[u,v]_1$ by

$$[u,v]_1 := \{w \in \mathbf{D}_1 : u \leq w \leq v\}.$$

**Definition 3.1.** We call $\alpha \in \mathbf{D}_1$ a *lower solution* of the SBVP (1)–(2) on $[a,b]$ provided

$$-(p\alpha^\Delta)^\Delta(t) + q(t)\alpha^\sigma(t) \leq f(t, \alpha^\sigma(t)) + g(t, \alpha^\sigma(t)) \quad \text{for all} \quad t \in [a,b]^{\kappa^2}$$

and

$$\alpha(a) \leq A, \quad \alpha(b) \leq B.$$

Similarly, $\beta \in \mathbf{D}_1$ is called an *upper solution* of the SBVP (1)–(2) on $[a, b]$ provided

$$-(p\beta^{\Delta})^{\Delta}(t) + q(t)\beta^{\sigma}(t) \geq f(t, \beta^{\sigma}(t)) + g(t, \beta^{\sigma}(t)) \quad \text{for all} \quad t \in [a, b]^{\kappa^2}$$

and

$$\beta(a) \geq A, \quad \beta(b) \geq B.$$

The following fundamental result from the calculus on time scales is very crucial in the proof of some results in the next section. A special case of the following lemma is proved by Akın [**3**].

**Lemma 3.1.** *Assume $h \in \mathbf{D}_1$. Suppose there exists $c \in (a, b)$ such that*

$$h(c) = \max\{h(t) : t \in [a, b]\}$$

*and*

$$h(t) < h(c) \quad \textit{for} \quad t \in (c, b].$$

*Then*

$$h^{\Delta}(c) \leq 0 \quad \text{and} \quad (ph^{\Delta})^{\Delta}(\rho(c)) \leq 0.$$

*Proof.* There are four cases as follows:

(i) $\rho(c) = c < \sigma(c)$;

(ii) $\rho(c) < c < \sigma(c)$;

(iii) $\rho(c) < c = \sigma(c)$;

(iv) $\rho(c) = c = \sigma(c)$.

As is shown in the proof of [**7**, Theorem 6.54], one can show that the first case is impossible and in the other cases $h^{\Delta}(c)$ and $(ph^{\Delta})^{\Delta}(\rho(c))$ are nonpositive. However, we only show the last case. From Case (iii) we have $h^{\Delta}(c) \leq 0$. If $h^{\Delta}(c) < 0$, then

$$\lim_{t \to c^-} h^{\Delta}(t) = h^{\Delta}(c) < 0.$$

This implies that there exists $\delta > 0$ such that $h^{\Delta}(t) < 0$ on $(c - \delta, c]$. Hence $h$ is strictly decreasing on $(c - \delta, c]$. But this contradicts the way

$c$ was chosen. Therefore $h^\Delta(c) = 0$. If $(ph^\Delta)^\Delta(\rho(c)) = (ph^\Delta)^\Delta(c) > 0$, then

$$\lim_{t\to c^+} (ph^\Delta)^\Delta(t) = (ph^\Delta)^\Delta(c) > 0.$$

This implies that there exists $\delta > 0$ such that $(ph^\Delta)^\Delta(t) > 0$ on $[c, c+\delta)$. Hence $ph^\Delta$ is strictly increasing on $[c, c+\delta)$. But $h^\Delta(c) = 0$, and hence $p(c)h^\Delta(c) = 0$. So $(ph^\Delta)(t) > 0$ on $(c, c+\delta)$ and therefore $h^\Delta(t) > 0$ on $(c, c+\delta)$. This implies that $h$ is strictly increasing on $(c, c+\delta)$. But this contradicts the way $c$ was chosen. Therefore $(ph^\Delta)^\Delta(c) \le 0$. □

*Remark* 3.1. The smoothness requirements on $\alpha$ and $\beta$ can be weakened in the following way as in [**7**, Theorem 6.54].

**Theorem 3.1.** *Assume that there exist a lower solution $\alpha$ and an upper solution $\beta$ of the* SBVP (1)–(2) *such that*

$$\alpha(t) \le \beta(t) \quad \textit{for all} \quad t \in [a, b].$$

*Then the* SBVP (1)–(2) *has a solution $x \in [\alpha, \beta]_1$ on $[a, b]$.*

We need one final fundamental result concerning the method of upper and lower solutions.

**Theorem 3.2.** *Assume that $f$ is strictly decreasing and $g$ is decreasing in $x$ for $t \in [a, b]^{\kappa^2}$. Moreover, assume that $\alpha$ and $\beta$ are lower and upper solutions of the* SBVP (1)–(2)*, respectively. Then*

$$\alpha(t) \le \beta(t) \quad \text{for all} \quad t \in [a, b].$$

*Proof.* Define $h := \alpha - \beta$. For the sake of contradiction assume that the result is not true. Hence there exists $t_1 \in [a, b]$ with $h(t_1) > 0$. Since $h(a) \le 0$ and $h(b) \le 0$, we can choose $t_0 \in (a, b)$ so that

$$h(t_0) = \max\{h(t) : t \in [a, b]\} > 0$$

and

$$h(t) < h(t_0) \quad \text{for all} \quad t \in (t_0, b].$$

Then by Lemma 3.1,

$$(ph^{\Delta})^{\Delta}(\rho(t_0)) \leq 0.$$

On the other hand, we have, note that $\sigma(\rho(t_0)) = t_0$ as the first case in Lemma 3.1 cannot occur,

$$\begin{aligned}(ph^{\Delta})^{\Delta}(\rho(t_0)) &= (p\alpha^{\Delta})^{\Delta}(\rho(t_0)) - (p\beta^{\Delta})^{\Delta}(\rho(t_0)) \\ &\geq -f(\rho(t_0), \alpha(t_0)) - g(\rho(t_0), \alpha(t_0)) + f(\rho(t_0), \beta(t_0)) \\ &\quad + g(\rho(t_0), \beta(t_0)) + q(\rho(t_0))[\alpha(t_0) - \beta(t_0)] > 0,\end{aligned}$$

where we used the definition of upper and lower solutions, the monotonicity conditions of $f$ and $g$, and the sign condition on $q$. □

Theorem 3.2 gives another approach for proving the uniqueness of solutions of the SBVP (1)–(2). The proof of the following theorem follows from the fact that every solution of the SBVP is also a lower and an upper solution.

**Theorem 3.3.** *Assume that the conditions on $f$, $g$, $\alpha$ and $\beta$ hold as in Theorem* 3.2. *Then the* SBVP (1)–(2) *has a unique solution on* $[a, b]$.

**4. The quasilinearization method.** In this section we let $f^{(i)}(t, x)$ be the $i$th derivative of $f$ with respect to $x$ for $i \geq 1$, and $f^{(0)}(t, x) = f(t, x)$. Note that these are the usual partial derivatives in $\mathbf{R}$. In this section we also assume that the point $b$ is right dense.

**Definition 4.2.** A function $f \in \mathrm{C}^{k+1}([a, b]^{\kappa^2} \times \mathbf{R})$ is called *$k$-hyperconvex* if $f^{(k+1)}(t, x) \geq 0$, $k \in \mathbf{N}$. Analogously, $f$ is called *$k$-hyperconcave* if the inequality is reversed.

