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q-TRIPLICATE INVERSE SERIES RELATIONS WITH APPLICATIONS TO q-SERIES

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ABSTRACT. q-triplicate inverse series relations are obtained and used to derive terminating summation formulas of q-series which include the generalization of Gessel and Stanton's result.

1. Introduction.

1.1 Notation and basic hypergeometric series. Here we recall some standard notation for q-series, and basic hypergeometric series [**6**].

Given a (fixed) complex number q with $|q| < 1$, the basic hypergeometric series is defined by

$$ {}_{r+1}\phi_r \left[\begin{matrix} a_1, a_2, \cdots, a_{r+1} \\ b_1, b_2, \cdots, b_r \end{matrix} ; q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1;q)_n \cdots (a_{r+1};q)_n}{(q;q)_n (b_1;q)_n \cdots (b_r;q)_n} z^n, $$

where, as before, the q-shifted factorial $(a;q)_n$ is given by

$$ \begin{aligned} (a;q)_n &:= (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad n \geq 1, \quad (a;q)_0 := 1. \\ (a;q)_{-n} &:= [(1-aq^{-1})(1-aq^{-2})\cdots(1-aq^{-n})]^{-1}, \quad n \geq 1. \end{aligned} $$

For brevity, we employ the usual notation

$$ \begin{aligned} (a_1, a_2, \dots, a_m; q)_n &= (a_1;q)_n (a_2;q)_n \cdots (a_m;q)_n, \\ (a_1, a_2, \dots, a_m; q)_\infty &= (a_1;q)_\infty (a_2;q)_\infty \cdots (a_m;q)_\infty. \end{aligned} $$

A basic hypergeometric series ${}_{r+1}\phi_r$ is called very well-poised if $a_i b_i = q a_0$ for $i = 1, 2, \dots, r$, and among the parameters a_i occur

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both $q\sqrt{a_0}$ and $-q\sqrt{a_0}$. If one of the numerator parameters $\{a_k\}$ is a negative integer, then the series becomes terminating. See [**6**, p. 25 and p. 125] for the criteria of when these series terminate, or, if not, when they converge.

A standard reference for basic hypergeometric series is Gasper and Rahman's text [**6**]. In our computations in the subsequent sections we frequently use some elementary identities of q-shifted factorials, listed in [**6**, Appendix I].

The following very well posed ${}_6\phi_5$ summation, cf., [**6**, (II. 20)],

$$(1.1)\quad {}_6\phi_5\left[\begin{matrix} a,\, q\sqrt{a},\, -q\sqrt{a},\, b,\, c,\, d \\ \sqrt{a},\, -\sqrt{a},\, aq/b,\, aq/c,\, aq/d \end{matrix}; q,\, \frac{aq}{bcd}\right] = \frac{(aq,\, aq/bc,\, aq/bd,\, aq/cd;\, q)_\infty}{(aq/b,\, aq/c,\, aq/d,\, aq/bcd;\, q)_\infty}$$

is an important summation in the theory of basic hypergeometric series which will be used in Section 3 to derive new summation formulae.

1.2 Inverse relations. In the following, we consider infinite upper triangular matrices $(a_{ij})_{0\le i\le j<\infty}$ and $(b_{ij})_{0\le i\le j<\infty}$ and infinite sequences $(f(n))_{0\le n<\infty}$ and $(g(n))_{0\le n<\infty}$.

We say that the infinite matrices $(a_{ij})_{0\le i\le j<\infty}$ and $(b_{ij})_{0\le i\le j<\infty}$ are inverses of each other if and only if the following orthogonality relation holds:

$$(1.2)\qquad \sum_{k=i}^{j} a_{ik}b_{kj} = \delta_{ij}, \quad i, j = 0, 1, 2\dots.$$

Since inverse matrices commute, we also have

$$(1.3)\qquad \sum_{k=i}^{j} b_{ik}a_{kj} = \delta_{ij}, \quad i, j = 0, 1, 2\dots.$$

