ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 35, Number 2, 2005

PHASE FUNCTIONS AND CENTRAL DISPERSIONS IN THE THEORY OF LINEAR FUNCTIONAL EQUATIONS

JITKA LAITOCHOVÁ

ABSTRACT. The terms introduced in this paper, like phase function, conjugate numbers induced by a phase function, fundamental numbers and fundamental orbits, fundamental central dispersion of the phase function and central dispersions of higher orders, were studied by Borůvka and Neuman in connection with properties of solutions of linear differential equations.

A new direction is taken in this paper in order to remove the explicit dependence of the ideas upon differential equations. The theory presented here begins by defining anew the terms aforementioned, based only on properties of continuous functions, rather than by means of solutions to differential equations. The central idea for the generalized definitions is in a cyclic group of continuous functions, which effectively replaces the differential equation, giving a new direction to the original ideas of Borůvka and Neuman. The direction is similar to, but different than, the *unrestricted n-parameter family* theory introduced by Hartman, which generalizes solution properties of nth order linear differential equations to an abstract setting devoid of differential equations.

Some applications are given for using phase function ideas to solve certain linear functional equations of higher order and special linear difference equations with constant coefficients. These examples do not have an underlying differential equation and therefore the Borůvka-Neuman theory does not apply.

1. Introduction. The theories of phase functions and central dispersions have been treated by Borůvka [**1**] for solution spaces of second-order homogeneous linear differential equations in the Jacobi form. In [**7**], Neuman considered the same theories for nth order linear differential equations. In [**3**], Hartman considered conjugate point theory for nth order linear differential equations and generalized

---

Key words and phrases. Determining group, phase function, conjugate number, central dispersion, functional equation.
AMS *Mathematics Subject Classification.* 34A30, 39B20.
Received by the editors on June 25, 2002.

to function spaces called *unrestricted n-parameter families*, some of the interesting properties found for solutions of differential equations. Additional solution properties for linear nth order differential equations appears in papers of Sherman [**8, 9**].

A new direction is taken in this paper, different than that of Hartman, Borůvka and Neuman, in order to generalize the notion of phase function and central dispersion to spaces of continuous functions. The setting, like Hartman's unrestricted n-parameter families, will be independent of differential equations. The theory applies directly to oscillation problems arising in purely functional equations or in difference equations, both settings lacking an underlying differential equation. But perhaps the central reason for the development of the theory is to strip off the purely differential equation ideas and leave exposed the algebraic structure.

The central, unifying idea is in an infinite cyclic group of functions, called the *determining group*, which makes this work distinct from the earlier studies in [**5**] and [**6**]. This cyclic group will replace the underlying differential equation found in earlier studies.

The *determining group* is defined in Section 1, the *phase function* in Section 2, *conjugate numbers* in Section 3, *fundamental numbers* and *fundamental orbits* in Section 4, *fundamental central dispersions* of the phase functions in Section 5 and *higher order central dispersions* in Section 6. An application of the theory from the earlier sections appears in Section 7 for the explicit expression of solutions of higher order linear functional and difference equations with constant coefficients.

Notation. The following notation is used. $\mathbf{N}$ is the set of natural numbers $1, 2, \dots,$ $\mathbf{Z}$ the set of integers $0, \pm 1, \pm 2, \dots,$ $\mathbf{R}$ the set of real numbers, $\mathcal{J}$ the ordered set of real numbers $(-\infty, \infty)$, $C_0(\mathcal{J})$ the set of continuous real functions defined on $\mathcal{J}$, $\alpha\phi$ the composite function $\alpha\phi(t) = \alpha[\phi(t)]$, $\phi(s+), \phi(s-)$ defined by $\lim_{t\to s+}\phi(t)$ and $\lim_{t\to s-}\phi(t)$, $\alpha(+\infty), \alpha(-\infty)$ defined by $\lim_{t\to+\infty}\alpha(t)$ and $\lim_{t\to-\infty}\alpha(t)$, $\{\phi_\mu(t)\}_{\mu=-\infty}^{+\infty}$, a sequence of functions in $C_0(\mathcal{J})$. Functions referenced here are assumed to be continuous on the interval $\mathcal{J}$, that is, they belong to $C_0(\mathcal{J})$.

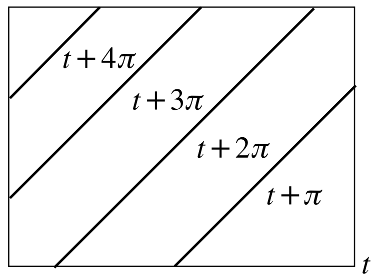

FIGURE 1. Determining group for $\phi(t) = t + \pi$.

1. Determining group. The infinite cyclic group to be called the *determining group* can be illustrated by the example $\phi(t) = t + \pi$. The determining group for $t + \pi$ turns out to be the set of all π-translates of ϕ, that is, the graphs of $t + n\pi$, $n = 0, \pm 1, \pm 2, \dots$, see Figure 1.1.

The *determining group* in the subsequent theory is used to define certain *crossings* of group element graphs with another monotonic function α, which is for the purpose of intuition thought of as the polar coordinate angle variable in an oscillation problem. In a classical differential equations setting, the crossings implicitly determine zeros of solutions, through some Prüfer transformation formula like $\alpha = \arctan(v/u)$. The project here is to extend the crossing ideas to settings which are outside the scope of application of differential equation theory, such as purely functional equations, or difference equations.

Definition 1.1. The sequence $\{f_\mu(t)\}_{\mu=-\infty}^{+\infty}$ is called *increasing* on $\mathcal{J}$ if, for any $t \in \mathcal{J}$ and any $\mu \in \mathbf{Z}$,

$$f_\mu(t) < f_{\mu+1}(t).$$

A sequence is *decreasing* if its negative is increasing.

Definition 1.2. The sequence $\{f_\mu(t)\}_{\mu=-\infty}^{+\infty}$ is called *normal* if it is increasing or decreasing on $\mathcal{J}$ and $\lim_{|\mu|\to\infty} |f_\mu(t)| = \infty$ for each $\bar{t} \in \mathcal{J}$.

Iterations. Let $f \in C_0(\mathcal{J})$ be an increasing function mapping the interval $\mathcal{J}$ onto itself. Then f has an inverse mapping $f^{-1} \in C_0(\mathcal{J})$

which maps $\mathcal{J}$ onto itself. Define $f_1 = f(t)$, $f_0 = t$ and for $n = 1, 2, \dots,$ let $f_{n+1} = f_1(f_n(t))$, $f_{-n} = f_n^{-1}$ where f_n^{-1} denotes the inverse function to the function f_n. The iterations $f_n(t) \in C_0(\mathcal{J})$ map the interval $\mathcal{J}$ onto itself.

Definition 1.3. A function ϕ satisfying the following properties is called a *p-function*:

$$\text{(p)} \qquad \begin{cases} 1.\ \phi \in C_0(\mathcal{J}), \\ 2.\ \phi \text{ increases on } \mathcal{J} \text{ from } -\infty \text{ to } +\infty, \\ 3.\ \phi(t) > t \text{ for any } t \in \mathcal{J}. \end{cases}$$

The *determining group* $\mathcal{G}(\phi)$ is the infinite cyclic group under composition of functions defined by the generator $\phi_1(t) = \phi(t)$ with identity element $\phi_0(t) = t$, see Theorem 1.1.