When $k = 1$, 1-hyperconvex (1-hyperconcave) function is the usual convex (concave) function.

Here we prove our main result when $k = 3$.

**Theorem 4.1.** *Assume that $\alpha_0$ and $\beta_0$ are respectively lower and upper solutions of the* SBVP (1)–(2) *on $[a, b]$ and $f, g \in \mathrm{C}^4([a, b]^{\kappa^2} \times [\alpha_0, \beta_0]_1)$. If*

(i) *$f$ is 3-hyperconvex such that $f^{(1)}(t,x) < 0$ and $f^{(i)}(t,x) \le 0$ on $[a,b]^{\kappa^2} \times [\alpha_0,\beta_0]_1$ for $i = 2,3$, and*

(ii) *$g$ is 3-hyperconcave such that $g^{(i)}(t,x) \le 0$ on $[a,b]^{\kappa^2} \times [\alpha_0,\beta_0]_1$ for $i = 1,2,3$,*

*then there exist monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$ converging uniformly in $[\alpha_0,\beta_0]_1$ on $[a,b]$ to the unique solution $x$ of the* SBVP (1)–(2).

*Proof.* Define $H$ and $L$ by

$$\begin{aligned} H(t,x^\sigma;\alpha_0,\beta_0) &= (f+g)(t,\alpha_0^\sigma) + (f+g)^{(1)}(t,\alpha_0^\sigma)(x^\sigma - \alpha_0^\sigma) \\ &\quad + \frac{1}{2!}(f+g)^{(2)}(t,\alpha_0^\sigma)(x^\sigma-\alpha_0^\sigma)^2 \\ &\quad + \frac{1}{3!}\left[f^{(3)}(t,\alpha_0^\sigma) + g^{(3)}(t,\beta_0^\sigma)\right](x^\sigma-\alpha_0^\sigma)^3 \end{aligned}$$

and

$$\begin{aligned} L(t,x^\sigma;\alpha_0,\beta_0) &= (f+g)(t,\beta_0^\sigma) + (f+g)^{(1)}(t,\beta_0^\sigma)(x^\sigma - \beta_0^\sigma) \\ &\quad + \frac{1}{2!}(f+g)^{(2)}(t,\beta_0^\sigma)(x^\sigma-\beta_0^\sigma)^2 \\ &\quad + \frac{1}{3!}\left[f^{(3)}(t,\alpha_0^\sigma) + g^{(3)}(t,\beta_0^\sigma)\right](x^\sigma-\beta_0^\sigma)^3. \end{aligned}$$

We now consider two SBVPs in addition to the SBVP (1)–(2) as follows:

$$-(p(t)x^\Delta)^\Delta + q(t)x^\sigma = H(t,x^\sigma;\alpha_0,\beta_0), \quad t \in [a,b]^{\kappa^2} \tag{5}$$

with the boundary conditions (2) and

$$-(p(t)x^\Delta)^\Delta + q(t)x^\sigma = L(t,x^\sigma;\alpha_0,\beta_0), \quad t \in [a,b]^{\kappa^2} \tag{6}$$

with the boundary conditions (2). First of all, we will show that $\alpha_0$ and $\beta_0$ are lower and upper solutions of the SBVPs (5)–(2) and (6)–(2). Since

$$H(t,\alpha_0^\sigma;\alpha_0,\beta_0) = (f+g)(t,\alpha_0^\sigma) \ge -(p\alpha_0^\Delta)^\Delta + q\alpha_0^\sigma,$$

$\alpha_0$ is trivially a lower solution of the SBVP (5)–(2). Applying Taylor's theorem, we obtain

$$\begin{aligned}
H(t,\beta_0^\sigma;\alpha_0,\beta_0) &= (f+g)(t,\alpha_0^\sigma)+(f+g)^{(1)}(t,\alpha_0^\sigma)(\beta_0^\sigma-\alpha_0^\sigma)\\
&\quad+\frac{1}{2!}(f+g)^{(2)}(t,\alpha_0^\sigma)(\beta_0^\sigma-\alpha_0^\sigma)^2\\
&\quad+\frac{1}{3!}\left[f^{(3)}(t,\alpha_0^\sigma)+g^{(3)}(t,\beta_0^\sigma)\right](\beta_0^\sigma-\alpha_0^\sigma)^3\\
&= f(t,\beta_0^\sigma)-\frac{1}{4!}f^{(4)}(t,\xi)(\beta_0^\sigma-\alpha_0^\sigma)^4+g(t,\beta_0^\sigma)\\
&\quad-\frac{1}{3!}g^{(3)}(t,\nu)(\beta_0^\sigma-\alpha_0^\sigma)^3+\frac{1}{3!}g^{(3)}(t,\beta_0^\sigma)(\beta_0^\sigma-\alpha_0^\sigma)^3\\
&\le (f+g)(t,\beta_0^\sigma)+\frac{1}{3!}\left[g^{(3)}(t,\beta_0^\sigma)-g^{(3)}(t,\nu)\right](\beta_0^\sigma-\alpha_0^\sigma)^3\\
&\le (f+g)(t,\beta_0^\sigma)\\
&\le -(p\beta_0^\Delta)^\Delta+q\beta_0^\sigma,
\end{aligned}$$

where $\alpha_0^\sigma \le \xi,\ \nu \le \beta_0^\sigma$ and we used the facts that $g$ is 3-hyperconcave and $f$ is 3-hyperconvex. Therefore $\beta_0$ is an upper solution of the SBVP (5)–(2). Theorem 3.1 assures the existence of a solution $\alpha_1$ of the SBVP (5)–(2) such that

$$\alpha_0(t)\le\alpha_1(t)\le\beta_0(t)\quad\text{for all}\quad t\in[a,b].$$

Similarly, $\beta_0$ is an upper solution of the SBVP (6)–(2) and $\alpha_0$ is a lower solution of the SBVP (6)–(2) since

$$L(t,\beta_0^\sigma;\alpha_0,\beta_0)=(f+g)(t,\beta_0^\sigma)\le-(p\beta_0^\Delta)^\Delta+q\beta_0^\sigma$$

and

$$\begin{aligned}
L(t,\alpha_0^\sigma;\alpha_0,\beta_0) &= (f+g)(t,\beta_0^\sigma)+(f+g)^{(1)}(t,\beta_0^\sigma)(\alpha_0^\sigma-\beta_0^\sigma)\\
&\quad+\frac{1}{2!}(f+g)^{(2)}(t,\beta_0^\sigma)(\alpha_0^\sigma-\beta_0^\sigma)^2\\
&\quad+\frac{1}{3!}\left[f^{(3)}(t,\alpha_0^\sigma)+g^{(3)}(t,\beta_0^\sigma)\right](\alpha_0^\sigma-\beta_0^\sigma)^3
\end{aligned}$$

$$
\begin{aligned}
&= f(t,\alpha_0^\sigma) - \frac{1}{3!}f^{(3)}(t,\xi)(\alpha_0^\sigma - \beta_0^\sigma)^3 \\
&\quad + \frac{1}{3!}f^{(3)}(t,\alpha_0^\sigma)(\alpha_0^\sigma - \beta_0^\sigma)^3 \\
&\quad + g(t,\alpha_0^\sigma) - \frac{1}{4!}g^{(4)}(t,\nu)(\alpha_0^\sigma - \beta_0^\sigma)^4 \\
&\geq (f+g)(t,\alpha_0^\sigma) + \frac{1}{3!}\left[f^{(3)}(t,\alpha_0^\sigma) - f^{(3)}(t,\xi)\right](\alpha_0^\sigma - \beta_0^\sigma)^3 \\
&\geq (f+g)(t,\alpha_0^\sigma) \\
&\geq -(p\alpha_0^\Delta)^\Delta + q\alpha_0^\sigma,
\end{aligned}
$$

where $\alpha_0^\sigma \leq \xi$, $\nu \leq \beta_0^\sigma$ and we used Taylor's theorem and the facts that $f$ is 3-hyperconvex and $g$ is 3-hyperconcave. Applying Theorem 3.1, there exists a solution $\beta_1$ of the SBVP (6)–(2) such that

$$\alpha_0(t) \leq \beta_1(t) \leq \beta_0(t) \quad \text{for all} \quad t \in [a,b].$$

