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# GOING-DOWN IMPLIES GENERALIZED GOING-DOWN

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ABSTRACT. It is proved that if a unital homomorphism $f$ of commutative rings satisfies the going-down property GD, then $f$ also satisfies the infinitistic variant, the generalized going-down property GGD (that is, $f$ exhibits going-down behavior for prime ideal chains of arbitrary cardinality).

**1. Introduction.** All rings considered below are commutative with identity; all ring homomorphisms are unital. Adapting notation for ring extensions in [**9**, p. 28], we let GU, GD, and LO denote the going-up, going-down and lying-over properties, respectively, for ring homomorphisms. As in [**5**], respectively [**6**], we let GGD, respectively GGU, denote the generalized going-down, respectively generalized going-up, property. Intuitively, GGD, respectively GGU, is the generalization of the usual GD, respectively GU, property when predicated for chains of prime ideals of arbitrary cardinality. Evidently, GGD $\Rightarrow$ GD, respectively GGU $\Rightarrow$ GU, and it is natural to ask if the converse is valid. A partial converse was obtained in [**2**, Remark (a)], respectively [**2**, Theorem], where it was shown that GD $\Rightarrow$ GGD, respectively GU $\Rightarrow$ GGU, for ring extensions $A \subseteq B$ such that each chain of prime ideals of $A$ is well-ordered by reverse inclusion, respectively by inclusion. Subsequently, [**5**] identified many other contexts in which GD $\Rightarrow$ GGD. Additional positive evidence appeared in [**3**, Theorem 2.4], in which the implications GD $\Rightarrow$ GGD and GU $\Rightarrow$ GGU were shown to hold for ring homomorphisms satisfying the strong going-between property SGB (a lifting property for chains that had been introduced by G. Picavet in [**11**]). It was also shown that SGB is equivalent to its infinitistic variant, GSGB [**3**, Corollary 2.3]. Then, at a conference in Venice in June 2002, Kang and Oh announced that GU $\Rightarrow$ GGU for unital ring extensions; it then follows via standard homomorphism theorems that GU $\Rightarrow$ GGU for arbitrary ring homomorphisms. We have benefited from access to the Kang-Oh result [**8**, Corollary 12] in preprint form.

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Indeed, the main purpose of this note is to resolve the nature of GGD by proving that GD $\Rightarrow$ GGD for arbitrary ring homomorphisms.

The proof of Theorem 2.2 shows, in effect, that the implications GD $\Rightarrow$ GGD and GU $\Rightarrow$ GGU are logically equivalent. The main agent that we use in establishing Theorem 2.2 is the connection between the Zariski topology and the flat topology, as in [**4**]. An interesting characterization of the flat topology identifies it as the result of applying the "opposite-order topology" of Hochster [**7**, Proposition 8] to the Zariski topology, cf. also [**10**, p. 88], [**4**, Remark 2.3 (a)]. However, order does not determine spectral topologies except in special cases, cf. [**7**, pp. 57–58], and so one needs a mixture of order-theoretic and topological reasoning to pass between going-down behavior and going-up behavior. The passage is effected in the proof of Theorem 2.2, with the key technical step isolated in Lemma 2.1 (c). For the sake of completeness, Lemma 2.1 contains ample supporting material on the flat topology.

We next summarize some notational conventions. Let $A$ be a ring. Then $X := \mathrm{Spec}\,(A)$ describes the *set* of all prime ideals of $A$; and $X_F$ denotes the topological space in which the set $X$ is endowed with the flat topology, as described in Lemma 2.1 (a). Following [**1**], we let $V(I) := V_A(I) := \{P \in X \mid I \subseteq P\}$ for any ideal $I$ of $A$; $X_g := \{P \in X \mid g \notin P\}$ for any $g \in A$; and ${}^a f\colon \mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$ the canonical function, $Q \mapsto f^{-1}(Q)$, for any ring homomorphism $f\colon A \to B$. As in [**7**], a *spectral space* is a topological space that is homeomorphic to $\mathrm{Spec}\,(A)$, endowed with the Zariski topology, for some ring $A$; and a *spectral map* is a continuous function between spectral spaces for which the inverse image of each quasi-compact open set of the codomain is quasi-compact in the domain.

## 2. Results.

We begin by collecting some elementary but useful facts about the flat topology. Lemma 2.1 (c) is needed for the key step in the proof of Theorem 2.2. Although Lemma 2.1 (c) may be known, its proof is included for lack of a convenient reference.

**Lemma 2.1.** (a) *Let $A$ be a ring and $X := \mathrm{Spec}\,(A)$. Then the sets of the form $X_g$, where $g \in A$, form a subbasis for the closed sets of $X_F$.*

(b) *Let $A$ be a ring and $X := \mathrm{Spec}\,(A)$. Then the quasi-compact open sets of $X_F$ are the sets of the form $\cup V(I_i)$, where $\{I_i\}$ is a finite set of finitely generated ideals of $A$.*

(c) *Let $f : A \to B$ be a ring homomorphism, with $X := \mathrm{Spec}\,(A)$ and $Y := \mathrm{Spec}\,(B)$. Then ${}^a f : Y_F \to X_F$ is a spectral map.*

*Proof.* (a) According to [**4**, Theorem 2.2], the flat topology on $X$ is the coarsest topology on $X$ for which $X_g$ is closed for each $g \in A$. Thus, the typical closed set in $X_F$ is the intersection of arbitrarily many sets of the form $X_{g_1} \cup \cdots \cup X_{g_n}$, for $g_i \in A$, and we have inferred a reformulation of the assertion.