It is immediate from the orthogonality relations (1.2) and (1.3) that the following inverse relations hold: Let $(a_{ij})_{0\le i\le j<\infty}$ and $(b_{ij})_{0\le i\le j<\infty}$ be infinite matrices which are inverses of each other. Then the system of equations

$$(1.4)\qquad f(n) = \sum_{k=0}^{n} a_{kn}g(k), \quad n = 0, 1, 2\dots$$

is equivalent to the system

$$g(n) = \sum_{k=0}^{n} b_{kn} f(k), \quad n = 0, 1, 2 \dots . \tag{1.5}$$

Inverse relations are a powerful tool for proving or deriving identities. For instance, given an identity in the form (1.5), we can immediately deduce (1.4), which may possibly be a new identity. After Gould and Hsu discovered the very general matrix inversions [**9**] and Carlitz found the q-analogue of Gould-Hsu inversions [**9**], Gessel and Stanton [**7, 8**] used matrix inversion to derive a number of basic hypergeometric summations and transformations. Chu also got hundreds of hypergeometric identities in papers [**2**–**5**]. Recently, Chu [**3**] presented the duplicate form of Gould-Hsu inversions and derived several balanced hypergeometric evaluations. Meanwhile, he also obtained the q-duplicate inverse relations and multiplicate inverse relations. But he did not give any applications. In [**12, 13**], we got some applications of Chu's q-duplicate inverse series relations on terminating basic hypergeometric series.

In this paper we establish another inverse relation, the q-triplicate inverse series relation, which is the q-analog of the triplicate form of Gould-Hsu inversions. After stating and proving the q-triplicate inverse series relations in Section 2, we combine our inverse relations with some basic hypergeometric summation formulas and transformation formulae to derive new formulas on terminating basic hypergeometric series in Sections 3, 4 and 5. In Section 6, we give the q-multiplicate inverse series relations and ask for applications of it.

2. q-triplicate inverse series relations. Before we present our main result, we define three parameters α, β, γ, which are used in following sections by $\alpha, \beta, \gamma \in \{0, -1, -2\}$ and α, β, γ, are distinct.

Let

$$\{A_{ij} | B_{ij}\}, \quad i = 0, 1, 2; \quad j = 0, 1, 2, \dots$$

be complex sequences and define the corresponding polynomials

$$\phi_i(x; 0) \equiv 1, \quad \phi_i(x; n) = \prod_{k=0}^{n-1} (A_{ik} + x B_{ik}), \quad n = 1, 2, \dots . \tag{2.1}$$

For convenience, products of the form $\prod_{j=u}^{u-1}$ are defined to be equal to 1, while for $u > v-1$ a product $\prod_{j=u}^{v-1}$ by definition is equal to 0.

Then the q-triplicate inverse series relations are:

Theorem 1. *With ϕ_i-polynomials defined by* (2.1), *the system of equations*

(2.2)

$$\begin{aligned}\Omega(n) = &\sum_{k\geq 0}(-1)^{3k}\begin{bmatrix} n \\ 3k \end{bmatrix}\frac{A_{2k}+q^{3k}B_{2k}}{\phi_0(q^n;k)\phi_1(q^n;k)\phi_2(q^n;k{+}1)}\,f(k)\\ &-\sum_{k\geq 0}(-1)^{3k}\begin{bmatrix} n \\ 1{+}3k \end{bmatrix}\frac{A_{1k}+q^{3k+1}B_{1k}}{\phi_0(q^n;k)\phi_1(q^n;k{+}1)\phi_2(q^n;k{+}1)}\,g(k)\\ &+\sum_{k\geq 0}(-1)^{3k}\begin{bmatrix} n \\ 2{+}3k \end{bmatrix}\frac{A_{0k}+q^{3k+2}B_{0k}}{\phi_0(q^n;k{+}1)\phi_1(q^n;k{+}1)\phi_2(q^n;k{+}1)}\,h(k)\end{aligned}$$

is equivalent to the system of equations

$$f(n) = \sum_{k=0}^{3n}(-1)^k\begin{bmatrix} 3n \\ k \end{bmatrix} q^{\binom{3n-k}{2}}\phi_0(q^k;n)\phi_1(q^k;n)\phi_2(q^k;n)\Omega(k), \tag{2.3}$$

$$\begin{aligned} g(n) = \sum_{k=0}^{3n+1}(-1)^k\begin{bmatrix} 3n+1 \\ k \end{bmatrix} q^{\binom{1+3n-k}{2}}\\ \times\, \phi_0(q^k;n)\phi_1(q^k;n)\phi_2(q^k;n+1)\Omega(k),\end{aligned} \tag{2.4}$$

$$\begin{aligned} h(n) = \sum_{k=0}^{3n+2}(-1)^k\begin{bmatrix} 3n+2 \\ k \end{bmatrix} q^{\binom{2+3n-k}{2}}\\ \times\, \phi_0(q^k;n)\phi_1(q^k;n+1)\phi_2(q^k;n+1)\Omega(k).\end{aligned} \tag{2.5}$$

Proof of Theorem 1. By proceeding as Chu Wenchang in the proof of [**2**, Theorem 3], to prove the equivalence between two systems of equations, it suffices to substitute one system into another and then verify the desired result.

