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NIMLIKE GAMES WITH GENERALIZED BASES

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ABSTRACT. In several earlier papers [**4**] and [**5**], we discussed single pile games of Nim in which the number of counters that can be removed varies during the play of the game. In [**6**] we showed how to effectively play the single-pile game in which the number of counters that can be removed is a function of the number removed on the previous move. In that paper we constructed a number base and showed that in the winning strategy, the winning player can reduce the number of summands in a certain representation of the current pile size. In this paper, we reverse the situation by starting with an arbitrary base, and then construct a game whose winning positions are determined by the base. In particular, the winning strategy for such games consists of reducing the number of summands in the representation, with respect to this base, of the current pile size. In the Appendix we have worked through an example that illustrates all of the concepts given in this paper.

1. Definition. A *number base* is a strictly increasing sequence $B = (b_0 = 1, b_1, b_2, \dots)$ of positive integers. B can be finite or infinite.

In this paper we will consider B to be infinite. The reader can prove analogous results when B is finite. The following theorem is well known. The proof is given for the sake of completeness.

Theorem. *Let B be an infinite number base. Then each positive integer N can be represented as $N = b_{i_1} + b_{i_2} + \cdots + b_{i_t}$, where $b_{i_1} \le b_{i_2} \le \cdots \le b_{i_t}$ and each b_{i_j} belongs to B, by the following recursive algorithm.*

First, we represent the number 1 by $1 = b_0$. If $1, 2, 3, \dots m-1$ have been represented by the algorithm, then m can be represented as follows: Let b_k denote the largest element of B not exceeding m. That is, $b_k \le m < b_{k+1}$. Then $m = (m - b_k) + b_k$ and $m - b_k < b_{k+1}$. If $m - b_k = 0$ then the algorithm is finished.

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If $1 \le m - b_k$ then, since $m - b_k$ is less than m, it follows that $m - b_k$ has been represented by the algorithm as $m - b_k = b_{i_1} + b_{i_2} + \cdots + b_{i_{t-1}}$, where $b_{i_1} \le b_{i_2} \le \cdots \le b_{i_{t-1}} \le b_k$ and each b_{i_j} belongs to B. Note that $b_{i_{t-1}} \le b_k$ since $b_{i_{t-1}} < b_{k+1}$ and b_k is the member of B that comes right before b_{k+1}. Then $m = b_{i_1} + b_{i_2} + \cdots + b_{i_t}$ where $b_{i_t} = b_k$ and $b_{i_1} \le b_{i_2} \le \cdots \le b_{i_t}$ and each b_{i_j} belongs to B.

As an example, let $B = (1, 5, 20, 60, 90, \dots)$. Then $89 = 1 + 1 + 1 + 1 + 5 + 20 + 60 = b_0 + b_0 + b_0 + b_0 + b_1 + b_2 + b_3$.

Definition. Let $B = (b_0 = 1, b_1, b_2, \dots)$ be an infinite number base. For any positive integer N, let $N = b_{i_1} + b_{i_2} + \cdots + b_{i_k}$, $b_{i_1} \le b_{i_2} \le \cdots \le b_{i_k}$, be the representation of N in B that is given by the above algorithm. Then we define $\bar{g}(N) = b_{i_1}$. Thus, in the example above, $\bar{g}(89) = 1$. For convenience, we define $\bar{g}(0) = \infty$. More examples are given in the Appendix.

Remark. The material that follows is also a solution to the problem we alluded to in the Introduction of the paper, namely the finding of a game for a given base. We state this problem in Section 3 of the paper because many readers will understand the concepts much better at that time. However, a few readers, depending on their background, may prefer to turn to the end of the paper and read the problem first.

Definition. Let Z^+ denote the set of positive integers, and B is any infinite number base. Let $f : Z^+ \to \{0\} \cup Z^+$ be any function that satisfies the following two conditions. For all $N \in Z^+$, let $N = b_{i_1} + b_{i_2} + \cdots + b_{i_t}, b_{i_1} \le b_{i_2} \le \cdots \le b_{i_t}$ be the representation of N in B that is computed by the algorithm. Then

1. $b_{i_1 - 1} \le f(N) < b_{i_2}$ when $t \ge 2$, and
2. $b_{i_1 - 1} \le f(N) < \infty$ when $t = 1$.

If $b_{i_1} = b_0 = 1$, then we agree that $b_{0-1} = b_{-1} = 0$. Note that $b_{i_1 - 1}$ is the member of B that comes just before $\bar{g}(N) = b_{i_1}$.

Thus in the base $B = (1, 5, 20, 60, 90, \dots)$ used earlier, we must have $0 = b_{-1} = b_{i_{1-1}} \le f(89) < b_{i_2} = b_0 = 1$ since $b_{i_1} = b_{i_2} = b_0 = 1$. This means that $f(89) = 0$. As another example, let $B = (1, 2, 4, 8, 16, \dots)$.