Theorem 1.1. *Let ϕ be a p-function and define the iterations of $\phi_1(t) = \phi(t)$ by $\phi_\mu = \phi_\mu(t)$, $\mu \in \mathbf{Z}$. Then the functions $\phi_\mu(t)$, $\mu \in \mathbf{Z}$, form a cyclic group $\mathcal{G}(\phi)$ under composition of functions. The generating element is $\phi_1 = \phi(t)$ and the identity element is $\phi_0(t) = t$. The sequence $\{\phi_\mu(t)\}_{\mu=-\infty}^{+\infty}$ has infinitely many elements and it is increasing in $\mathcal{J}$.*

Proof. It is routinely verified that, for each $\mu \in \mathbf{Z}$, the μth iteration ϕ_μ of the function $\phi_1(t) = \phi(t)$ satisfies (p). Therefore, the set of functions $\phi_\mu(t)$, $\mu \in \mathbf{Z}$, under composition of functions is a cyclic group with generating element $\phi_1 = \phi(t)$ and identity element $\phi(t) = t$. Property 3 gives

$$\phi_1(t) > \phi_0(t) = t \tag{1.1}$$

and replacing t by $\phi_1(t)$ repeatedly gives $\phi_{n+1}(t) > \phi_n(t) > \cdots > \phi_0(t) = t$, $n \in \mathbf{N}$. From (1.1) we have $\phi_{-1}(t) < \phi_0(t) = t$ and therefore $\phi_{-(n+1)}(t) < \phi_{-n}(t) < \cdots < \phi_0(t) = t$, $n \in \mathbf{N}$. Thus $\{\phi_\mu(t)\}_{\mu=-\infty}^{+\infty}$ is an increasing sequence in $\mathcal{J}$. The increasing property implies that the cyclic group is infinite. □

2. Phase function. A *phase function* in Borůvka [**1**] is given by $\alpha(t) = \arctan(v(t)/u(t))$ where the functions u, v form a basis for the solution space of the second order linear differential equation. In this classical differential equations setting, oscillation on $[a, b]$ is detected by the angle variable α having an image interval $\alpha([a, b])$ of length at least π.

In the theory to be developed here, differential equations will be stripped from the theory and replaced by some essential monotonicity and continuity properties of α, see below. Nevertheless, for the purpose of intuition, it is useful to think of an illustration like $\alpha(t) = \arctan(v(t)/u(t))$ and $\phi(t) = t + \pi$.

Definition 2.1. Let $\mathcal{G}(\phi)$ be a determining group. A *phase function* in $C_0(\mathcal{J})$ for $\mathcal{G}(\phi)$, or briefly a phase function in $C_0(\mathcal{J})$, is a function $\alpha(t)$ defined on $\mathcal{J}$ such that $\alpha \in C_0(\mathcal{J})$ and α is strictly monotonic on $\mathcal{J}$.

The continuity and monotonicity of phase functions imply that they have limits at all points of $\mathcal{J}$ and limits in the extended sense at the endpoints.

Definition 2.2. The extended numbers $c_\alpha = \alpha(-\infty)$ and $d_\alpha = \alpha(\infty)$ are called the *left* and *right boundary values* of the phase function α in $\mathcal{J}$. Define ω_α to be the left endpoint of the image interval $\alpha(\mathcal{J})$; then $\omega_\alpha = c_\alpha$ for α increasing and $\omega_\alpha = d_\alpha$ for α decreasing.

The extended number $O(\alpha) = |c_\alpha - d_\alpha|$ is called the *oscillation value* of the phase function α on the interval $\mathcal{J}$.

Theorem 2.1. *Let ϕ be a p-function and $\mathcal{G}(\phi)$ its determining infinite cyclic group. The set M of all phase functions on $\mathcal{J}$ can be decomposed into the following classes, organized by the values of the oscillation function $O(\alpha)$.*

Class I. *Finite oscillation value $O(\alpha)$.* For some index $m \in \mathbf{N}$,

I(a) $\phi_{m-1}(\omega_\alpha) - \phi_0(\omega_\alpha) < O(\alpha) < \phi_m(\omega_\alpha) - \phi_0(\omega_\alpha)$

I(b) $O(\alpha) = \phi_m(\omega_\alpha) - \phi_0(\omega_\alpha)$.

Class II. *Infinite oscillation value* $O(\alpha)$. Here, $\pm$ means $+$ when α increases and $-$ when α decreases on $\mathcal{J}$.

II(a) c_α is finite and $d_\alpha = \pm\infty$

II(b) d_α is finite and $-c_\alpha = \pm\infty$

II(c) $-c_\alpha = \pm\infty$ and $d_\alpha = \pm\infty$.

Proof. The routine details are omitted. □

We will use the following notation for the classes of phase functions for $\mathcal{G}(\phi)$.

Definition 2.3. Let M denote the set of all phase functions. For each $m \in \mathbf{N}$, denote by M_{mg} and M_{ms} the phase functions satisfying I(a) and I(b), respectively. Denote by M_{rg}, M_{lg} and M_g the phase functions satisfying II(a), II(b) and II(c), respectively. In more detail,

$$\begin{aligned} M &= \text{All phase functions } \alpha \in C_0(\mathcal{J}) \\ M_{mg} &= \{\alpha \in M : \phi_{m-1}(\omega_\alpha) < \phi_0(\omega_\alpha) + O(\alpha) < \phi_m(\omega_\alpha)\} \\ M_{ms} &= \{\alpha \in M : O(\alpha) = \phi_m(\omega_\alpha) - \phi_0(\omega_\alpha)\} \\ M_{rg} &= \{\alpha \in M : |c_\alpha| \neq \infty,\ d_\alpha = \pm\infty\} \\ M_{lg} &= \{\alpha \in M : |d_\alpha| \neq \infty,\ -c_\alpha = \pm\infty\} \\ M_g &= \{\alpha \in M : -c_\alpha = d_\alpha = \pm\infty\} \end{aligned}$$

where $\pm$ means $+$ when α increases and $-$ when α decreases. The subscripts other than m are acronyms: $l =$ left, $r =$ right, $g =$ general, $s =$ special, as in [**1**].

3. Conjugate numbers induced by a phase function. The notion of conjugate numbers in the interval $\mathcal{J}$ for a phase function, studied by Borůvka and Neuman for differential equations, will be extended to phase functions on the space of continuous functions $C_0(\mathcal{J})$.

Definition 3.1. Let $\mathcal{G}(\phi)$ be a determining group. Let $\alpha \in M$ be a phase function. Let $t_0 \in \mathcal{J}$ be an arbitrary number. The μth *conjugate* *number* t_μ to the number t_0 for phase function α, provided it exists, is defined by the equation

$$(3.1)\qquad \begin{aligned} \alpha(t_\mu) &= \phi_\mu \alpha(t_0), \quad \alpha \text{ increasing}, \\ \alpha(t_\mu) &= \phi_{-\mu} \alpha(t_0), \quad \alpha \text{ decreasing}. \end{aligned}$$

If t_0 has at least one conjugate number different from t_0 in the interval $\mathcal{J}$, then we say that the phase function *induces conjugate numbers* in $\mathcal{J}$. When counting conjugate numbers to the number t_0, we also count t_0 itself.