(b) By [**4**, Theorem 2.2], the typical open set $U$ in $X_F$ is of the form $U = \cup V(I_i)$, where $\{I_i\}$ is a set of finitely generated ideals of $A$. As each $V(I_i)$ is open in $X_F$, it follows that if $U$ is quasi-compact in $X_F$, the open subcover of $U$ by $\{V(I_i)\}$ has a finite subcover, thus yielding finitely many indexes $i_1, \dots, i_n$ such that $U = V(I_{i_1}) \cup \cdots \cup V(I_{i_n})$. Conversely, to show that any such finite union is quasi-compact in $X_F$, we need only consider the case $n = 1$, for the union of finitely many quasi-compact subsets is necessarily quasi-compact. We shall show, in fact, that if $I$ is any ideal of $A$, then $V(I)$ is quasi-compact in $X_F$.

Consider the ring $\overline{A} := A/I$, the set $Z := \mathrm{Spec}\,(\overline{A})$, and the bijection $\sigma : V_A(I)_F \to Z_F$, given by $\sigma(Q) = Q/I$ for all $Q \in V(I)$. (The notation $V_A(I)_F$ refers to the subspace topology inherited by $V_A(I)$ from $X_F$.) It is enough to show that $\sigma$ is a homeomorphism, for $Z_F$ is quasi-compact, cf. [**7**, Proposition 8], [**4**, Remark 2.3 (a)]. Thus, it suffices to prove that $\sigma$ sends the typical basic closed set $K$ in $V_A(I)_F$ to the typical basic closed set $L$ in $Z_F$.

Now, by (a) and the definition of the subspace topology, $K = V_A(I) \cap (\cup X_{g_i})$, where $\{g_i\}$ is a finite subset of $A$. On the other hand, [**4**, Theorem 2.2] allows us to write $L = Z \setminus V_{\overline{A}}(\sum \overline{A} h_i)$, for some finite set $\{h_i\} \subseteq \overline{A}$. For each $i$, write $h_i = g_i + I$, with $g_i \in A$. By [**1**, (2) p. 98], $L = Z \setminus \cap V_{\overline{A}}(\overline{A} h_i) = \cup (Z \setminus V_{\overline{A}}(\overline{A} h_i)) = \cup Z_{h_i}$. To show that $\sigma(K) = L$, it suffices (since $\sigma$ commutes with unions) to observe that $\sigma(V_A(I) \cap X_{g_i}) = \{Q/I \in Z \mid Q \in X, I \subseteq Q, g_i \notin Q\} = \{\mathfrak{Q} \in Z \mid h_i \notin \mathfrak{Q}\} = Z_{h_i}$, for each $i$.

(c) Recall, cf. [**7**, Proposition 8], [**10**, p. 88], [**4**, Remark 2.3 (a)], that $Y_F$ and $X_F$ are spectral spaces. Moreover, it is easy to see that

${}^a f : Y_F \to X_F$ is continuous, cf. [**4**, Remark 2.3 (e)]. Thus, it remains to show that if $U$ is a quasi-compact open set in $X_F$, then $({}^a f)^{-1}(U)$ is a quasi-compact subset of $Y_F$. As $({}^a f)^{-1}$ commutes with unions, (b) permits us to assume that $U = V(I)$, where $I = \sum Ag_i$ for some finite set $\{g_i\} \subseteq A$. Put $h_i := f(g_i) \in B$. It now suffices to repeat the essential calculation in the proof of [**1**, Proposition 13, p. 101]. Indeed, observe that

$$\begin{aligned}({}^a f)^{-1}(U) &= \Big\{Q \in Y \mid ({}^a f)(Q) \in V_A\Big(\sum Ag_i\Big)\Big\}\\ &= \Big\{Q \in Y \mid \sum Ag_i \subseteq f^{-1}(Q)\Big\}\\ &= \{Q \in Y \mid f(g_i) \in Q \text{ for all } i\} = V_B\Big(\sum Bh_i\Big),\end{aligned}$$

which is quasi-compact by (b). The proof is complete. $\square$

We next present our main result.