Now $12 = 4 + 8 = b_2 + b_3 = b_{i_1} + b_{i_2}$. Therefore, $b_1 = b_{i_1 - 1} \leq f(12) < b_{i_2} = b_3$ which means $2 \leq f(12) < 8$.

2. General framework of the game. Let $B = (b_0 = 1, b_1, b_2, \dots)$ be an infinite number base, and let $f : Z^+ \to \{0\} \cup Z^+$ be a function satisfying the two conditions above. Two players alternate removing a positive number of counters from a single pile of counters with the winner being the player who makes the last move in the game. An ordered pair of nonnegative integers (N, x) is called a position in the game. The number N represents the current pile size, and x represents the greatest number of counters that can be removed on the next, i.e., current, move. Thus if the moving player were facing a position (10, 4), then he could remove $1, 2, 3$ or 4 counters. If the moving player were facing (3, 7), then he could remove 1, 2, or 3 counters. A position (N, x) is a terminal position if and only if $N = 0$ or $x = 0$, and the winner is the first player to move to a terminal position.

Specific rules of the game. A move in this game is defined as an ordered pair of positions $(N, x) \mapsto (N - k, f(N))$, where $1 \leq k \leq \min(N, x)$. If we call the move after the current move the succeeding move, then we see that f determines the maximum size of the succeeding move in terms of the current pile size. The initial position, (N_0, x_0) will be given.

As an example, suppose $(N_0, x_0) = (100, 79)$. If the player moving first removes $k = 15$ counters and $f(100) = 5$, then the new position becomes $(100, 79) \mapsto (100 - 15, f(100)) = (85, 5)$. This means the next player can remove from the 85 counter pile either 1, 2, 3, 4 or 5 counters. Suppose the next player removes 4 counters and $f(85) = 3$. Then the new position becomes $(85, 5) \mapsto (85 - 4, f(85)) = (81, 3)$. The next player can remove 1, 2 or 3 counters from the 81 counter pile. This continues until a player cannot move, and the winner is the player who makes the last move.

Suppose we play the game that we have just specified.

Definition. A position (N, x) is called *unsafe* if it is unsafe to move to it. Since the player who moves to an unsafe position loses with best play, the player who moves from an unsafe position can always win.

Similarly, a position is *safe* if it is safe to move to it. For every positive integer N, we define $g(N)$ to be the positive integer such that the position (N, x) is unsafe if $g(N) \leq x$ and (N, x) is safe if $0 \leq x < g(N)$. Also, we agree that $g(0) = \infty$. This means that for a pile size of $N \geq 1$ counters, $g(N)$ is the smallest winning move size. That is, $g(N)$ is the smallest number of counters the moving player can remove to get to a safe position. Of course, this means the removal of any x counters, $1 \leq x < g(N)$, would result in an unsafe position. It is obvious that $1 \leq g(N) \leq N$.

Note. Some authors use the terminology "*P-positions*," "*N-positions*," respectively, for the terms *safe* and *unsafe* positions.

Observation. Note that the function $g : Z^+ \to Z^+, g(0) = \infty$ can be computed recursively as follows. First note that $g(1) = 1$. Also, suppose $g(1), g(2), \ldots, g(N-1), N \geq 2$ have been computed. Then $g(N)$ is the smallest $i \in \{1, 2, 3, \ldots, N\}$ such that $f(N) < g(N-i)$. Note that $g(N-N) = g(0) = \infty$. We deal with this more in the Appendix.

Theorem. *Let* $B = (b_0 = 1, b_1, b_2, \ldots)$, *and* $f : Z^+ \to \{0\} \cup Z^+$ *satisfy conditions* 1 *and* 2 *above. Let* $\bar{g}$ *and* g *be the functions defined earlier. Then for all* $N \in Z^+, g(N) = \bar{g}(N)$.

Proof. We will prove this by mathematical induction on N. Now $g(1) = 1$ and also $\bar{g}(1) = 1$. So the induction is started. Suppose that, for all $k \in \{1, 2, 3, \ldots, N-1\}$, $g(k) = \bar{g}(k)$ where $N - 1 \geq 1$. We now show that $g(N) = \bar{g}(N)$.

Let $N = b_{i_1} + b_{i_2} + \cdots + b_{i_t}, b_{i_1} \leq b_{i_2} \leq \cdots \leq b_{i_t}$ be the representation of N in B that is computed by the algorithm. We need to show that $g(N) = \bar{g}(N) = b_{i_1}$. By the observation, we know that $g(N) = g(b_{i_1} + b_{i_2} + \cdots + b_{i_t})$ is the smallest $i \in \{1, 2, 3, \ldots, N\}$ such that $f(N) < g(N-i) = g(b_{i_1} + b_{i_2} + \cdots + b_{i_t} - i) = \bar{g}(b_{i_1} + b_{i_2} + \cdots + b_{i_t} - i)$ since, by induction, $g(N-i) = \bar{g}(N-i)$ when $i \in \{1, 2, 3, \ldots, N\}$. Now when $i \in \{1, 2, 3, \ldots, b_{i_1} - 1\}$, it follows from the definition of the algorithm that $\bar{g}(N-i) = \bar{g}(b_{i_1} + b_{i_2} + \cdots + b_{i_t} - i) = \bar{g}(b_{i_1} - i)$. The reader might like to use a specific example to see this more easily.