Theorem 3.1. *If $\mu \neq 0$, $\mu \in \mathbf{Z}$, then the μth conjugate number to the number t_0 lies on the right of t_0 for $\mu > 0$ and to the left for $\mu < 0$.*

Proof. Let α increase on $\mathcal{J}$. Then (3.1) yields $t_\mu = \alpha^{-1}\phi_\mu\alpha(t_0)$ and therefore $t_\mu < \alpha^{-1}\phi_{\mu+1}\alpha(t_0) = t_{\mu+1}$. Hence, for $\mu = 0, 1, 2, \ldots$, the numbers $t_1, t_2, \ldots$ lie on the right of t_0 and for $\mu = -1, -2, \ldots$, they lie on the left of t_0. The details when α decreases on $\mathcal{J}$ are similar. $\square$

Definition 3.2. Let t_μ, $\mu \in \mathbf{Z}$, be the μth conjugate number to a number $t_0 \in \mathcal{J}$ for the phase function α. If $\mu > 0$, then t_μ is called the μth *right conjugate number* to the number t_0 in the interval $\mathcal{J}$ for the phase function α. Similarly, if $\mu < 0$, then t_μ is called the $|\mu|$th *left conjugate number.*

Theorem 3.2. *Let $\alpha \in M$ be a phase function, and let $\mathcal{G}(\phi)$ be a determining group. If S is a set of conjugate points to the point t_0, then no subset of S can have a finite limit point. In particular, each conjugate point in S is isolated and the only possible cluster points of S are at $\pm\infty$.*

Proof. Assume S has a finite limit point A and that α increases; the decreasing case is similar. Then in S there is a sequence of points t_n and increasing indices μ_n such that $\alpha(t_n) = \phi_{\mu_n}\alpha(t_0)$, $n \geq 1$, and $\lim_{n\to\infty} t_n = A$. Continuity of α gives $\lim_{n\to\infty}\alpha(t_n) = \alpha(A)$, which implies that the sequence $\{\phi_{\mu_n}\alpha(t_0)\}_{n=1}^{\infty}$ is bounded above. The properties of ϕ imply that the entire sequence $\{\phi_n\alpha(t_0)\}_{n=1}^{\infty}$ is bounded above and increasing, hence convergent. Limiting gives

$$\alpha(A) = \lim_{n\to\infty} \phi\phi_{\mu_n - 1}\alpha(t_0) = \phi(\alpha(A)).$$

This equality contradicts the property $\phi(t) > t$, hence S has no finite cluster point. $\square$

Theorem 3.3. *Let $\alpha \in M_g$ be a phase function, and let $\mathcal{G}(\phi)$ be a determining group. Then, for each $t_0 \in \mathcal{J}$, α has in $\mathcal{J}$ infinitely many left conjugate numbers and infinitely many right conjugate numbers, which cluster at $\pm\infty$, respectively.*

Proof. Assume α increases; the decreasing case is similar. Since $\alpha(\mathcal{J}) = (-\infty, \infty)$, then $\phi_\mu(\alpha(t_0)) \in \alpha(\mathcal{J})$, hence the μth conjugate exists. Apply Theorem 3.2. $\square$

4. Fundamental numbers and orbits. It is assumed in this section that the phase function α has finite oscillation $O(\alpha)$: α belongs to *Class* I in Theorem 2.1. Defined here are *fundamental numbers* and *fundamental orbits of numbers* induced by the phase function α.

Theorem 4.1. *Let $\mathcal{G}(\phi)$ be a determining group. Let α belong to M_{mg} or M_{ms} for some $m \geq 2$. Let c_α and d_α be the left and right boundary values of the phase function α. Then there exist numbers α_μ and $b_{-\mu}$, $\mu = 1, 2, \dots, m-1$, in $\mathcal{J}$ such that*

$$\begin{aligned} &\alpha(a_\mu) = \phi_\mu(c_\alpha), && \alpha \text{ increasing},\\ &\alpha(a_\mu) = \phi_{-\mu}(c_\alpha), && \alpha \text{ decreasing}, \end{aligned} \tag{4.1}$$

$$\begin{aligned} &\alpha(b_{-\mu}) = \phi_{-\mu}(d_\alpha), && \alpha \text{ increasing},\\ &\alpha(b_{-\mu}) = \phi_\mu(d_\alpha), && \alpha \text{ decreasing}. \end{aligned} \tag{4.2}$$

Proof. The proof is completed by applying the intermediate value theorem for continuous functions after showing that the values on the right in equations (4.1) and (4.2) belong to image integral $\mathcal{J} = \alpha(\mathcal{J})$, the set of all values of the phase function α on the interval $\mathcal{J}$.

When α increases on $\mathcal{J}$, then

$$c_\alpha = \phi_0(c_\alpha) < \phi_1(c_\alpha) < \cdots < \phi_{m-1}(c_\alpha) < d_\alpha \leq \phi_m(c_\alpha),$$
$$\phi_{-m}(d_\alpha) \leq c_\alpha < \phi_{-m+1}(d_\alpha) < \cdots < \phi_{-1}(d_\alpha) < \phi_0(d_\alpha) = d_\alpha.$$

When α decreases on $\mathcal{J}$, then

$$c_\alpha = \phi_0(c_\alpha) > \phi_{-1}(c_\alpha) > \cdots > \phi_{-m+1}(c_\alpha) > d_\alpha \geq \phi_{-m}(c_\alpha),$$
$$\phi_m(d_\alpha) \geq c_\alpha > \phi_{m-1}(d_\alpha) > \cdots > \phi_1(d_\alpha) > \phi_0(d_\alpha) = d_\alpha.$$

These relations show that the right sides of equations (4.1) and (4.2) are points in the image $\alpha(\mathcal{J})$. Continuity and monotonicity of the phase function α imply that the interval $\mathcal{J}$ contains exactly $m-1$ points α_μ, $\mu = 1, 2, \ldots, m-1$, which satisfy (4.1). Similarly, there are exactly $m-1$ points $b_{-\mu}$, $\mu = 1, 2, \ldots, m-1$, which satisfy (4.2).

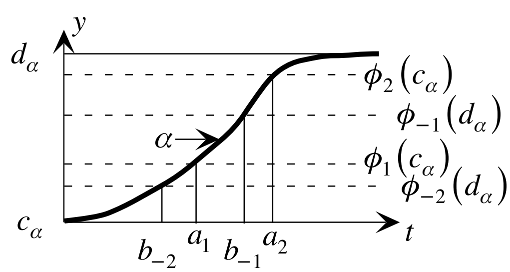

FIGURE 4.2. The ordering of the points $a_\mu, b_{-\mu}$ for $m = 3$ and α increasing.

There is considerably more detailed information in the proof of Theorem 4.1. Some of this detail appears in Figure 4.2, which is just for the special case $m = 3$.

In particular, Figure 4.2. shows the finiteness of the endpoints c_α, d_α of the image interval $\alpha(\mathcal{J})$. The numbers a_μ and $b_{-\mu}$ are found from the *crossings* of the curve $y = \alpha(t)$ with the horizontal lines $y =$ constant, where the constant is one of $\phi_1(c_\alpha), \phi_2(c_\alpha), \phi_{-1}(d_\alpha)$ or $\phi_{-2}(d_\alpha)$. □

Definition 4.1. Let $\mathcal{G}(\phi)$ be a determining group. If α belongs to M_{mg} or M_{ms} for some $m \geq 2$, then the finite sequence of numbers $\{\alpha_\mu\}_{\mu=1}^{m-1}$ defined by (4.1) is called the *left fundamental orbit* of numbers induced by the phase function α in the interval $\mathcal{J}$. Similarly, $\{b_{-\mu}\}_{\mu=1}^{m-1}$ defined by (4.2) is called the *right fundamental orbit* of numbers induced by the phase function α in the interval $\mathcal{J}$.

Definition 4.2. The number $r = a_1$ defined by (4.1) is called the *left fundamental number* induced by the phase function α in the interval $\mathcal{J}$. The number $s = b_{-1}$ defined by (4.2) is called the *right fundamental number* induced by the phase function α in the interval $\mathcal{J}$. See [**2**].