Next, we show that

$$\alpha_1(t) \leq \beta_1(t) \quad \text{for all} \quad t \in [a,b]. \tag{7}$$

To see this, we apply Taylor's theorem

$$
\begin{aligned}
-(p\alpha_1^\Delta)^\Delta + q\alpha_1^\sigma &= H(t,\alpha_1^\sigma;\alpha_0,\beta_0) \\
&= (f+g)(t,\alpha_0^\sigma) + (f+g)^{(1)}(t,\alpha_0^\sigma)(\alpha_1^\sigma - \alpha_0^\sigma) \\
&\quad + \frac{1}{2!}(f+g)^{(2)}(t,\alpha_0^\sigma)(\alpha_1^\sigma - \alpha_0^\sigma)^2 \\
&\quad + \frac{1}{3!}\left[f^{(3)}(t,\alpha_0^\sigma) + g^{(3)}(t,\beta_0^\sigma)\right](\alpha_1^\sigma - \alpha_0^\sigma)^3 \\
&= f(t,\alpha_1^\sigma) - \frac{1}{4!}f^{(4)}(t,\xi)(\alpha_1^\sigma - \alpha_0^\sigma)^4 + g(t,\alpha_1^\sigma) \\
&\quad - \frac{1}{3!}g^{(3)}(t,\nu)(\alpha_1^\sigma - \alpha_0^\sigma)^3 \\
&\quad + \frac{1}{3!}g^{(3)}(t,\beta_0^\sigma)(\alpha_1^\sigma - \alpha_0^\sigma)^3 \\
&\leq (f+g)(t,\alpha_1^\sigma) + \frac{1}{3!}\left[g^{(3)}(t,\beta_0^\sigma) - g^{(3)}(t,\nu)\right](\alpha_1^\sigma - \alpha_0^\sigma)^3 \\
&\leq (f+g)(t,\alpha_1^\sigma),
\end{aligned}
$$

where $\alpha_0^\sigma \le \xi,\ \nu \le \alpha_1^\sigma$ and we used the facts that $f$ is 3-hyperconvex and $g$ is 3-hyperconcave. Hence, $\alpha_1$ is a lower solution of the SBVP (1)–(2). Again we apply Taylor's theorem to obtain

$$\begin{aligned}
-(p\beta_1^\Delta)^\Delta + q\beta_1^\sigma &= L(t,\beta_1^\sigma;\alpha_0,\beta_0)\\
&= (f+g)(t,\beta_0^\sigma) + (f+g)^{(1)}(t,\beta_0^\sigma)(\beta_1^\sigma-\beta_0^\sigma)\\
&\quad + \frac{1}{2!}(f+g)^{(2)}(t,\beta_0^\sigma)(\beta_1^\sigma-\beta_0^\sigma)^2\\
&\quad + \frac{1}{3!}\left[f^{(3)}(t,\alpha_0^\sigma)+g^{(3)}(t,\beta_0^\sigma)\right](\beta_1^\sigma-\beta_0^\sigma)^3\\
&= f(t,\beta_1^\sigma) - \frac{1}{3!}f^{(3)}(t,\xi)(\beta_1^\sigma-\beta_0^\sigma)^3\\
&\quad + \frac{1}{3!}f^{(3)}(t,\alpha_0^\sigma)(\beta_1^\sigma-\beta_0^\sigma)^3\\
&\quad + g(t,\beta_1^\sigma) - \frac{1}{4!}g^{(4)}(t,\nu)(\beta_1^\sigma-\beta_0^\sigma)^4\\
&\ge (f+g)(t,\beta_1^\sigma) + \frac{1}{3!}\left[f^{(3)}(t,\alpha_0^\sigma) - f^{(3)}(t,\xi)\right](\beta_1^\sigma-\beta_0^\sigma)^3\\
&\ge (f+g)(t,\beta_1^\sigma),
\end{aligned}$$

where $\beta_1^\sigma \le \xi,\ \nu \le \beta_0^\sigma$ and we used the facts that $f$ is 3-hyperconvex and $g$ is 3-hyperconcave. Therefore $\beta_1$ is an upper solution of the SBVP (1)–(2). Applying Theorem 3.2, we have the inequality in (7). Continuing this process by induction, we obtain sequences $\{\alpha_n\}_{n\in\mathbf{N}_0}$ and $\{\beta_n\}_{n\in\mathbf{N}_0}$ with

$$\alpha_n(t) \le \alpha_{n+1}(t) \le \beta_{n+1}(t) \le \beta_n(t) \quad \text{for all} \quad t\in[a,b], n\in\mathbf{N}_0$$

where for each $n\in\mathbf{N}_0$, $\alpha_{n+1}$ and $\beta_{n+1}$ satisfy the SBVPs

$$-(p(t)x^\Delta)^\Delta + q(t)x^\sigma = H(t,x^\sigma;\alpha_n,\beta_n), \quad t\in[a,b]^{\kappa^2}$$

and

$$-(p(t)x^\Delta)^\Delta + q(t)x^\sigma = L(t,x^\sigma;\alpha_n,\beta_n), \quad t\in[a,b]^{\kappa^2}$$

with the boundary conditions (2), respectively.

Since $[a,b]$ is compact and the convergence is monotone and bounded, $\{\alpha_n\}$ converges uniformly to some function $x$. Erbe and Peterson [**10**] have constructed the Green's function $G(t,s)$ associated with the SBVP (1)–(2) and shown the positivity property of $G(t,s)$. We employ $G(t,s)$ to solve nonlinear dynamic equations through the following observation: $x$ is a solution of the SBVP (1)–(2) satisfying

$$x(t) = z_1(t) + \int_a^{\rho(b)} G(t,s)(f+g)(s,x^\sigma(s))\Delta s, \quad t \in [a,b], \tag{8}$$

where $z_1$ is the solution of the SBVP

$$\begin{cases} -(p(t)z_1^\Delta)^\Delta + q(t){z_1}^\sigma = 0, & t \in [a,b]^{\kappa^2} \\ z_1(a) = A, \quad z_1(b) = B. \end{cases}$$

Now, $\{\alpha_n\}$ converges monotonically and uniformly to some function $x$ and

$$\alpha_{n+1}(t) = z_1(t) + \int_a^{\rho(b)} G(t,s)H(s,\alpha_{n+1}^\sigma(s);\alpha_n,\beta_n)\Delta s, \quad t \in [a,b].$$