Now, substituting (2.2) into the righthand side of (2.3), we have

(2.6)

$$\begin{aligned}
&\sum_{k=0}^{3n}(-1)^k \begin{bmatrix} 3n \\ k \end{bmatrix} q^{\binom{3n-k}{2}} \phi_0(q^k;n)\phi_1(q^k;n)\phi_2(q^k;n) \\
&\quad\times \sum_{m\geq 0}(-1)^{3m} \begin{bmatrix} k \\ 3m \end{bmatrix} \frac{A_{2m}+q^{3m}B_{2m}}{\phi_0(q^k;m)\phi_1(q^k;m)\phi_2(q^k;m+1)} f(m) \\
&\quad- \sum_{k=0}^{3n}(-1)^k \begin{bmatrix} 3n \\ k \end{bmatrix} q^{\binom{3n-k}{2}} \phi_0(q^k;n)\phi_1(q^k;n)\phi_2(q^k;n) \\
&\quad\times \sum_{m\geq 0}(-1)^{3m} \begin{bmatrix} k \\ 1+3m \end{bmatrix} \frac{A_{1m}+q^{3m+1}B_{1m}}{\phi_0(q^k;m)\phi_1(q^k;m+1)\phi_2(q^k;m+1)} g(m) \\
&\quad+ \sum_{k=0}^{3n}(-1)^k \begin{bmatrix} 3n \\ k \end{bmatrix} q^{\binom{3n-k}{2}} \phi_0(q^k;n)\phi_1(q^k;n)\phi_2(q^k;n) \\
&\quad\times \sum_{m\geq 0}(-1)^{3m} \begin{bmatrix} k \\ 2+3m \end{bmatrix} \frac{A_{0m}+q^{3m+2}B_{0m}}{\phi_0(q^k;m+1)\phi_1(q^k;m+1)\phi_2(q^k;m+1)} h(m).
\end{aligned}$$

Denote the first sum as $S_f(n)$, then

(2.7)

$$\begin{aligned}
S_f(n) &= \sum_{k=0}^{3n}(-1)^k \begin{bmatrix} 3n \\ k \end{bmatrix} q^{\binom{3n-k}{2}} \phi_0(q^k;n)\phi_1(q^k;n)\phi_2(q^k;n) \\
&\quad\times \sum_{m\geq 0}(-1)^{3m} \begin{bmatrix} k \\ 3m \end{bmatrix} \frac{A_{2m}+q^{3m}B_{2m}}{\phi_0(q^k;m)\phi_1(q^k;m)\phi_2(q^k;m+1)} f(m) \\
&= \sum_{m=0}^{n}(-1)^{3m} \begin{bmatrix} 3n \\ 3m \end{bmatrix} (A_{2m}+q^{3m}B_{2m}) f(m) \\
&\quad\times \sum_{k=3m}^{3n}(-1)^k \begin{bmatrix} 3n-3m \\ k-3m \end{bmatrix} q^{\binom{3n-k}{2}} \\
&\quad\times \frac{\phi_0(q^k;n)\phi_1(q^k;n)\phi_2(q^k;n)}{\phi_0(q^k;m)\phi_1(q^k;m)\phi_2(q^k;m+1)}.
\end{aligned}$$

When $n > m$, the fraction

$$P(q^k) := \frac{\phi_0(q^k;n)\phi_1(q^k;n)\phi_2(q^k;n)}{\phi_0(q^k;m)\phi_1(q^k;m)\phi_2(q^k;m+1)}$$

is a polynomial of degree $3n-3m-1$ in q^k, therefore it can be written as

$$P(q^k) := \sum_{t=0}^{3n-3m-1} c_t q^{kt}.$$

Considering that

$$(x;q)_n = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} x^k,$$

then we have

$$\begin{aligned} S_f(n) &= f(n) + \sum_{m=0}^{n-1} (-1)^{3m} \begin{bmatrix} 3n \\ 3m \end{bmatrix} (A_{2m} + q^{3m} B_{2m}) f(m) \\ &\quad \times \sum_{t=0}^{3n-3m-1} c_t q^{3mt + \binom{3n-3m}{2}} (q^{t-3n+3m+1}; q)_{3n-3m} \\ &= f(n). \end{aligned} \tag{2.8}$$

Similarly, we can assert that the second sum and the third sum in (2.6) all equal zero. So (2.6) equals $f(n)$.