As shown in Figure 4.2, a_1 is the infimum of all points in $\mathcal{J}$ which have a right conjugate number and b_{-1} is the supremum of all points in $\mathcal{J}$ which have a left conjugate number.

Definition 4.3. Let α belong to M_{mg} or M_{ms} for some $m \geq 2$. Denote by $\{a_\mu\}_{\mu=1}^{m-1}$ and $\{b_{-\mu}\}_{\mu=1}^{m-1}$ the left and right fundamental orbits of numbers induced by the phase function α. The orbit elements a_μ and $b_{-\mu}$ divide the interval $\mathcal{J}$ into subintervals $\mathcal{J}_\mu$ and $\mathcal{I}_\nu$, where

$$\mathcal{J}_\mu = (a_\mu, b_{-(m-1-\mu)}), \quad \mu = 0, 1, 2, \dots, m-1, \tag{4.3}$$

$$\mathcal{I}_\nu = (b_{-(m-\nu)}, a_\nu), \quad \nu = 1, 2, \dots, m-1. \tag{4.4}$$

The ordered sequence

$$\mathcal{J}_0, \mathcal{I}_1, \mathcal{J}_1, \dots, \mathcal{I}_{m-1}, \mathcal{J}_{m-1}$$

is called the *fundamental decomposition* of the interval $\mathcal{J}$.

Properties of conjugate numbers. Let α belong to M_{mg} or M_{ms} for some $m \geq 2$. Denote by $\{a_\mu\}_{\mu=1}^{m-1}$ and $\{b_{-\mu}\}_{\mu=1}^{m-1}$ the left and right fundamental orbits of numbers induced by the phase function α. The following assertions are routinely verified:

1. The left orbit and the right orbit each have exactly $m-1$ elements.
2. The orbit elements $a_1, \dots, a_{m-1}$ are conjugate numbers. The same is true for $b_{-1}, \dots, b_{-(m-1)}$.

3. Orbit elements a_μ and $b_{-\nu}$ satisfy the inequalities

$$a_1 < a_2 < \cdots < a_{m-1}, \tag{4.5}$$

$$b_{-1} > b_{-2} > \cdots > b_{-(m-1)}. \tag{4.6}$$

4. Let $a_0 = -\infty$, $b_0 = +\infty$, then orbit elements are ordered as follows:

$$a_0 < b_{-(m-1)} \leq a_1 < b_{-(m-2)} \leq \cdots < b_{-1} \leq a_{m-1} < b_0.$$

   If $\alpha \in M_{mg}$, then in (4.7) all inequalities are strict.

   If $\alpha \in M_{ms}$, then in (4.7) all inequalities are equalities and the fundamental sequences of numbers are identical.

5. The left and right fundamental numbers $r = a_1$ and $s = b_{-1}$ are conjugate for $\alpha \in M_{ms}$ and not conjugate for $\alpha \in M_{mg}$.

6. If $\alpha \in M_{mg}$, then for $\mu = 1, 2, \ldots, m-1$,

$$0 < |\alpha(b_{-(m-\mu)}) - \alpha(a_\mu)| < |\phi_{\mu-1}(c_\alpha) - \phi_\mu(c_\alpha)|, \quad \alpha \text{ increasing,}$$
$$0 < |\alpha(b_{-(m-\mu)}) - \alpha(a_\mu)| < |\phi_{-(\mu-1)}(c_\alpha) - \phi_{-\mu}(c_\alpha)|, \quad \alpha \text{ decreasing.}$$

   If $\alpha \in M_{ms}$, then for $\mu = 1, 2, \ldots, m-1$,

$$|\alpha(b_{-(m-\mu)}) - \alpha(a_\mu)| = 0.$$

7. If $\alpha \in M_{ms}$, then the fundamental intervals $\mathcal{I}_\nu$, $\nu = 1, 2, \ldots, m-1$ are empty sets and for $a_0 = -\infty$, $b_0 = \infty$ the fundamental intervals $\mathcal{J}_\mu$ are given by

$$\mathcal{J}_\mu = (a_\mu, a_{\mu+1}), \quad 0 \leq \mu \leq m-2, \tag{4.8}$$

$$\mathcal{J}_{m-1} = (a_{m-1}, b_0). \tag{4.9}$$

8. Let $\alpha \in M_{mg}$, $m \geq 2$. Let $t_0 \in \mathcal{J}_\mu$ for one of the indices $\mu = 0, 1, \ldots, m-1$. Each other interval $\mathcal{J}_\mu$ contains just one conjugate number to the number t_0, making together with t_0 precisely m conjugate numbers in $\mathcal{J}$.

Let $t_0 \in \mathcal{I}_\nu$ for one of the indices $\nu = 1, 2, \dots, m-1$. Each other interval $\mathcal{I}_\nu$ contains just one conjugate number to the number t_0, making together with t_0 exactly $m-1$ conjugate numbers in $\mathcal{J}$.

9. Let $\alpha \in M_{ms}$, $m \geq 2$. Let $t_0 \in \mathcal{J}_\mu$ for one of the indices $\mu = 0, 1, \dots, m-1$. Each other interval $\mathcal{J}_\mu$ contains just one conjugate number to the number t_0, making together with the number t_0 exactly m conjugate numbers in $\mathcal{J}$.

Theorem 4.2. *If $\alpha \in M_{mg}$ or $\alpha \in M_{ms}$ and $O(\alpha) \leq \phi_1(\omega_\alpha) - \omega_\alpha$, then in the interval $\mathcal{J}$ there are no conjugate numbers induced by the phase function α.*

Proof. Apply the definition of conjugate numbers. □

Theorem 4.3. *Let $\alpha \in M_{mg}$, $m \geq 2$. Then to each $t_0 \in \mathcal{J}$ there are together with t_0 either exactly $m-1$ conjugate numbers or else t_0 belongs to an open interval in which each point has exactly m conjugate numbers.*

Proof. The assertion follows directly from 8 and 2. If we choose $t_0 \in \mathcal{J}_\mu$ for one of the indices $\mu = 0, 1, \dots, m-1$, then α induces exactly m conjugate numbers in $\mathcal{J}$. If we choose $t_0 \in \mathcal{I}_\nu$ for one of the indices $\nu = 1, 2, \dots, m-1$, or t_0 is an endpoint of one of these intervals, then α induces exactly $m-1$ conjugate numbers in $\mathcal{J}$. □

Theorem 4.4. *Let $\alpha \in M_{ms}$, $m \geq 2$, $t_0 \in \mathcal{J}$. If t_0 is not one of the orbit numbers $a_1, \dots, a_{m-1}$, then in $\mathcal{J}$ there are together with t_0 exactly m conjugate numbers to the number t_0. If t_0 equals one of $a_1, \dots, a_{m-1}$, then there are exactly $m-1$ conjugate numbers to the number t_0.*

Proof. The assertion follows directly from 9 and 2. If we choose $t_0 \in \mathcal{J}_\mu$ for one of the indices $\mu = 0, 1, \dots, m-1$, then α induces m conjugate numbers in $\mathcal{J}$. If t_0 is one of $a_1, \dots, a_{m-1}$ (they are identical to $b_{-1}, \dots, b_{-(m-1)}$), then α induces for t_0 exactly $m-1$ conjugate numbers in $\mathcal{J}$. □

5. Fundamental central dispersion. The objective is to define a function called the *fundamental central dispersion* of the phase function. It could also be called the *first conjugate function*, according to its history in differential equation literature.

Definition 5.1. Let $\mathcal{G}(\phi)$ be a determining group. Let $\alpha \in M$ be a phase function. Let $\mathcal{J}^*$ be the set of all numbers $t_0 \in \mathcal{J}$ which have a conjugate number on the right. The *fundamental central dispersion* Φ is the function on $\mathcal{J}^*$ whose value $\Phi(t_0)$ is the *first* conjugate number to the right of t_0.