Also, when $i \in \{1, 2, 3, \dots, b_{i_1} - 1\}$ we see that $\bar{g}(b_{i_1} - i) \le b_{i_1 - 1}$ since $b_{i_1} - i < b_{i_1}$. Of course, when $i = b_{i_1} - b_{i_1 - 1}$ we have $\bar{g}(b_{i_1} - i) = b_{i_1 - 1}$. This means equality can occur, and this fact is needed in the proof of the converse theorem. Now by the definition of f, we know that $f(N) = f(b_{i_1} + b_{i_2} + \cdots + b_{i_t}) \ge b_{i_1 - 1}$. Therefore, when $i \in \{1, 2, 3, \dots, b_{i_1} - 1\}$, we have $f(N) \ge b_{i_1 - 1} \ge \bar{g}(b_{i_1} - i) = \bar{g}(N - i) = g(N - i)$. That is, $f(N) \ge g(N - i)$. Let us next suppose $t \ge 2$. Now when $i = b_{i_1}$, we have by induction $g(N - i) = \bar{g}(N - i) = \bar{g}(b_{i_2} + b_{i_3} + \cdots + b_{i_t}) = b_{i_2}$.

Now $f(N) = f(b_{i_1} + b_{i_2} + \cdots + b_{i_t}) < b_{i_2}$. Therefore, when $i = b_{i_1}$, $f(N) < b_{i_2} = g(N - i) = g(N - b_{i_1})$. Therefore, $g(N) = b_{i_1} = \bar{g}(N)$. Of course, if $N = b_{i_1}$, the proof is trivial to finish. □

3. The converse theorem.

Converse theorem. *Suppose $B = (b_0 = 1, b_1, b_2, \dots)$ is an infinite number base, $\bar{g}$ is defined for B as above, and $f : Z^+ \to \{0\} \cup Z^+$ is a function. Suppose we play this game with f and define g as we did previously. Finally, suppose $g = \bar{g}$. Then f must satisfy conditions* 1 *and* 2 *above. We leave this proof to the reader.*

The material in this paper is a solution to the following problem.

Problem. Let B denote an infinite number base $(b_0 = 1, b_1, b_2, \dots)$. For every positive integer $N, \bar{g}(N)$ is defined for N using this base B. We wish to specify the rules of a nontrivial game that has the following properties. Two players alternate removing a positive number of counters from a single pile of counters with the winner being the player who makes the last move in the game. An ordered pair of nonnegative integers (N, x) is called a position in the game. The number N represents the current size of the pile of counters, and x represents the greatest number of counters that can be removed on the next, i.e., current, move. Thus a move in the game is an ordered pair of positions $(N, x) \mapsto (N - k, \bar{x})$, where $1 \le k \le \min(N, x)$ and where $\bar{x}$ is specified by the *rules* of the game. A position (N, x) is a terminal position if and only if $N = 0$ or $x = 0$, and the winner is the first player to move to a terminal position. We wish to devise the rules of the game so that for any positive integer N, the position (N, x) is unsafe if $\bar{g}(N) \leq x$ and is safe if $0 \leq x < \bar{g}(N)$. Also, when $\bar{g}(N) \leq x$, we would like the removal of $\bar{g}(N)$ counters from a pile of N counters to be a winning move.

Acknowledgement. The game studied in this paper, although different from, was nevertheless influenced by a game studied by James Rudzinski, a UNC Charlotte undergraduate. The authors also acknowledge the referee, who made some valuable suggestions.

Appendix

If $x \leq y$ are integers, then $[x, y) = \{x, x+1, x+2, \dots, y-1\}$. Thus $[x, x)$ is empty. Let $B = (1, 3, 7, \dots)$. Then $1 = 1, 2 = 1+1, 3 = 3, 4 = 1+3, 5 = 1+1+3, 6 = 3+3, 7 = 7$. As always, $g(N)$ is the smallest $i \in \{1, 2, 3, \dots, N\}$ such that $f(N) < g(N-i)$,

N	0	1	2	3	4	5	6	7
$f(N)$		$[0, \infty)$	0	$[1, \infty)$	[0,3)	0	[1,3)	$[3, \infty)$
$\bar{g}(N)$	∞	1	1	3	1	1	3	7
$g(N)$	∞	1	1	3	1	1	3	7

The table and the results of the paper show, for example, that the positions $(7, i)$ are safe for $1 \leq i \leq 6$. Similarly, $(6, i)$ is safe for $i \in \{1, 2\}$, and unsafe for $i \geq 3$. It follows that the only winning move from $(6, 3)$ is to $(3, f(6))$.

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