Note that

$$H(s,\alpha_{n+1}^\sigma;\alpha_n,\beta_n) \longrightarrow (f+g)(s,x^\sigma)$$

and the convergence is uniform on $[a,b]$ since $[a,b]$ is compact. It is now straightforward to show that (8) holds. A similar argument can be used for $\{\beta_n\}$. The conclusion of the theorem follows from the fact that the SBVP (1)–(2) has a unique solution, namely by Theorem 3.3. □

**Corollary 4.1.** *The convergence of each sequence* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *is cubic.*

*Proof.* First we show that the convergence of $\{\alpha_n\}$ is cubic. There exists $n_0 \in \mathbf{N}$ such that

$$
\begin{aligned}
&-(p(x-\alpha_{n+1})^{\Delta})^{\Delta} + q(x^{\sigma} - \alpha_{n+1}^{\sigma}) \\
&\quad = (f+g)(t, x^{\sigma}) - H(t, \alpha_{n+1}^{\sigma}; \alpha_n, \beta_n) \\
&\quad = (f+g)(t, x^{\sigma}) - (f+g)(t, \alpha_n^{\sigma}) - (f+g)^{(1)}(t, \alpha_n^{\sigma})(\alpha_{n+1}^{\sigma} - \alpha_n^{\sigma}) \\
&\qquad - \frac{1}{2!}(f+g)^{(2)}(t, \alpha_n^{\sigma})(\alpha_{n+1}^{\sigma} - \alpha_n^{\sigma})^2 \\
&\qquad - \frac{1}{3!}\left[f^{(3)}(t, \alpha_n^{\sigma}) + g^{(3)}(t, \beta_n^{\sigma})\right](\alpha_{n+1}^{\sigma} - \alpha_n^{\sigma})^3 \\
&\quad = (f+g)(t, \alpha_n^{\sigma}) + (f+g)^{(1)}(t, \alpha_n^{\sigma})(x^{\sigma} - \alpha_n^{\sigma}) \\
&\qquad + \frac{1}{2!}(f+g)^{(2)}(t, \alpha_n^{\sigma})(x^{\sigma} - \alpha_n^{\sigma})^2 \\
&\qquad + \frac{1}{3!}f^{(3)}(t, \alpha_n^{\sigma})(x^{\sigma} - \alpha_n^{\sigma})^3 + \frac{1}{4!}f^{(4)}(t, \xi)(x^{\sigma} - \alpha_n^{\sigma})^4 \\
&\qquad + \frac{1}{3!}g^{(3)}(t, \nu)(x^{\sigma} - \alpha_n^{\sigma})^3 \\
&\qquad - (f+g)(t, \alpha_n^{\sigma}) - (f+g)^{(1)}(t, \alpha_n^{\sigma})(\alpha_{n+1}^{\sigma} - \alpha_n^{\sigma}) \\
&\qquad - \frac{1}{2!}(f+g)^{(2)}(t, \alpha_n^{\sigma})(\alpha_{n+1}^{\sigma} - \alpha_n^{\sigma})^2 \\
&\qquad - \frac{1}{3!}\left[f^{(3)}(t, \alpha_n^{\sigma}) + g^{(3)}(t, \nu)\right](\alpha_{n+1}^{\sigma} - \alpha_n^{\sigma})^3 \\
&\quad = (f+g)^{(1)}(t, \alpha_n^{\sigma})(x^{\sigma} - \alpha_{n+1}^{\sigma}) \\
&\qquad + \frac{1}{2!}(f+g)^{(2)}(t, \alpha_n^{\sigma})\left[(x^{\sigma} - \alpha_n^{\sigma})^2 - (\alpha_{n+1}^{\sigma} - \alpha_n^{\sigma})^2\right] \\
&\qquad + \frac{1}{3!}f^{(3)}(t, \alpha_n^{\sigma})\left[(x^{\sigma} - \alpha_n^{\sigma})^3 - (\alpha_{n+1}^{\sigma} - \alpha_n^{\sigma})^3\right] \\
&\qquad + \frac{1}{4!}f^{(4)}(t, \xi)(x^{\sigma} - \alpha_n^{\sigma})^4 + \frac{1}{3!}g^{(3)}(t, \nu)(x^{\sigma} - \alpha_n^{\sigma})^3 \\
&\qquad - \frac{1}{3!}g^{(3)}(t, \beta_n^{\sigma})(\alpha_{n+1}^{\sigma} - \alpha_n^{\sigma})^3 \\
&\quad = (f+g)^{(1)}(t, \alpha_n^{\sigma})(x^{\sigma} - \alpha_{n+1}^{\sigma}) \\
&\qquad + \frac{1}{2!}(f+g)^{(2)}(t, \alpha_n^{\sigma})(x^{\sigma} - 2\alpha_n^{\sigma} + \alpha_{n+1}^{\sigma})(x^{\sigma} - \alpha_{n+1}^{\sigma}) \\
&\qquad + \frac{1}{3!}f^{(3)}(t, \alpha_n^{\sigma})\left(\sum_{j=0}^{2}(x^{\sigma} - \alpha_n^{\sigma})^{2-j}(\alpha_{n+1}^{\sigma} - \alpha_n^{\sigma})^j\right)(x^{\sigma} - \alpha_{n+1}^{\sigma}) \\
&\qquad + \frac{1}{3!}g^{(3)}(t, \nu)(x^{\sigma} - \alpha_n^{\sigma})^3 - \frac{1}{3!}g^{(3)}(t, \beta_n^{\sigma})(\alpha_{n+1}^{\sigma} - \alpha_n^{\sigma})^3 \\
&\qquad + \frac{1}{4!}f^{(4)}(t, \xi)(x^{\sigma} - \alpha_n^{\sigma})^4 \\
&\quad \leq -\frac{1}{3!}g^{(3)}(t, \beta_n^{\sigma})(\alpha_{n+1}^{\sigma} - \alpha_n^{\sigma})^3 + \frac{1}{4!}f^{(4)}(t, \xi)(x^{\sigma} - \alpha_n^{\sigma})^4
\end{aligned}
$$

$$\leq \left( -\frac{1}{3!} g^{(3)}(t, \beta_n^\sigma) + \frac{1}{4!} f^{(4)}(t, \xi) \right) (x^\sigma - \alpha_n^\sigma)^3$$
$$\leq M_3 (x^\sigma - \alpha_n^\sigma)^3,$$

for all $n \geq n_0$, where $\alpha_n^\sigma \leq \nu, \xi \leq x^\sigma$, $-(1/3!) g^{(3)}(t, \beta_n^\sigma) + (1/4!) f^{(4)}(t, \xi) \leq M_3$ and we used Taylor's theorem and the sign conditions (i) and (ii) of Theorem 4.1. Since $\alpha_{n+1}(a) = x(a)$ and $\alpha_{n+1}(b) = x(b)$, we have

$$(x - \alpha_{n+1})(t) = \int_a^{\rho(b)} G(t,s)[-\left(p(x - \alpha_{n+1})^\Delta\right)^\Delta + \left(q(x - \alpha_{n+1})^\sigma\right)](s)\Delta s, \quad t \in [a,b].$$