Similarly, by substituting (2.2) into the righthand side of (2.4), or replacing the righthand side of (2.5) by (2.2), we can also demonstrate that the corresponding sums reduce to $g(n)$ or $h(n)$. □

In Theorem 1, by taking

$$\phi_0(q^n;k) = (aq^{n+3+\alpha};q^3)_k, \phi_1(q^n;k) = (aq^{n+3+\beta};q^3)_k$$

and

$$\phi_2(q^n;k) = (aq^{n+\gamma};q^3)_k,$$

after simplification, we can get the following result.

Corollary 2. *The solutions of the system of equations*

(2.9)

$$\begin{aligned}\Omega(n) &= \sum_{k\geq 0}(-1)^{3k}\begin{bmatrix} n \\ 3k\end{bmatrix}\frac{1-aq^{6k+\gamma}}{(aq^{n+3+\alpha};q^3)_k(aq^{n+3+\beta};q^3)_k(aq^{n+\gamma};q^3)_{k+1}}\,f(k)\\ &\quad -\sum_{k\geq 0}(-1)^{3k}\begin{bmatrix} n \\ 1+3k\end{bmatrix}\\ &\qquad\times\frac{1-aq^{6k+4+\beta}}{(aq^{n+3+\alpha};q^3)_k(aq^{n+3+\beta};q^3)_{k+1}(aq^{n+\gamma};q^3)_{k+1}}\,g(k)\\ &\quad +\sum_{k\geq 0}(-1)^{3k}\begin{bmatrix} n \\ 2+3k\end{bmatrix}\\ &\qquad\times\frac{1-aq^{6k+5+\alpha}}{(aq^{n+3+\alpha};q^3)_{k+1}(aq^{n+3+\beta};q^3)_{k+1}(aq^{n+\gamma};q^3)_{k+1}}\,h(k)\end{aligned}$$

are given by the following summations

(2.10)

$$f(n)=q^{\binom{3n}{2}}\sum_{k=0}^{3n}\frac{(q^{-3n};q)_k q^k}{(q;q)_k}\frac{(aq^{k+1};q)_{3n}(aq^{k+\gamma};q^3)_n}{(aq^{k+3+\gamma};q^3)_n}\,\Omega(k),$$

(2.11)

$$g(n)=q^{\binom{1+3n}{2}}\sum_{k=0}^{3n+1}\frac{(q^{-3n-1};q)_k q^k}{(q;q)_k}\,(aq^{k+1};q)_{3n}(1-aq^{k+\gamma})\,\Omega(k),$$

(2.12)

$$\begin{aligned}h(n)=q^{\binom{2+3n}{2}}\sum_{k=0}^{3n+2}\frac{(q^{-3n-2};q)_k q^k}{(q;q)_k}\,(aq^{k+1};q)_{3n}&\\ \times(1-aq^{3n+k+3+\beta})(1-aq^{k+\gamma})\Omega(k).&\end{aligned}$$

Similar to Corollary 2, if, in Theorem 1, putting

$$\begin{aligned}\phi_0(q^n;k)&=(aq^{n+\alpha};q^3)_k,\phi_1(q^n;k)=(aq^{n+\beta};q^3)_k,\phi_2(q^n;k)\\ &=(aq^{n+\gamma};q^3)_k,\end{aligned}$$

then by straightforward calculations we can obtain

Corollary 3. *The solutions of the system of equations*

(2.13)

$$\Omega(n)=\sum_{k\geq 0}(-1)^{3k}\begin{bmatrix} n\\ 3k\end{bmatrix}\frac{1-aq^{6k+\gamma}}{(aq^{n+\alpha};q^3)_k(aq^{n+\beta};q^3)_k(aq^{n+\gamma};q^3)_{k+1}}f(k)$$
$$-\sum_{k\geq 0}(-1)^{3k}\begin{bmatrix} n\\ 1+3k\end{bmatrix}$$
$$\times\frac{1-aq^{6k+1+\beta}}{(aq^{n+\alpha};q^3)_k(aq^{n+\beta};q^3)_{k+1}(aq^{n+\gamma};q^3)_{k+1}}g(k)$$
$$+\sum_{k\geq 0}(-1)^{3k}\begin{bmatrix} n\\ 2+3k\end{bmatrix}$$
$$\times\frac{1-aq^{6k+2+\alpha}}{(aq^{n+\alpha};q^3)_{k+1}(aq^{n+\beta};q^3)_{k+1}(aq^{n+\gamma};q^3)_{k+1}}h(k)$$

are given by the following summations

(2.14) $$f(n)=q^{\binom{3n}{2}}\sum_{k=0}^{3n}\frac{(q^{-3n};q)_k q^k}{(q;q)_k}(aq^{k-2};q)_{3n}\,\Omega(k),$$

