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HELLY'S SELECTION PRINCIPLE FOR FUNCTIONS OF BOUNDED P-VARIATION

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ABSTRACT. The classical Helly's selection principle states that a uniformly bounded sequence of functions with uniform bounded variation admits a subsequence which converges pointwise to a function of bounded variation. Helly's selection principle for metric space-valued functions of bounded p-variation is proven answering a question of Chistyakov and Galkin.

1. Introduction. Jordan introduced the concept of *variation* of a function and characterized functions of bounded variation as differences of nondecreasing functions. Helly [**10**, p. 222] used this decomposition to prove a compactness theorem for functions of bounded variation which has become known as Helly's selection principle, a uniformly bounded sequence of functions with uniform bounded variation has a pointwise convergent subsequence.

The interest in Helly's selection principle is natural since it provides an effective means of proving existence theorems in analysis. For some examples see [**3**] and [**9**]. A problem of importance is proving Helly type selection theorems for functions of generalized variation. For example, Helly's Selection Principle has been proven by Fleischer and Porter [**7**] for metric-space valued BV functions, Waterman [**11**] for functions of bounded Λ-variation, and Cyphert and Kelingos [**4**] for functions of bounded χ-variation.

The p-variation, $p \geq 1$, may be defined for a metric space-valued function $f : E \to X$ of a real variable as

$$V_p(f, E) = \sup \sum_{i=1}^{m} d(f(t_i), f(t_{i-1}))^p$$

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where the supremum is taken over all finite $t_0 \leq \cdots \leq t_i \leq \cdots \leq t_m$ in $E \subset \mathbf{R}$ (d is the metric on X). When finite, the function is said to have bounded p-variation. The p-variation function is defined by $\phi(x) = V_p(f, E \cap (-\infty, x])$ which is a nondecreasing function.

The notion of p-variation for $p = 2$ was first introduced by Wiener [**12**]. Young [**14**] later studied functions of bounded p-variation for $p \geq 1$. Functions of bounded p-variation for $p = 1$ are often referred to as functions of bounded Jordan variation and were studied by Chistyakov [**1**]. Recently, functions of bounded p-variation have been applied to problems in stochastic differential equations [**13**] and integral equations [**8**]. For an excellent list of papers on functions of bounded p-variation see [**5**].

Chistyakov and Galkin [**2**] thoroughly studied the properties of functions with bounded p-variation in which they proved the following Helly type selection principle:

Theorem 1.1. *Let K be a compact subset of the metric space X, and let $\mathcal{F} \subset \mathcal{C}([a, b]; K)$ be an infinite family of continuous maps from the interval $[a, b]$ into K of uniformly bounded p-variation, that is, $\sup_{f \in \mathcal{F}} V_p(f, [a, b]) < \infty$. Then there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of maps from $\mathcal{F}$ which converges pointwise on $[a, b]$ to a map $f : [a, b] \to K$ of bounded p-variation. Moreover, if X is a Banach space, then the assumption of continuity of the family $\mathcal{F}$ is redundant.*

The second section is devoted to extending this theorem, dispensing with continuity, to arbitrary real subsets and lighten compactness of the range to pointwise precompactness, which answers Remark 6.1 in [**2**].

2. Helly's selection principle. Recall that a map $f : E \to X$ is *Hölderian of exponent* $0 < \gamma \leq 1$ if there exists a positive number C such that $d(f(t), f(s)) \leq C|t - s|^{\gamma}$ for all $t, s \in E$. The least number C satisfying the above inequality is called the *Hölder constant* of f and is denoted by $H(f)$. The argument is based on the following structure theorem proved in [**2**]:

Lemma 2.1. *A map $f : E \to X$ has bounded p-variation if and only if it factors as $g \circ \phi$ where $\phi : E \to \mathbf{R}$ is its total p-variation and $g : \phi(E) \to X$ is a Hölderian map of exponent $\gamma = 1/p$ and $H(g) \leq 1$. Moreover, if X is a Banach space, the map $g : \phi(E) \to X$ can be extended to a Hölderian map $\bar{g} : \mathbf{R} \to X$ of the same exponent $\gamma = 1/p$ and Hölderian constant $H(\bar{g}) \leq 3^{1-\gamma} H(g)$.*

In proving Helly's Selection Principle for mappings of bounded p-variation, it suffices to prove convergence of the factors of a function of bounded p-variation. Note that a family of Hölderian functions with uniformly bounded Hölderian constants is equicontinuous. For convergence of the non-decreasing factors, the following lemma is needed.