Now we derive explicit formulas for the fundamental central dispersion of the phase function in the classes M_g, M_{mg}, M_{ms}. For these classes, the domain of Φ will be shown to be a nonvoid open subinterval of $\mathcal{J}$.

Theorem 5.1. *Let $\mathcal{G}(\phi)$ be a determining group and let α be a phase function in M_g. The fundamental central dispersion Φ is defined for all $t \in \mathcal{J}$ and it is given by*

$$\begin{aligned} &\Phi(t) = \alpha^{-1}\phi_1\alpha(t), \quad t \in \mathcal{J}, \quad \alpha \text{ increasing},\\ &\Phi(t) = \alpha^{-1}\phi_{-1}\alpha(t), \quad t \in \mathcal{J}, \quad \alpha \text{ decreasing}. \end{aligned} \tag{5.1}$$

Proof. For $\alpha \in M_g$, there are infinitely many right conjugate numbers for each point $t \in \mathcal{J}$; therefore, the domain of ϕ is $\mathcal{J}$ itself. Apply equation (3.1) for $\mu = 1$ to get, for each $t \in \mathcal{J}$,

$$\begin{aligned} &\alpha\Phi(t) = \phi_1\alpha(t), \quad \alpha \text{ increasing},\\ &\alpha\Phi(t) = \phi_{-1}\alpha(t), \quad \alpha \text{ decreasing}. \end{aligned} \tag{5.2}$$

Since $\alpha \in M_g$, it is continuous, strictly monotonic and maps $\mathcal{J}$ onto $\mathcal{J}$. Thus there is an inverse function α^{-1} to the function α which maps the interval $\mathcal{J}$ onto $\mathcal{J}$. Applying the inverse function to (5.2) gives (5.1). □

Theorem 5.2. *Let $\mathcal{G}(\phi)$ be a determining group, and let α be a phase function in M_g. The fundamental central dispersion Φ exists on all of $\mathcal{J}$ and has the following properties*:

1. $\Phi \in C_0(\mathcal{J})$,
2. Φ *increases in* $\mathcal{J}$,
3. $\Phi(-\infty) = -\infty$, $\Phi(+\infty) = +\infty$,
4. $\Phi(t) > t$ *for* $t \in \mathcal{J}$.

Proof. As argued in the previous proof, α has an inverse function α^{-1}.

Property 1 is valid because the continuous functions α, α^{-1} and $\phi_1\alpha$ (or $\phi_{-1}\alpha$) map $\mathcal{J}$ onto $\mathcal{J}$. Thus the composite function Φ given by (5.1) is continuous on $\mathcal{J}$.

Property 2 is valid because α, α^{-1} and $\phi_{\pm 1}\alpha$ simultaneously increase or decrease. Thus the composite function Φ given by (5.1) increases on $\mathcal{J}$.

Property 3 will be verified. Assume α increases, then

$$\lim_{t\to-\infty} \alpha(t) = -\infty, \quad \lim_{t\to-\infty} \phi_1\alpha(t) = -\infty, \quad \lim_{t\to-\infty} \alpha^{-1}(t) = -\infty,$$
$$\lim_{t\to+\infty} \alpha(t) = +\infty, \quad \lim_{t\to+\infty} \phi_1\alpha(t) = +\infty, \quad \lim_{t\to+\infty} \alpha^{-1}(t) = +\infty.$$

Thus the composite function Φ given by (5.1) satisfies

$$\lim_{t\to-\infty} \Phi(t) = -\infty, \quad \lim_{t\to+\infty} \Phi(t) = +\infty.$$

Assume next that α decreases, then

$$\lim_{t\to-\infty} \alpha(t) = +\infty, \quad \lim_{t\to-\infty} \phi_{-1}\alpha(t) = +\infty, \quad \lim_{t\to-\infty} \alpha^{-1}(t) = +\infty,$$
$$\lim_{t\to+\infty} \alpha(t) = -\infty, \quad \lim_{t\to+\infty} \phi_{-1}\alpha(t) = -\infty, \quad \lim_{t\to+\infty} \alpha^{-1}(t) = -\infty.$$

Thus the composite function Φ given by (5.1) satisfies

$$\lim_{t\to-\infty} \Phi(t) = -\infty, \quad \lim_{t\to\infty} \Phi(t) = +\infty.$$

Property 4 will be verified. Since $\phi_1 = \phi(t) > 1$, then also $\phi_1\alpha(t) > \alpha(t)$. Since $\phi_{-1}(t) < t$, then also $\phi_{-1}\alpha(t) < \alpha(t)$. Let α increase, then α^{-1} increases and (5.1) yields $\Phi(t) = \alpha^{-1}\phi_1\alpha(t) > \alpha^{-1}\alpha(t) = t$. Let

α decrease, then α^{-1} decreases and (5.1) yields $\Phi(t) = \alpha^{-1}\phi_{-1}\alpha(t) > \alpha^{-1}\alpha(t) = t$. □

Theorem 5.3. *Let $\mathcal{G}(\phi)$ be a determining group. Let $\alpha \in M_{mg}$ or $\alpha \in M_{ms}$, $m \geq 2$, and denote by b_{-1} the right fundamental number induced by α on $\mathcal{J}$.*

Then the fundamental central dispersion Φ has domain $-\infty < t < b_{-1}$ and it is given for such t by

$$\Phi(t) = \begin{cases} \alpha^{-1}\phi_1\alpha(t) & \alpha \text{ increasing}, \\ \alpha^{-1}\phi_{-1}\alpha(t) & \alpha \text{ decreasing}. \end{cases} \tag{5.3}$$

Proof. Assume first that $\alpha \in M_{mg}$. Let α be increasing, $m \geq 2$. The case for α decreasing is similar.

The fundamental decomposition of the interval $\mathcal{J}$ is defined in terms of the fundamental orbits of numbers $\{a_\mu\}_{\mu=1}^{m-1}$ and $\{b_{-\mu}\}_{\mu=1}^{m-1}$ given in formulas (4.1) and (4.2) and the completions $a_0 = -\infty$, $b_0 = \infty$:

$$\begin{aligned} \mathcal{J}_\mu &= (a_\mu, b_{-(m-1-\mu)}), & \mu &= 0, 1, \ldots, m-1, \\ \mathcal{I}_\nu &= (b_{-(m-\nu)}, a_\nu), & \nu &= 1, 2, \ldots, m-1. \end{aligned} \tag{5.4}$$

The idea of the proof is to examine Φ on the subintervals in (5.4). The details will be explained for $\mathcal{J}_0$; the other cases are similar. Given $t \in \mathcal{J}_0$, then *define* $x = \phi_1(\alpha(t))$; it will be established that $x \in \alpha(\mathcal{J}_1)$; thus, $\alpha(t^*) = x$ for some $t^* \in \mathcal{J}_1$. In this way, $\Phi(t) = t^*$ is shown to exist, belong to $\mathcal{J}_1$ and satisfy $\alpha(\Phi(t)) = x = \phi_1(\alpha(t))$.