(2.15) $$g(n)=q^{\binom{1+3n}{2}}\sum_{k=0}^{3n+1}\frac{(q^{-3n-1};q)_k q^k}{(q;q)_k}$$
$$\times(aq^{k-2};q)_{3n}(1-aq^{3n+k+\gamma})\,\Omega(k),$$

(2.16) $$h(n)=q^{\binom{2+3n}{2}}\sum_{k=0}^{3n+2}\frac{(q^{-3n-2};q)_k q^k}{(q;q)_k}(aq^{k-2};q)_{3n}$$
$$\times(1-aq^{3n+k+\beta})(1-aq^{3n+k+\gamma})\,\Omega(k).$$

In the following section, we shall give some terminating summation formulas by using the q-inverse series relations.

3. Summation formulas. By replacing q by q^3 and then setting $b = q^{-n}$, $c = q^{-n+1}$ and $d = q^{-n+2}$ in (1.1), we establish that

$$\sum_{k\geq 0} \frac{(a, q^3\sqrt{a}, -q^3\sqrt{a}, q^{-n}, q^{-n+1}, q^{-n+2}; q^3)_k}{(q^3, \sqrt{a}, -\sqrt{a}, aq^{n+1}, aq^{n+2}, aq^{n+3}; q^3)_k} (aq^{3n})^k = \frac{(a;q^3)_n (aq;q)_n}{(a;q)_{2n}}.$$

If, setting

$$\omega(n) = \frac{(a;q^3)_n (aq;q)_n}{(a;q)_{2n}}, \quad T(n) = \frac{(q, q^2, a; q^3)_n a^n}{(1-a)},$$

then we have

$$\omega(n) = \sum_{k\geq 0} \begin{bmatrix} n \\ 3k \end{bmatrix} \frac{(-1)^{3k}}{(aq^{n+1}, aq^{n+2}, aq^{n+3}; q^3)_k} (1-aq^{6k}) q^{\binom{3k}{2}} T(k). \tag{3.1}$$

Since (3.1) can be reformulated as

$$\frac{\omega(n)}{(1-aq^{n+\gamma})} = \sum_{k\geq 0} \begin{bmatrix} n \\ 3k \end{bmatrix} \frac{(-1)^{3k}(1-aq^{6k+\gamma})}{(aq^{n+3+\alpha}, aq^{n+3+\beta}; q^3)_k (aq^{n+\gamma}; q^3)_{k+1}} \times \frac{(1-aq^{6k})}{(1-aq^{6k+\gamma})} q^{\binom{3k}{2}} T(k).$$

by (2.10), (2.11) and (2.12), we obtain

$$\sum_{k=0}^{3n} \frac{(q^{-3n};q)_k q^k}{(q;q)_k} \frac{(aq^{k+1};q)_{3n} (aq^{k+3+\gamma};q^3)_{n-1}}{(aq^{k+3+\gamma};q^3)_n} \omega(k) = \frac{(1-aq^{6n})}{(1-aq^{6n+\gamma})} T(n),$$

$$\sum_{k=0}^{3n+1} \frac{(q^{-3n-1};q)_k q^k}{(q;q)_k} (aq^{k+1};q)_{3n} \, \omega(k) = 0,$$

$$\sum_{k=0}^{3n+2} \frac{(q^{-3n-2};q)_k q^k}{(q;q)_k} (aq^{k+1};q)_{3n} (1-aq^{3n+k+3+\beta}) \, \omega(k) = 0.$$

They can be simplified to give the formulas

$$\sum_{k=0}^{3n} \frac{(q^{-3n}, aq^{3n+1}, aq^{3n+\gamma}; q)_k (a; q^3)_k q^k}{(q, aq^{3n+1+\gamma}; q)_k (a; q)_{2k}} = \frac{(1-aq^{6n})(1-aq^{3n+\gamma})}{(1-a)(1-aq^{6n+\gamma})} \frac{(q, q^2, a; q^3)_n a^n}{(aq; q)_{3n}},$$

$$\sum_{k=0}^{3n+1} \frac{(q^{-3n-1}, aq^{3n+1}; q)_k (a; q^3)_k q^k}{(q; q)_k (a; q)_{2k}} = 0,$$

$$\sum_{k=0}^{3n+2} \frac{(q^{-3n-2}, aq^{3n+1}, aq^{3n+4+\beta}; q)_k (a; q^3)_k q^k}{(q, aq^{3n+3+\beta}; q)_k (a; q)_{2k}} = 0.$$