It follows that

$$0 \leq (x - \alpha_{n+1})(t) \leq L_1 M_3 ||x - \alpha_n||^3, \quad t \in [a,b]$$

where
$$L_1 := \max_{t \in [a,b]} \int_a^{\rho(b)} G(t,s)\Delta s.$$

For the cubic convergence of $\{\beta_n\}$, we apply Taylor's theorem

$$\begin{aligned}
-(p(\beta_{n+1} - x)^\Delta)^\Delta &+ q(\beta_{n+1}^\sigma - x^\sigma) \\
&= L(t, \beta_{n+1}^\sigma; \alpha_n, \beta_n) - (f+g)(t, x^\sigma) \\
&= (f+g)(t, \beta_n^\sigma) + (f+g)^{(1)}(t, \beta_n^\sigma)(\beta_{n+1}^\sigma - \beta_n^\sigma) \\
&\quad + \frac{1}{2!}(f+g)^{(2)}(t, \beta_n^\sigma)(\beta_{n+1}^\sigma - \beta_n^\sigma)^2 \\
&\quad + \frac{1}{3!}\left[ f^{(3)}(t, \alpha_n^\sigma) + g^{(3)}(t, \beta_n^\sigma) \right] (\beta_{n+1}^\sigma - \beta_n^\sigma)^3 - (f+g)(t, x^\sigma) \\
&= (f+g)(t, \beta_n^\sigma) + (f+g)^{(1)}(t, \beta_n^\sigma)(\beta_{n+1}^\sigma - \beta_n^\sigma) \\
&\quad + \frac{1}{2!}(f+g)^{(2)}(t, \beta_n^\sigma)(\beta_{n+1}^\sigma - \beta_n^\sigma)^2 \\
&\quad + \frac{1}{3!}\left[ f^{(3)}(t, \alpha_n^\sigma) + g^{(3)}(t, \beta_n^\sigma) \right] (\beta_{n+1}^\sigma - \beta_n^\sigma)^3 - (f+g)(t, \beta_n^\sigma) \\
&\quad - (f+g)^{(1)}(t, \beta_n^\sigma)(x^\sigma - \beta_n^\sigma) - \frac{1}{2!}(f+g)^{(2)}(t, \beta_n^\sigma)(x^\sigma - \beta_n^\sigma)^2 \\
&\quad - \frac{1}{3!} g^{(3)}(t, \beta_n^\sigma)(x^\sigma - \beta_n^\sigma)^3 - \frac{1}{4!} g^{(4)}(t, \nu)(x^\sigma - \beta_n^\sigma)^4
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{3!} f^{(3)}(t,\xi)(x^\sigma - \beta_n^\sigma)^3 \\
&= (f+g)(t,\beta_n^\sigma)(\beta_{n+1}^\sigma - x^\sigma) \\
&\quad + \frac{1}{2!}(f+g)^{(2)}(t,\beta_n^\sigma)\left[(\beta_{n+1}^\sigma - \beta_n^\sigma)^2 - (x^\sigma - \beta_n^\sigma)^2\right] \\
&\quad + \frac{1}{3!} g^{(3)}(t,\beta_n^\sigma)\left[(\beta_{n+1}^\sigma - \beta_n^\sigma)^3 - (x^\sigma - \beta_n^\sigma)^3\right] \\
&\quad + \frac{1}{3!} f^{(3)}(t,\alpha_n^\sigma)(\beta_{n+1}^\sigma - \beta_n^\sigma)^3 \\
&\quad - \frac{1}{3!} f^{(3)}(t,\xi)(x^\sigma - \beta_n^\sigma)^3 - \frac{1}{4!} g^{(4)}(t,\nu)(x^\sigma - \beta_n^\sigma)^4 \\
&= (f+g)^{(1)}(t,\beta_n^\sigma)(\beta_{n+1}^\sigma - x^\sigma) \\
&\quad + \frac{1}{2!}(f+g)^{(2)}(t,\beta_n^\sigma)(\beta_{n+1}^\sigma - 2\beta_n^\sigma + x^\sigma)(\beta_{n+1}^\sigma - x^\sigma) \\
&\quad + \frac{1}{3!} g^{(3)}(t,\beta_n^\sigma)\left[\sum_{j=0}^{2}(\beta_{n+1} - \beta_n^\sigma)^{2-j}(x^\sigma - \beta_n^\sigma)^j\right](\beta_{n+1}^\sigma - x^\sigma) \\
&\quad + \frac{1}{3!} f^{(3)}(t,\alpha^\sigma)(\beta_{n+1}^\sigma - \beta_n^\sigma)^3 - \frac{1}{3!} f^{(3)}(t,\xi)(x^\sigma - \beta_n^\sigma)^3 \\
&\quad - \frac{1}{4!} g^{(4)}(t,\nu)(x^\sigma - \beta_n^\sigma)^4 \\
&\le [(f+g)^{(1)}(t,\beta_n^\sigma) + \frac{1}{2!}(f+g)^{(2)}(t,\beta_n^\sigma)(\beta_{n+1}^\sigma - 2\beta_n^\sigma + x^\sigma) \\
&\quad + \frac{1}{3!} g^{(3)}(t,\beta_n^\sigma) C_n](\beta_{n+1}^\sigma - x^\sigma) - \frac{1}{3!} f^{(3)}(t,\alpha_n^\sigma)(\beta_n^\sigma - \beta_{n+1}^\sigma)^3 \\
&\quad - \frac{1}{4!} g^{(4)}(t,\nu)(\beta_n^\sigma - x^\sigma)^4 \\
&\le [(f+g)^{(1)}(t,\beta_n^\sigma) + \frac{1}{2!}(f+g)^{(2)}(t,\beta_n^\sigma)(\beta_{n+1}^\sigma - 2\beta_n^\sigma + x^\sigma) \\
&\quad + \frac{1}{3!} g^{(3)}(t,\beta_n^\sigma) C_n](\beta_{n+1}^\sigma - x^\sigma) \\
&\quad - \left[\frac{1}{3!} f^{(3)}(t,\alpha_n^\sigma) + \frac{1}{4!} g^{(4)}(t,\nu)\right](\beta_n^\sigma - x^\sigma)^3 \\
&\le [(f+g)^{(1)}(t,\beta_n^\sigma) + \frac{1}{2!}(f+g)^{(2)}(t,\beta_n^\sigma)(\beta_{n+1}^\sigma - 2\beta_n^\sigma + x^\sigma) \\
&\quad + \frac{1}{3!} g^{(3)}(t,\beta_n^\sigma) C_n](\beta_{n+1}^\sigma - x^\sigma) + N_3(\beta_n^\sigma - x^\sigma)^3,
\end{aligned}
$$

where $C_n = \sum_{j=0}^{2}(\beta_{n+1}^\sigma - \beta_n^\sigma)^{2-j}(x^\sigma - \beta_n^\sigma)^j$, $-(1/3!)f^{(3)}(t,\alpha_n^\sigma) -$