The inequality $\alpha(t) < \alpha(b_{m-1}) = \phi_{-(m-1)}(d_\alpha)$ follows from α increasing on $\mathcal{J}_0$, hence equations (4.1)–(4.2) imply $x = \phi_1(\alpha(t)) < \phi_{-(m-2)}(d_\alpha) = \alpha(b_{(m-2)})$. Similarly, $c_\alpha \leq \alpha(t)$ implies $\alpha(a_1) = \phi_1(c_\alpha) < \phi_1(\alpha(t))$. Hence x belongs to the range interval $\alpha(\mathcal{J}_1)$. By continuity of α, choose $t^* \in \mathcal{J}_1$ such that $\alpha(t^*) = x$. Then t^* satisfies $\alpha(t^*) = \phi_1(\alpha(t))$ and t^* is the first right conjugate number to number t. Hence $\Phi(t) = t^*$ is defined, belongs to $\mathcal{J}_1$, and satisfies $\alpha(\Phi(t)) = \phi_1(\alpha(t))$.

The endpoint $t = b_{-(m-1)}$ of $\mathcal{J}_0$ is treated separately by showing that $t^* = b_{-(m-2)}$ is its right conjugate number, $m \geq 3$ to make sense.

Indeed, equation (4.2) gives $\alpha(t) = \phi_{-(m-1)}(d_\alpha)$, hence $\phi_1(\alpha(t)) = \phi_1(\phi_{-(m-1)}(d_\alpha)) = \alpha(t^*)$, which says that t^* is the right conjugate number of t.

The above reasoning applies for intervals $\mathcal{J}_n$, $n = 0, \dots, m-2$. No points in the last interval $\mathcal{J}_{m-1}$ have right conjugates because $b_{-1} < a_{m-1}$ and b_{-1} is the supremum of all points t_0 in $\mathcal{J}$ which have a right conjugate. Therefore, $\Phi(t)$ is *undefined* for $t \geq b_{-1}$ and *a fortiori* for $t \geq a_{m-1}$.

A similar analysis can be carried out for intervals $\mathcal{I}_1, \dots, \mathcal{I}_{m-2}$, showing that Φ is defined on $\mathcal{I}_n$ with values in $\mathcal{I}_{n+1}$, $1 \leq n \leq m-2$, and in fact Φ maps endpoints to endpoints. There are no right conjugates to t for points $t \in \mathcal{I}_{m-1}$.

Applying such details to each subinterval, the fundamental central dispersion Φ is shown to exist for $-\infty < t < b_{-1}$ and map points and intervals as follows:

$$\begin{gathered} a_1 \to a_2, \dots, a_{m-2} \to a_{m-1}, \\ b_{-(m-1)} \to b_{-(m-2)}, \dots, b_{-2} \to b_{-1} \end{gathered} \tag{5.5}$$

$$\begin{gathered} \mathcal{J}_0 \to \mathcal{J}_1, \mathcal{J}_1 \to \mathcal{J}_2, \dots, \mathcal{J}_{m-2} \to \mathcal{J}_{m-1}, \\ \mathcal{I}_1 \to \mathcal{I}_2, \mathcal{I}_2 \to \mathcal{I}_3, \dots, \mathcal{I}_{m-2} \to \mathcal{I}_{m-1}. \end{gathered} \tag{5.6}$$

Assume now that $\alpha \in M_{ms}$, $m \geq 2$. The preceding proof applies, except that the intervals $\mathcal{I}_\mu$ are empty and therefore they are not considered in the proof. □

Theorem 5.4. *Let $\mathcal{G}(\phi)$ be a determining group and let $\alpha \in M_{mg}$ or $\alpha \in M_{ms}$, $m \geq 2$. Define $r = a_1$ and $s = b_{-1}$ to be the left and right fundamental numbers for α. The fundamental central dispersion Φ given by* (5.3) *has the following properties*:

1. *The domain of Φ is $\mathcal{J}^* = (-\infty, s)$ and $\Phi \in C_0(\mathcal{J}^*)$,*
2. *Φ increases in $(-\infty, s)$,*
3. $\Phi(-\infty) = r$, $\Phi(s-) = +\infty$,
4. *$\Phi(t) > t$ for $t \in (-\infty, s)$.*

Proof. The details will be done for $\alpha \in M_{mg}$; details for $\alpha \in M_{ms}$ are similar.

Property 1 follows from formula (5.3) which yields the definition of the function Φ.

Property 2 follows from the fact that both components of the composite function Φ, given by (5.3), simultaneously increase or decrease on the relevant interval.

Property 3 can be derived from equation (5.3), properties of α and ϕ, plus reference to Figure 4.2.

Property 4 will be verified. If α increases, then $\Phi(t) = \alpha^{-1}\phi_1\alpha(t) > \alpha^{-1}\alpha(t) = t$ holds on $(-\infty, s)$ because α^{-1} increases and $\phi_1(t) > t$. If α decreases, then $\Phi(t) = \alpha^{-1}\phi_{-1}\alpha(t) > \alpha^{-1}\alpha(t) = t$ holds on $(-\infty, s)$ because α^{-1} decreases and $\phi_{-1}(t) < t$. $\square$

6. Central dispersions of higher orders.

Definition 6.1. Let $\mathcal{G}(\phi)$ be a determining group and $\alpha \in M_g$ a phase function. The μth *central dispersion* is the function $\Phi_\mu = \Phi_\mu(t)$, $t \in \mathcal{J}$, $\mu \in \mathbf{Z}$, defined implicitly by the *Abel functional equations*

$$\begin{aligned} &\alpha\Phi_\mu(t) = \phi_\mu\alpha(t), \quad \alpha \text{ increasing},\\ &\alpha\Phi_\mu(t) = \phi_{-\mu}\alpha(t), \quad \alpha \text{ decreasing}. \end{aligned} \tag{6.1}$$

For differential equations, the Abel functional equation for central dispersions is defined in references [**1**] and [**5**]. Functions $\alpha \in M_g$ are invertible; therefore, (6.1) can be rewritten as

$$\begin{aligned} &\Phi_\mu(t) = \alpha^{-1}\phi_\mu\alpha(t), \quad \alpha \text{ increasing},\\ &\Phi_\mu(t) = \alpha^{-1}\phi_{-\mu}\alpha(t), \quad \alpha \text{ decreasing}. \end{aligned} \tag{6.2}$$

Theorem 6.1. *Let $\mathcal{G}(\phi)$ be a determining group and $\alpha \in M_g$. The central dispersion $\Phi_\mu = \Phi_\mu(t)$, $t \in \mathcal{J}$, $\mu \in \mathbf{Z}$, forms an infinite cyclic group with generating element Φ_1 and unit element Φ_0 with group operation composition of functions.*

Proof. Let $*$ denote the group operation of composition of functions. Assume α is increasing. Equation (6.2) yields that $\Phi_\mu * \Phi_\nu = \alpha^{-1}\phi_\mu\alpha\alpha^{-1}\phi_\nu\alpha(t) = \alpha^{-1}\phi_{\mu+\nu}\alpha(t)$, $\Phi_1 = \alpha^{-1}\phi_1\alpha(t)$, $\Phi_0 =$

$\alpha^{-1}\phi_0\alpha(t) = t$. Thus $\Phi_\mu(t)*\Phi_0(t) = \alpha^{-1}\phi_\mu\alpha\alpha^{-1}\phi_0\alpha(t) = \alpha^{-1}\phi_\mu\alpha(t) = \Phi_\mu(t)$, $\mu \in \mathbf{Z}$. The proof for α decreasing is analogous. □

Definition 6.2. The group $\mathcal{G}(\Phi) = \{\Phi_\mu(t)\}_{\mu=-\infty}^{+\infty}$ consists of the central dispersions Φ_μ given by (6.2).