**Theorem 2.2.** *Let $f : A \to B$ be a ring homomorphism. Then $f$ satisfies* GGD *if and only if $f$ satisfies* GD.

*Proof.* The "only if" assertion is trivial. Conversely, assume that $f$ satisfies GD. Let $\mathcal{C} := \{P_\alpha \mid \alpha \in I\}$ be an increasing chain in $X := \operatorname{Spec}(A)$; that is, $P_\alpha \subseteq P_\beta$ whenever $\alpha \leq \beta$ in $I$. Let $P \in X$ be such that $P_\alpha \subseteq P$ for all $\alpha \in I$; let $Q \in Y := \operatorname{Spec}(B)$ be such that $f^{-1}(Q) = P$. Our task is to find an increasing chain $\mathcal{D} := \{Q_\alpha \mid \alpha \in I\}$ in $Y$ such that $f^{-1}(Q_\alpha) = P_\alpha$ and $Q_\alpha \subseteq Q$ for each $\alpha \in I$.

First, we reduce to the case in which $f$ also satisfies LO. To do so, replace $f : A \to B$ with the canonically induced ring homomorphism $A_P \to B_Q$. Of course, the latter map satisfies LO since $A \to B$ satisfies GD and $Q$ lies over $P$, cf. [**9**, Exercise 38, p. 45]; moreover, $A_P \to B_Q$ evidently inherits GD from $A \to B$. If we show that a suitable increasing chain $\mathcal{E}$ in $\operatorname{Spec}(B_Q)$ covers $\{P_\alpha \mid \alpha \in I\}$, then it suffices to take $\mathcal{D}$ to be the canonical chain in $Y$ that is the image of $\mathcal{E}$. By *abus de langage*, we replace $(A, B)$ with $(A_P, B_Q)$ and, thus, suppose henceforth that $f$ satisfies (GD and) LO.

By Lemma 2.1 (c), ${}^a f : Y_F \to X_F$ is a spectral map. Since $f$ satisfies LO, ${}^a f$ is also surjective. Thus, a realization result of Hochster

[**7**, Theorem 6 (a)] yields a ring homomorphism $g : C \to D$ such that, when $\mathrm{Spec}\,(C)$ and $\mathrm{Spec}\,(D)$ are each endowed with the Zariski topology, one has homeomorphims $h_1 : X_F \to \mathrm{Spec}\,(C)$ and $h_2 : Y_F \to \mathrm{Spec}\,(D)$ satisfying $({}^a g) \circ h_2 = h_1 \circ ({}^a f)$.

Recall from [**7**, p. 53] that any $T_0$-topological space $Z$ has an intrinsic partial order $\leq_Z$ defined as follows: for $u, v \in Z, u \leq_Z v \Leftrightarrow v$ is in the closure of $\{u\}$. (For instance, if $Z = \mathrm{Spec}\,(R)$ with the usual Zariski topology, then $\leq_Z$ is just inclusion, $\subseteq$, on $Z$.) It follows easily that any homeomorphism of spectral spaces induces an order isomorphism of posets. We claim that it now follows that $g : C \to D$ satisfies GU. To prove this claim, we need only observe that "going-up" behavior is exhibited (with respect to $\leq_{Y_F}$ and $\leq_{X_F}$) in ${}^a f : Y_F \to X_F$; and this, in turn, follows from the hypothesis that $f$ satisfies GD, in view of the fact [**7**, Proposition 8] that the intrinsic partial order $\leq_{Y_F}$, respectively $\leq_{X_F}$, is the opposite of the Zariski-induced intrinsic partial order on $Y$, respectively $X$.

Observe that $\{P_\alpha\}$ is a decreasing chain with respect to $\leq_{X_F}$; that is, $P_\alpha \geq_{X_F} P_\beta$ whenever $\alpha \leq \beta$ in $I$. Hence, $\{h_1(P_\alpha) \mid \alpha \in I\}$ forms a decreasing chain in $\mathrm{Spec}\,(C)$; in particular, $h_1(P_\alpha) \supseteq h_1(P_\beta) \supseteq h_1(P)$ whenever $\alpha \leq \beta$ in $I$. Moreover, since $g$ satisfies GU, the result of Kang-Oh [**8**, Corollary 12] that was mentioned in the Introduction ensures that $g$ satisfies GGU. Therefore, there exists a decreasing chain $\{W_\alpha \mid \alpha \in I\}$ in $\mathrm{Spec}\,(D)$ such that $g^{-1}(W_\alpha) = h_1(P_\alpha)$ and $W_\alpha \supseteq h_2(Q)$ for each $\alpha \in I$. Using the homeomorphism $h_2^{-1}$, we may construct the chain $\{h_2^{-1}(W_\alpha) \mid \alpha \in I\}$. Observe that $({}^a f)(h_2^{-1}(W_\alpha)) = (h_1^{-1} \circ ({}^a g))(W_\alpha) = P_\alpha$ for each $\alpha \in I$; since $\leq_{Y_F}$ is $\supseteq$, that $h_2^{-1}(W_\alpha) \subseteq Q$ for each $\alpha \in I$; and, similarly, that $h_2^{-1}(W_\alpha) \subseteq h_2^{-1}(W_\beta)$ whenever $\alpha \leq \beta$ in $I$. Therefore, it suffices to take $\mathcal{E} := \{h_2^{-1}(W_\alpha) \mid \alpha \in I\}$, viewed as a chain in $Y$. □

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