$(1/4!)g^{(4)}(t,\nu) \le N_3$, $x^\sigma \le \xi, \nu \le \beta_n^\sigma$ and we used the fact that $f^{(3)}(t,x) \le 0$. It follows that

$$-(p(\beta_{n+1}-x)^\Delta)^\Delta + Q_n(\beta_{n+1}^\sigma - x^\sigma) \le N_3(\beta_n^\sigma - x^\sigma)^3,$$

where $Q_n = q - (f+g)^{(1)}(t,\beta_n^\sigma) - (1/2!)(f+g)^{(2)}(t,\beta_n^\sigma)(\beta_{n+1}^\sigma - 2\beta_n^\sigma + x^\sigma) - (1/3!)g^{(3)}(t,\beta_n^\sigma)C_n$. Since $\beta_n$ converges to $x$ uniformly, (i) and (ii) imply that there exist $n_0 \in \mathbf{N}$ and $Q \ge 0$ such that $Q_n(t) \ge Q$ for all $n \ge n_0$. Hence, there exists a continuous function $h_n \le 0$ on $[a,b]$ such that

$$-(p(\beta_{n+1}-x)^\Delta)^\Delta + Q(\beta_{n+1}^\sigma - x^\sigma) = N_3(\beta_n^\sigma - x^\sigma)^3 + h_n,$$
$$(\beta_n - x)(a) = 0, \quad (\beta_{n+1}-x)(b) = 0$$

holds and it is equivalently,

$$(\beta_{n+1}-x)(t) = \int_a^{\rho(b)} G(t,s,Q)(N_3(\beta_n^\sigma - x^\sigma)^3 + h_n)(s)\Delta s$$

for every $t \in [a,b]$. And for any $n \ge n_0$ we have that

$$0 \le (\beta_{n+1}-x)(t) \le L_1^* N_3 \max_{t\in[a,b]}\{\beta_n(t) - x(t)\}^3,$$

where $L_1^* := \max_{t\in[a,b]} \int_a^{\rho(b)} G(t,s,Q)\Delta s$. Therefore

$$||\beta_{n+1} - x||_\infty \le L_1^* N_3 ||\beta_n - x||_\infty^3, \quad \text{for all} \quad n \ge n_0. \qquad \square$$

*Remark* 4.1. If $g = 0$, then the order of convergence of $\{\alpha_n\}$ is 4.

*Remark* 4.2. In [**14**] the main result has one monotone sequence converging from below to the unique solution of the SBVP when $\mathbf{T} = \mathbf{R}$. Here Theorem 4.1 gives an improvement to their result. The method developed here provides that the lower and upper solutions serve as bounds of solutions of the nonlinear problem in a traditional way.

We generalize Theorem 4.1 and Corollary 4.1 where $f$ and $g$ are $k$-hyperconvex and $k$-hyperconcave functions, respectively.

**Theorem 4.2.** *Assume that* $\alpha_0$ *and* $\beta_0$ *are respectively lower and upper solutions of the* SBVP (1)–(2) *on* $[a,b]$ *and assume that* $f, g \in \mathrm{C}^{k+1}([a,b]^{\kappa^2} \times [\alpha_0, \beta_0]_1)$, $k \in \mathbf{N}$. *If*

(i) $f$ *is* $k$*-hyperconvex such that* $f^{(1)}(t,x) < 0$ *and* $f^{(i)}(t,x) \leq 0$ *on* $[a,b]^{\kappa^2} \times [\alpha_0, \beta_0]_1$ *for* $2 \leq i \leq k$, *and*

(ii) $g$ *is* $k$*-hyperconcave such that* $g^{(i)}(t,x) \leq 0$ *on* $[a,b]^{\kappa^2} \times [\alpha_0, \beta_0]_1$ *for* $1 \leq i \leq k$,

*then there exist monotone sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *converging uniformly in* $[\alpha_0, \beta_0]_1$ *on* $[a,b]$ *to the unique solution* $x$ *of the* SBVP (1)–(2).

*Proof.* We define

$$H(t, x^\sigma; \alpha_0, \beta_0) = \sum_{i=0}^{k-1} (f+g)^{(i)}(t, \alpha_0^\sigma)(x^\sigma - \alpha_0^\sigma)^i + \frac{1}{k!}\left[f^{(k)}(t, \alpha_0^\sigma) + g^{(k)}(t, \beta_0^\sigma)\right](x^\sigma - \alpha_0^\sigma)^k$$

and

$$L(t, x^\sigma; \alpha_0, \beta_0) = \sum_{i=0}^{k-1} (f+g)^{(i)}(t, \beta_0^\sigma)(x^\sigma - \beta_0^\sigma)^i + (-1)^{k+1}\frac{1}{k!}\left[f^{(k)}(t, \alpha_0^\sigma) + g^{(k)}(t, \beta_0^\sigma)\right](x^\sigma - \beta_0^\sigma)^k.$$

The proof is similar to that of Theorem 4.1 and hence is omitted. □

**Corollary 4.2.** *The order of convergence of each sequence* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *is* $k$.

**Corollary 4.3.** *If* $g = 0$, *then the order of convergence of* $\{\alpha_n\}$ *is* $k+1$.

In the following examples we will apply Theorem 4.2 when $k = 2$.

**Example 4.1.** Consider the conjugate BVP

$$\begin{cases} -\left(\sigma(t)x^{\Delta}\right)^{\Delta}+(1/t)x^{\sigma}=-x^{\sigma}\left(x^{\sigma}+1\right), & t\in[1,2]^{\kappa^2} \\ x(1)=1/2, \quad x(2)=1/4, \end{cases} \tag{9}$$

where $f(t,x)=-x$ and $g(t,x)=-x^2$. Since

$$0=-\left(\sigma(t)\alpha_0^{\Delta}\right)^{\Delta}+\frac{1}{t}\,\alpha_0^{\sigma}\le 0$$

on $[1,2]^{\kappa^2}$ and $\alpha_0(1)\le 1/2$, $\alpha_0(2)\le 1/4$, $\alpha_0(t)\equiv 0$ is a lower solution of the SBVP (9) on $[1,2]$. Similarly, $\beta_0(t)\equiv \frac{1}{t}$ is an upper solution of SBVP (9) on $[1,2]$ since

$$0=-(\sigma(t)\beta_0^{\Delta})^{\Delta}+\frac{1}{t}\,\beta_0^{\sigma}\ge -\frac{1}{\sigma(t)}\left(\frac{1}{\sigma(t)}+1\right)$$

on $[1,2]^{\kappa^2}$ and $\beta_0(1)=1\ge 1/2$, $\beta_0(2)=1/2\ge 1/4$. By Theorem 3.1, we conclude that there is a solution in $[0,\frac{1}{t}]_1$ for $t\in[1,2]$. Moreover, since $f$ and $g$ satisfy the conditions (i) and (ii) in Theorem 4.2, we also conclude that there are monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$ converging uniformly in $[0,\frac{1}{t}]_1$ on $[1,2]$ to the unique solution of the SBVP (9).