Theorem 6.2. *Let $\mathcal{G}(\phi)$ be a determining group and $\alpha \in M_g$ a phase function. The μth central dispersion $\Phi_\mu = \Phi_\mu(t)$ given by* (6.2) *has the following properties*:

1. $\Phi_\mu \in C_0(\mathcal{J})$,

2. Φ_μ *increases on* $\mathcal{J}$,

3. $\Phi_\mu(-\infty) = -\infty$, $\Phi_\mu(+\infty) = +\infty$,

4. $\Phi_\mu(t) > 1$ *for* $t \in \mathcal{J}$ *and* $\mu = 1, 2, 3, \dots$, $\Phi_\mu(t) < t$ *for* $t \in \mathcal{J}$ *and* $\mu = -1, -2, -3, \dots$.

5. *If α increases on $\mathcal{J}$, then $\{\Phi_\mu(t)\}_{\mu=-\infty}^{+\infty}$ is an* increasing sequence *on $\mathcal{J}$. If α decreases on $\mathcal{J}$, then it is a* decreasing sequence.

6. *The sequence $\{\Phi_\mu(t)\}_{\mu=-\infty}^{+\infty}$ of central dispersions induced by the phase function $\alpha \in M_g$ for the determining group $\mathcal{G}(\phi)$ is* normal.

Proof. Property 1. The functions $\Phi_\mu = \Phi_\mu(t)$, $\mu \in \mathbf{Z}$, expressed by formula (6.2), are compositions of continuous functions, each one mapping the interval $\mathcal{J}$ into itself. Hence the same is true of each Φ_μ.

Property 2. On interval $\mathcal{J}$, the functions α, α^{-1} simultaneously either increase or decrease and the functions ϕ_μ and $\phi_{-\mu}$ both increase. Therefore, the composite functions Φ_μ, $\mu \in \mathbf{Z}$, expressed by formula (6.2) increase on the interval $\mathcal{J}$.

Property 3. If α increases on $\mathcal{J}$, then $\Phi_\mu(-\infty) = \alpha^{-1}\phi_\mu\alpha(-\infty) = -\infty$, since $\alpha(-\infty) = -\infty$, $\phi_\mu(-\infty) = -\infty$ and $\alpha^{-1}(-\infty) = -\infty$. Furthermore, $\Phi_\mu(+\infty) = \alpha^{-1}\phi_\mu\alpha(+\infty) = +\infty$ because $\alpha(+\infty) = +\infty$, $\phi_\mu(+\infty) = +\infty$ and $\alpha^{-1}(+\infty) = +\infty$. If α decreases on $\mathcal{J}$, then $\Phi_\mu(-\infty) = \alpha^{-1}\phi_{-\mu}\alpha(-\infty) = -\infty$ since $\alpha(-\infty) = \infty$, $\phi_{-\mu}(\infty) = \infty$ and $\alpha^{-1}(-\infty) = \infty$.

Property 4. Given $\Phi(t) > t$, then $\Phi_2(t) = \Phi\Phi(t) > \Phi(t) > t$ and, in general, $\Phi_\mu(t) > t$ for $\mu = 1, 2, 3, \dots$. Inequality $\Phi(t) > t$ implies

$\Phi\Phi^{-1}(t) > \Phi^{-1}(t)$ and hence $\Phi_{-1}(t) < t$, $\Phi_{-2}(t) = \Phi_{-1}\Phi_{-1}(t) < \Phi_{-1}(t) < t$ and, in general, $\Phi_\mu(t) < t$ for $\mu = -1, -2, \ldots$.

Property 5. Let $\{\phi_\mu(t)\}_{\mu=-\infty}^{+\infty}$ be increasing on $\mathcal{J}$. Let α increase on $\mathcal{J}$. Since $\phi_{\mu-1}(t) < \phi_\mu(t)$, $t \in \mathcal{J}$, $\mu \in \mathbf{Z}$, then $\phi_{\mu-1}\alpha(t) < \phi_\mu\alpha(t)$. Because α^{-1} increases, then $\alpha^{-1}\phi_{\mu-1}\alpha(t) < \alpha^{-1}\phi_\mu\alpha(t)$ or $\Phi_{\mu-1}(t) < \Phi_\mu(t)$. The sequence $\{\Phi_\mu(t)\}_{\mu=-\infty}^{+\infty}$ is thus increasing on $\mathcal{J}$. If α decreases on $\mathcal{J}$, then the details are similar, except α and α^{-1} decreasing causes the reversal of inequalities.

Property 6. According to Property 5, the sequence $\{\Phi_\mu\}_{\mu=-\infty}^{+\infty}$ is either an increasing or decreasing sequence. If it increases and $\bar{t} \in \mathcal{J}$ is given, then the sequence of functional values $\{\Phi_\mu(\bar{t})\}_{\mu=-\infty}^{+\infty}$ is increasing and unbounded both above and below. For instance, if it was bounded above, then it would have a finite limit A. Then $A = \lim_{\mu\to\infty}\Phi_\mu(\bar{t}) = \Phi(\lim\Phi_{\mu-1}(\bar{t})) = \Phi(A)$, which contradicts $\Phi(t) > t$, see Property 4. The other details follow similarly. Thus the sequence $\{\Phi_\mu(t)\}_{\mu=-\infty}^{+\infty}$ is normal. □

7. Linear functional equations. Given constants $a_0, \ldots, a_n$ and distinct points $t_0, \ldots, t_n$, the equation

$$a_0 f(t_n) + a_1 f(t_{n-1}) + \cdots + a_n f(t_0) = 0 \tag{7.1}$$

is called a *linear functional equation*. The objective is to find the unknown function $f(t)$. Included in (7.1) are *linear difference equations*, in which only the values $f(t_i)$, $i = 0, \ldots, n$, are required to be found.

Euler's method for nth order linear differential equations can be applied to (7.1), which assumes $f(t) = e^{pt}$ for some p. Then (7.1) holds provided

$$a_0\omega^{t_n} + a_1\omega^{t_{n-1}} + \cdots + a_n\omega^{t_0} = 0, \quad \omega = e^p. \tag{7.2}$$

Equation (7.2) is a *characteristic equation* for variable ω. Given a solution ω of (7.2), then a solution of (7.1) is given by

$$f(t) = \omega^t. \tag{7.3}$$

General dispersion theory as developed in this paper contributes to this problem by offering a different solution pair

$$a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n\lambda^0 = 0, \tag{7.4}$$

$$f(t) = \lambda^{\alpha(t)}, \tag{7.5}$$

for some increasing function $\alpha(t)$. The contribution is *nontrivial*, because (7.4) is technically routine to solve, compared to (7.2) and (7.5) is a more general solution than (7.3). The tradeoff made is that *something* has to be assumed about the distinct points $t_0, t_1, \dots, t_n$, and that is where central dispersions enter the discussion.

Determining group. The group $\mathcal{G}(\phi)$ with group operation composition of functions will have generator $\phi(t) = t + 1$. Then the identity in $\mathcal{G}(\phi)$ is $\phi_0(t) = t$ and the iterates are defined by $\phi_\mu(t) = t + \mu$, $\mu \in \mathbf{Z}$. This choice produces the *polynomial equation* (7.4) as will be seen below.

Phase function. Choose an increasing phase function $\alpha \in M_g$. The determining group $\mathcal{G}(\phi)$ with phase function α induces successive conjugate points $t_0, t_1, \dots, t_n$ that satisfy for $i = 0, 1, \dots, n$ the relations

$$\begin{aligned} \alpha(t_i) &= \phi_i(\alpha(t_0)) \\ &= \alpha(t_0) + i. \end{aligned}$$

The last calculation $\phi_i(\alpha(t_0)) = \alpha(t_0) + i$ is justified below in Theorem 7.1.