Lemma 2.2. *A uniformly bounded sequence of nondecreasing real-valued functions has a pointwise convergent subsequence.*

Proof. The proof is identical to Lemma 2 in [**10**, pp. 221–222] starting with any countable dense subset of E.

Before we begin the proof of Theorem 2.4, we need one more lemma.

Lemma 2.3. *Let $\{\phi_n(t)\}_{n=1}^{\infty}$ be a sequence of real-valued functions such that $\phi_n(t) \to \phi(t)$ pointwise on $E \subset \mathbf{R}$. Let $\{g_n(t)\}_{n=1}^{\infty}$ be a sequence of Hölderian functions of exponent $0 < \gamma \leq 1$ from the reals into a metric space X such that $H(g_n) \leq C < \infty$ for all n. Then $\{g_n \circ \phi_n\}_{n=1}^{\infty}$ converges pointwise on E if and only if $\{g_n\}_{n=1}^{\infty}$ converges pointwise on $\phi(E)$.*

Proof. ($\Rightarrow$). Suppose $\{g_n \circ \phi_n\}_{n=1}^{\infty}$ converges pointwise on E. Let $t \in E$, and let $y = \lim_{n\to\infty} g_n(\phi_n(t))$. Then

$$\begin{aligned} d(g_n(\phi(t)), y) &\leq d(g_n(\phi(t)), g_n(\phi_n(t))) + d(g_n(\phi_n(t)), y) \\ &\leq C|\phi_n(t) - \phi(t)|^{\gamma} + d(g_n(\phi_n(t)), y). \end{aligned}$$

Since the terms in the last sum tend to zero as $n \to \infty$, $\{g_n\}_{n=1}^{\infty}$ converges pointwise on $\phi(E)$.

($\Leftarrow$). Suppose $\{g_n\}_{n=1}^{\infty}$ converges pointwise on $\phi(E)$. Let $t \in E$, and let $y = \lim_{n\to\infty} g_n(\phi(t))$. Then

$$\begin{aligned} d(g_n(\phi_n(t)), y) &\leq d(g_n(\phi_n(t)), g_n(\phi(t))) + d(g_n(\phi(t)), y) \\ &\leq C|\phi_n(t) - \phi(t)|^{\gamma} + d(g_n(\phi(t)), y). \end{aligned}$$

Since the terms in the last sum tend to zero as $n \to \infty$, $\{g_n \circ \phi_n\}_{n=1}^{\infty}$ converges pointwise on E.

Theorem 2.4. *Let $\mathcal{F}$ be a sequence of functions of uniform bounded p-variation from $E \subset \mathbf{R}$ to a metric space X, that is, $\sup_{f\in\mathcal{F}} V_p(f,E) < \infty$, such that $\mathcal{F}$ is pointwise precompact, i.e., the closure of $\{f(t) : f \in \mathcal{F}\}$ is compact for every $t \in E$. Then there exists a subsequence $\{f_n\} \subset \mathcal{F}$, pointwise convergent on E to a function $f : E \to X$, hence of bounded p-variation with $V_p(f,E) \leq sup_{f^*\in\mathcal{F}} V_p(f^*,E)$.*

Proof. Without loss of generality, we can assume X is a Banach space since every metric space can be embedded isometrically in a Banach space. Represent each $f \in \mathcal{F}$ as a composite $f = g_f \circ \phi_f$ where ϕ is the p-variation function of f and $g : \phi(E) \to X$ is a Hölderian map of exponent $\gamma = 1/p$ and $H(g) \leq 1$. Note that $\{\phi_f : f \in \mathcal{F}\}$ is a uniformly bounded sequence of nondecreasing real-valued functions since $\mathcal{F}$ has uniform bounded p-variation. By Lemma 2.2, $\{\phi_f : f \in \mathcal{F}\}$ has a subsequence $\{\phi_n\}$ which converges pointwise to a nondecreasing function $\phi : E \to \mathbf{R}$. Extend each g_n to Hölderian map $\overline{g_n} : \mathbf{R} \to X$ such that $H(\overline{g}) \leq 3^{1-\gamma} H(g)$. Since $\{f_n = g_n \circ \phi_n\}_{n=1}^{\infty}$ is pointwise precompact on E, $\{\overline{g_n}\}$ is pointwise precompact on $\phi(E)$ by Lemma 2.3. By the Arzela-Ascoli theorem, see [**6**], there exists a subsequence g_{n_k} which converges on $\phi(E)$, and again by Lemma 2.3, $f_{n_k} = g_{n_k} \circ \phi_{n_k}$ converges pointwise on E. The inequality $V_p(f,E) \leq \sup_{f^*\in\mathcal{F}} V_p(f^*,E)$ follows from (P7) in [**2**].

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