**Example 4.2.** Consider the SBVP

$$\begin{cases} -x^{\Delta\Delta}+x^{\sigma}=4\sin\left((2\pi/3)x^{\sigma}\right)-\left(x^{\sigma}\right)^3, & t\in[a,b]^{\kappa^2} \\ x(a)=1, \quad x(b)=3/2, \end{cases} \tag{10}$$

where $f(t,x)=4\sin\left((2\pi/3)x\right)$ and $g(t,x)=-x^3$. Since

$$1=-\alpha_0^{\Delta\Delta}+\alpha_0^{\sigma}\le 2\sqrt{3}-1$$

on $[a,b]^{\kappa^2}$ and $\alpha_0(a)=1$, $\alpha_0(b)\le 3/2$, $\alpha_0(t)\equiv 1$ is a lower solution of the SBVP (10) on $[a,b]$. Similarly, $\beta_0(t)\equiv 3/2$ is an upper solution of the SBVP (10) on $[a,b]$ since

$$\frac{3}{2}=-\beta_0^{\Delta\Delta}+\beta_0^{\sigma}\ge -\frac{27}{8}$$

on $[a,b]^{\kappa^2}$ and $\beta_0(a) \geq 1$, $\beta_0(b) \equiv 3/2$. By Theorem 3.1, we conclude that there is a solution in $[1, 3/2]_1$ for $t \in [a,b]$. Moreover, since $f$ and $g$ satisfy the conditions (i) and (ii) in Theorem 4.2, we also conclude that there are monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$ converging uniformly in $[1, 3/2]_1$ on $[a,b]$ to the unique solution of the SBVP (10).

**5. Mixed derivative.** In this section, we are concerned with the SBVP (3)–(4), where $p, q \in \mathrm{C}([a,b])$ such that $p(t) > 0$ and $q(t) \geq 0$ for each $t \in [a,b]$ and $f, g \in \mathrm{C}([a,b] \times \mathbf{R})$ with respect to the standard topology on $\mathbf{T} \times \mathbf{R}$. We shall only state results for this SBVP whose proofs can be obtained using analogous arguments.

We define the set

$$\begin{aligned}\mathbf{D}_2 := \big\{x \in \mathbf{B} : \, & x^{\Delta} \text{ is continuous on } [\rho(a), b], \\ & px^{\Delta} \text{ is nabla-differentiable on } [a,b] \\ & \text{and } (px^{\Delta})^{\nabla} \text{ is continuous on } [a,b] \, \big\}\end{aligned}$$

where $\mathbf{B}$ is the Banach space $\mathbf{B} = \mathrm{C}([\rho(a), \sigma(b)])$ of all real-valued, in the topology of $\mathbf{T}$, functions $x$ defined on $[\rho(a), \sigma(b)]$ with the norm

$$\|x\| = \max_{t \in [\rho(a), \sigma(b)]} |x(t)|.$$

A function $x : [a,b] \mapsto \mathbf{R}$ is said to be a solution of the equation $-(p(t)y^{\Delta})^{\nabla} + q(t)y = 0$ on $[a,b]$ provided $x \in \mathbf{D}_2$ and the equation $-(px^{\Delta})^{\nabla}(t) + q(t)x(t) = 0$ holds for all $t \in [a,b]$. For any $u, v \in \mathbf{D}_2$, we define the sector $[u,v]_2$ by

$$[u,v]_2 := \{w \in \mathbf{D}_2 : u \leq w \leq v\}.$$

**Definition 5.1.** Real valued functions $\alpha, \beta \in \mathbf{D}_2$ on $[\rho(a), \sigma(b)]$ are called *lower and upper solutions* for the SBVP (3)–(4) if

$$\begin{gathered}-(p\alpha^{\Delta})^{\nabla}(t) + q(t)\alpha(t) \leq f(t, \alpha(t)) + g(t, \alpha(t)) \quad \text{for all} \quad t \in [a,b], \\ \alpha(\rho(a)) \leq A, \quad \alpha(\sigma(b)) \leq B\end{gathered}$$

and

$$\begin{gathered}-(p\beta^{\Delta})^{\nabla}(t) + q(t)\beta(t) \geq f(t, \beta(t)) + g(t, \beta(t)) \quad \text{for all} \quad t \in [a,b], \\ \beta(\rho(a)) \geq A, \quad \beta(\sigma(b)) \geq B\end{gathered}$$

hold, respectively.

Next we define

$$L_2 := \max_{t\in[\rho(a),\sigma(b)]} \int_{\rho(a)}^{b} G(t,s)\nabla s,$$

where $G(t,s)$ is the Green's function for the SBVP

$$\begin{cases} -(p(t)x^{\Delta})^{\nabla} + q(t)x = 0, & t \in [a,b], \\ x(\rho(a)) = 0, \ x(\sigma(b)) = 0. \end{cases}$$

The positivity property of this Green's function has been obtained in [**6**].

Some results concerning monotone methods and the method of quasilinearization for second order dynamic equations require the use of second derivative test. The next lemma deals with the sign of the delta and the delta-nabla derivatives of a function at a point of local maximum. The proof of the lemma follows from Lemma 3.1 and Theorem 2.3 which gives the relationship between delta and nabla derivatives.

**Lemma 5.1.** *Assume $h \in \mathbf{D}_2$. Suppose there exists $c \in (a,b)$ such that*

$$h(c) = \max\{h(t) : t \in [a,b]\} \quad \textit{and} \quad h(t) < h(c) \quad \textit{for} \quad t \in (c,b].$$

*Then*

$$h^{\Delta}(c) \le 0 \quad \textit{and} \quad (ph^{\Delta})^{\nabla}(c) \le 0.$$

The proof of the following theorem follows from Schauder fixed point theorem.

**Theorem 5.1.** *If $M > 0$ satisfies $M \ge \max\{|A|,|B|\}$ and $L_2Q \le M$ where $Q > 0$ satisfies*

$$Q \ge \max_{||x||\le 2M}|f(t,x) + g(t,x)|, \quad t \in [a,b],$$

*then the* SBVP (3)–(4) *has a solution.*

**Theorem 5.2.** *Assume that* $\alpha$ *and* $\beta$ *are lower and upper solutions for the* SBVP (3)–(4) *such that* $\alpha(t) \leq \beta(t)$ *for all* $t \in [\rho(a), \sigma(b)]$. *Then the* SBVP (3)–(4) *has a solution* $x \in [\alpha, \beta]_2$ *on* $[\rho(a), \sigma(b)]$.

**Theorem 5.3.** *Assume that*

(i) $\alpha$ *and* $\beta$ *are lower and upper solutions of the* SBVP (3)–(4) *on* $[\rho(a), \sigma(b)]$;

(ii) $f$ *is strictly decreasing in* $x$ *for* $t \in [a, b]$ *and;*

(iii) $g$ *is decreasing in* $x$ *for* $t \in [a, b]$.

*Then* $\alpha(t) \leq \beta(t)$ *for all* $t \in [\rho(a), \sigma(b)]$.

**Corollary 5.1.** *Under the hypotheses of Theorem* 5.3*, solutions of the* SBVP (3)–(4) *are unique.*

**Theorem 5.4.** *Assume that* $\alpha_0$ *and* $\beta_0$ *are respectively lower and upper solutions of the* SBVP (3)–(4) *on* $[\rho(a), \sigma(b)]$ *and assume that* $f, g \in C^{k+1}([a, b] \times [\alpha_0, \beta_0]_2)$, $k \in \mathbf{N}$. *If*

(i) $f$ *is k-hyperconvex such that* $f^{(1)}(t, x) < 0$ *and* $f^{(i)}(t, x) \leq 0$ *on* $[a, b] \times [\alpha_0, \beta_0]_2$ *for* $2 \leq i \leq k$, *and*

(ii) $g$ *is k-hyperconcave such that* $g^{(i)}(t, x) \leq 0$ *on* $[a, b] \times [\alpha_0, \beta_0]_2$ *for* $1 \leq i \leq k$,

*then there exist monotone sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *converging uniformly in* $[\alpha_0, \beta_0]_2$ *on* $[\rho(a), \sigma(b)]$ *to the unique solution of the* SBVP (3)–(4).