Central dispersion solution. The function $f(t) = \lambda^{\alpha(t)}$ satisfies for the constant $C = \lambda^{\alpha(t_0)}$ the equations

$$\begin{aligned} f(t_i) &= \lambda^{\alpha(t_i)} \\ &= \lambda^{\phi_i \alpha(t_0)} \\ &= \lambda^{\alpha(t_0)+i} && \text{see Theorem 7.1,} \\ &= C\lambda^i && \text{because } C = e^{\alpha(t_0)}. \end{aligned}$$

Therefore, $f(t) = \lambda^{\alpha(t)}$ will solve (7.1) provided (7.4) holds; see the proof of Theorem 7.2 for details.

Extensions. It may be possible to solve (7.1) for a function $f(t) = \lambda^{\alpha(t)}$ where α is increasing and has finite oscillation $O(\alpha)$. All of this depends upon representation of the points $t_0, \dots, t_n$ as successive conjugate points in group $\mathcal{G}(\phi)$ for some increasing function $\alpha \in M_{ms}$. Some other directions are given in Theorems 7.3 and 7.4.

Theorem 7.1. *Let $\alpha \in M_g$ be an increasing phase function. Let $\Phi = \Phi(t)$ be a solution of the Abel functional equation* (6.1) *for $\phi_1 = t + 1$, which is a solution of the functional equation*

$$\alpha\Phi(t) = \alpha(t) + 1. \tag{7.6}$$

The μth central dispersion $\Phi_\mu(t)$ induced by the phase function $\alpha = \alpha(t)$ for the group $\mathcal{G}(\phi)$, $\phi(t) = t + 1$, satisfies

$$\alpha\Phi_\mu(t) = \alpha(t) + \mu, \quad t \in \mathcal{J}, \quad \mu \in \mathbf{Z}. \tag{7.7}$$

Proof. The assertion follows from formula (6.1). □

Theorem 7.2. *Let $\Phi_\mu = \Phi_\mu(t)$, $\mu \in \mathbf{Z}$, be the μth central dispersion induced by an increasing phase function $\alpha \in M_g$ for the group $\mathcal{G}(\phi)$, $\phi(t) = t + 1$. Consider the homogeneous linear functional equation of the nth order with constant coefficients $a_i \in R$, $i = 0, 1, \ldots, n$, to be solved for unknown $f(t)$, given by*

$$a_0 f\Phi_n(t) + a_1 f\Phi_{n-1}(t) + \cdots + a_{n-1} f\Phi_1(t) + \alpha_n f\Phi_0(t) = 0. \tag{7.8}$$

Let λ be a root of the characteristic equation

$$a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0. \tag{7.9}$$

Then (7.8) *has a continuous solution*

$$f(t) = \lambda^{\alpha(t)}. \tag{7.10}$$

Proof. If we substitute (7.10) into (7.8) then, according to (7.7), we obtain $a_0\lambda^{\alpha\Phi_n(t)} + a_1\lambda^{\alpha\Phi_{n-1}} + \cdots + a_{n-1}\lambda^{\alpha\Phi_1(t)} + a_n\lambda^{\alpha\Phi_0(t)} = a_0\lambda^{\alpha(t)+n} + a_1\lambda^{\alpha(t)+n-1} + \cdots + a_{n-1}\lambda^{\alpha(t)+1} + a_n\lambda^{\alpha(t)} = \lambda^{\alpha(t)}(a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n) = 0$, since λ in the formula (7.10) is a root of the characteristic equation (7.9), see [**4**]. □

Theorem 7.3. *Let $\mathcal{G}(\phi)$ and $\mathcal{G}(\psi)$ be determining groups and $\alpha \in M_g$ an increasing phase function. Let Φ_μ, $\mu \in \mathbf{Z}$, be central dispersions* *induced by the phase function* α *for the group* $\mathcal{G}(\phi)$*. Let the generating functions* ϕ *and* ψ *of the determining groups be connected by the Abel functional equation*

$$\beta\phi = \psi\beta, \tag{7.11}$$

where β *is an arbitrary phase function from* M_g*. Then*

$$\beta\alpha\Phi_\mu(t) = \psi_\mu\beta\alpha, \quad \mu \in \mathbf{Z}. \tag{7.12}$$

Proof. We know that central dispersions Φ_μ satisfy the Abel equation

$$\alpha\Phi_\mu = \phi_\mu\alpha, \quad \mu \in \mathbf{Z}. \tag{7.13}$$

Equations (7.11) and (7.13) imply the following steps for $\mu \in \mathbf{Z}$:

$$\begin{aligned} \beta\phi_\mu &= \psi_\mu\beta \\ \phi_\mu\alpha &= \beta_{-1}\psi_\mu\beta\alpha \\ \alpha\Phi_\mu &= \phi_\mu\alpha \\ &= \beta_{-1}\psi_\mu\beta\alpha \\ \beta\alpha\Phi_\mu &= \psi_\mu\beta\alpha. \end{aligned}$$

The last equation is (7.12). □

If in Theorem 7.3 the determining group $\mathcal{G}(\psi)$ has generator $\psi(t) = t+1$, then (7.12) is of the form

$$\beta\alpha\Phi_\mu(t) = \beta\alpha(t) + \mu, \quad \mu \in \mathbf{Z}. \tag{7.14}$$

This formula can be used for explicit solution of a linear functional equation (7.1). Techniques of the previous theorems can be applied to prove the following result.

Theorem 7.4. *Let* $\Phi_\mu = \Phi_\mu(t)$*,* $\mu \in \mathbf{Z}$*, be the* μ*th central dispersion induced by an increasing phase function* $\alpha \in M_g$ *for the group* $\mathcal{G}(\phi)$*. Let* β *be a solution of the Abel functional equation*

$$\beta\phi(t) = \beta(t) + 1.$$

Then the homogeneous linear functional equation of the nth order with constant coefficients $a_i \in \mathbf{R}$, $i = 0, 1, \dots, n$,

$$a_0 f\Phi_n(t) + a_1 f\Phi_{n-1}(t) + \cdots + a_{n-1} f\Phi_1(t) + a_n f\Phi_0(t) = 0,$$

has a solution in the form

$$f(t) = \lambda^{\beta\alpha(t)},$$

where λ is a root of the characteristic equation

$$a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0.$$

REFERENCES

1. O. Borůvka, *Linear differential transformations of the second order*, The English University Press, London, 1971.

2. A.O. Gelfond, *The method of finite differences*, State Publisher of Technical Literature, Moscow-Leningrad, 1952.

3. P. Hartman, *Unrestricted n-parameter families*, Rend. Circ. Mat. Palermo (2) **7** (1958), 123–142.

4. M. Laitoch, *To the theory of linear difference equations*, Acta Univ. Palacky Olomouc, F.R.N. **97** (1984), 11–24.

5. J. Laitochová, *On a functional central dispersion of the first kind and the Abel functional equation in strongly regular spaces of continuous functions*, Acta Math. Palackianae Olomucensis, F.R.N., Math. 28, vol. 94, 1989, 165–175.

6. ———, *Algebraic structure of phase functions*, Acta Math. Hungar. **73** (1996), 301–318.

7. F. Neumann, *Global properties of linear ordinary differential equations*, Academia, Prague, 1991.

8. T.L. Sherman, *Properties of solutions of Nth order linear differential equations*, Pacific J. Math. **15** (1965), 1045–1060.

9. ———, *Conjugate points and simple zeros for ordinary linear differential equations*, Trans. Amer. Math. Soc. **146** (1969), 397–411.

Math. Dept. PdF, Palacký University, Žižkovo nám. 5, Olomouc, Czech Republic
E-mail address: jitka.laitochova@upol.cz