**Corollary 5.2.** *The order of convergence of each sequence* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *is* $k$.

**6. Remark.** The following proof of the next theorem shows that we can define $H$ and $L$ different than the ones in the proof of Theorem 4.2. So this implies that there exist other sequences of lower and upper solutions converging to unique solution of SBVP (1)–(2).

**Theorem 6.1.** *Assume that $\alpha_0$ and $\beta_0$ are respectively lower and upper solutions of the* SBVP (1)–(2) *on $[a,b]$ and assume that $f, g \in \mathrm{C}^{k+1}([a,b]^{\kappa^2} \times [\alpha_0, \beta_0]_1)$, $k \in \mathbf{N}$. If*

(i) *$f$ is $k$-hyperconvex such that $f^{(1)}(t,x) < 0$ and $f^{(i)}(t,x) \le 0$ on $[a,b]^{\kappa^2} \times [\alpha_0, \beta_0]_1$ for $2 \le i \le k$, and*

(ii) *$g$ is $k$-hyperconcave such that $g^{(i)}(t,x) \le 0$ on $[a,b]^{\kappa^2} \times [\alpha_0, \beta_0]_1$ for $1 \le i \le k$,*

*then there exist monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$ converging uniformly in $[\alpha_0, \beta_0]_1$ on $[a,b]$ to the unique solution $x$ of the* SBVP (1)–(2).

*Proof.* For any $(t,x), (t,y) \in [a,b]^{\kappa^2} \times [\alpha_0, \beta_0]_1$ with $x \ge y$ we obtain that

$$f(t,x) \ge \sum_{i=0}^{k} f^{(i)}(t,y)(x-y)^i \tag{11}$$

and

$$g(t,x) \ge \sum_{i=0}^{k-1} g^{(i)}(t,y)(x-y)^i + g^{(k)}(t,x)(x-y)^k, \tag{12}$$

which follow from the assumptions (i) and (ii) and we used the mean value theorem. We now define

$$\begin{aligned} H(t, x^\sigma; \alpha_0, \beta_0) = &\sum_{i=0}^{k-1} (f+g)^{(i)}(t, \alpha_0^\sigma)(x^\sigma - \alpha_0^\sigma)^i \\ &+ [f^{(k)}(t, \alpha_0^\sigma) + g^{(k)}(t, \beta_0^\sigma)](x^\sigma - \alpha_0^\sigma)^k \end{aligned}$$

and

$$\begin{aligned} L(t, x^\sigma; \alpha_0, \beta_0) = (f+g)(t, \beta_0^\sigma) &- \sum_{i=1}^{k-1} (f+g)^{(i)}(t, \beta_0^\sigma)(\beta_0^\sigma - x^\sigma)^i \\ &- [f^{(k)}(t, \alpha_0^\sigma) + g^{(k)}(t, \beta_0^\sigma)](\beta_0^\sigma - x^\sigma)^k. \end{aligned}$$

The proof is similar to that of Theorem 4.1 and hence is omitted. □

**Corollary 6.1.** *The convergence of each sequence* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *is quadratic when* $k = 2$.

*Proof.* Set $u_n = x - \alpha_n$ and $v_n = \beta_n - x$, where $x$ denotes the unique solution of the SBVP (1)–(2). We only show that the convergence of $\{\beta_n\}$ is quadratic. Similarly, the quadratic convergence of $\{\alpha_n\}$ can be seen. Note that $v_n \geq 0$ follows from the monotone convergence of $\{\beta_n\}$ to $x$. Applying mean value theorem, we obtain

$$\begin{aligned}
-(pv_{n+1}^{\Delta})^{\Delta} + qv_{n+1}^{\sigma} &= -(p\beta_{n+1}^{\Delta})^{\Delta} + (px^{\Delta})^{\Delta} + q\beta_{n+1}^{\sigma} - qx^{\sigma} \\
&= L(t, \beta_{n+1}^{\sigma}; \alpha_n, \beta_n) - (f+g)(t, x^{\sigma}) \\
&= (f+g)(t, \beta_n^{\sigma}) + (f+g)^{(1)}(t, \beta_n^{\sigma})(\beta_{n+1}^{\sigma} - \beta_n^{\sigma}) \\
&\quad - [f^{(2)}(t, \alpha_n^{\sigma}) + g^{(2)}(t, \beta_n^{\sigma})](\beta_{n+1}^{\sigma} - \beta_n^{\sigma})^2 \\
&\quad - (f+g)(t, x^{\sigma}) \\
&= (f+g)^{(1)}(t, \xi)(\beta_n^{\sigma} - x^{\sigma}) \\
&\quad - (f+g)^{(1)}(t, \beta_n^{\sigma})(\beta_{n+1}^{\sigma} - \beta_n^{\sigma}) \\
&\quad - \left[f^{(2)}(t, \alpha_n^{\sigma}) + g^{(2)}(t, \beta_n^{\sigma})\right](\beta_{n+1}^{\sigma} - \beta_n^{\sigma})^2 \\
&\leq \left[(f+g)^{(1)}(t, \xi) - (f+g)^{(1)}(t, \beta_n^{\sigma})\right](\beta_n^{\sigma} - x^{\sigma}) \\
&\quad - \left[f^{(2)}(t, \alpha_n^{\sigma}) + g^{(2)}(t, \beta_n^{\sigma})\right](\beta_{n+1}^{\sigma} - \beta_n^{\sigma})^2 \\
&= (f+g)^{(2)}(t, \nu)(\xi - \beta_n^{\sigma})(\beta_n^{\sigma} - x^{\sigma}) \\
&\quad - \left[f^{(2)}(t, \alpha_n^{\sigma}) + g^{(2)}(t, \beta_n^{\sigma})\right](\beta_{n+1}^{\sigma} - \beta_n^{\sigma})^2 \\
&\leq -(f+g)^{(2)}(t, \nu)(\beta_n^{\sigma} - x^{\sigma})^2 \\
&\quad - \left[f^{(2)}(t, \alpha_n^{\sigma}) + g^{(2)}(t, \beta_n^{\sigma})\right](\beta_{n+1}^{\sigma} - x^{\sigma})^2 \\
&\leq (M+N)||v_n||^2,
\end{aligned}$$

where $x^{\sigma} \leq \xi, \nu \leq \beta^{\sigma}$, $|f^{(2)}(t,x)| \leq M/2$, $|g^{(2)}(t,x)| \leq N/2$ for $(t,x) \in [a,b]^{\kappa^2} \times [\alpha_0, \beta_0]_1$. Since $v_{n+1}(a) = \beta_{n+1}(a) - x(a) = 0$ and $v_{n+1}(b) = \beta_{n+1}(b) - x(b) = 0$, we have

$$v_{n+1}(t) = \int_a^{\rho(b)} G(t,s)[-(pv_{n+1}^{\Delta})^{\Delta}(s) + (qv_{n+1}^{\sigma})(s)]\Delta s, \quad t \in [a,b].$$

It follows that

$$0 \leq v_{n+1}(t) \leq L_1(M+N)||v_n||^2, \quad t \in [a,b]. \qquad \square$$

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