Considering that

$$\begin{aligned}(a; q^3)_k &= (a^{1/3}, \omega\, a^{1/3}, \omega^2 a^{1/3}; q)_k, \\ (a; q)_{2k} &= (\sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}; q)_k,\end{aligned}$$

where $\omega := e^{2\pi i/3}$, we have

$$(3.2) \quad {}_6\phi_5\left[\begin{matrix} q^{-3n}, aq^{3n+1}, aq^{3n+\gamma}, a^{1/3}, \omega a^{1/3}, \omega^2 a^{1/3} \\ aq^{3n+1+\gamma}, \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq} \end{matrix}; q, q\right] = \frac{(1-aq^{6n})(1-aq^{3n+\gamma})}{(1-aq^{6n+\gamma})(1-a^{3n})} \frac{a^n (q, q^2, a; q^3)_n}{(a; q)_{3n}},$$

$$(3.3) \quad {}_5\phi_4\left[\begin{matrix} q^{-3n-1}, aq^{3n+1}, a^{1/3}, \omega a^{1/3}, \omega^2 a^{1/3} \\ \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq} \end{matrix}; q, q\right] = 0,$$

$$(3.4) \quad {}_6\phi_5\left[\begin{matrix} q^{-3n-2}, aq^{3n+1}, aq^{3n+4+\beta}, a^{1/3}, \omega a^{1/3}, \omega^2 a^{1/3} \\ aq^{3n+3+\beta}, \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq} \end{matrix}; q, q\right] = 0.$$

The special case $\gamma = 0$, $\beta = -2$ of (3.2) is of particular interest to us, and we state it as a separate result.

$$(3.5)\quad {}_5\phi_4\left[\begin{matrix} q^{-n}, aq^n, a^{1/3}, \omega\, a^{1/3}, \omega^2 a^{1/3} \\ \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq} \end{matrix}; q, q\right] = \begin{cases} a^{n/3}(q, q^2, a; q^3)_n/(a;q)_n, & \text{if } n = 0 \pmod 3, \\ 0, & \text{if } n \neq 0 \pmod 3, \end{cases}$$

which appears previously as (4.32) of [**7**].

4. Further summation formulas. We begin this section by giving a useful formula

$$(4.1)\quad (1-d)(1-ea)(1-fa) + d(1-a)(1-b)(1-c) = (1-a)(1-bd)(1-cd) + a(1-d)(1-e)(1-f),$$

where $efa = bcd$. Formula (4.1) can be derived by setting $n = 1$ in Sear's transformations of terminating balanced ${}_4\phi_3$ series, [**6**, (III.15)] and replacing e, f, b and c by ea, fa, bd, cd, respectively.

If we set $f = c$ in (4.1) and replace a, b, c, d, by $aq^{n+3k+\alpha}$, $q^{\alpha-\beta+1}$, $q^{n-3k}, aq^{6k+\beta}$, respectively, then we have

(4.2)

$$1 - aq^{2n+\alpha} = \frac{1}{1-aq^{6k+\beta}}(1-aq^{n+3k+\alpha})(1-aq^{n+3k+\beta}) + \frac{aq^{6k+\alpha+1}(1-q^{\beta-\alpha-1})}{(1-aq^{6k+\beta})(1-aq^{6k+\alpha+1})}(1-q^{n-3k})(1-aq^{n+3k+\alpha}) - \frac{aq^{6k+\alpha+1}}{1-aq^{6k+\alpha+1}}(1-q^{n-3k})(1-q^{n-3k-1}),$$

which can be used in (3.1) to get

$$
\begin{aligned}
&\frac{(1-aq^{2n+\alpha})\omega(n)}{(1-aq^{n+\alpha})(1-aq^{n+\beta})(1-aq^{n+\gamma})}\\
&\quad=\sum_{k\geq 0}\begin{bmatrix} n\\ 3k\end{bmatrix}\frac{(-1)^{3k}(1-aq^{6k+\gamma})}{(aq^{n+\alpha},aq^{n+\beta};q^3)_k(aq^{n+\gamma};q^3)_{k+1}}\\
&\qquad\times\frac{(1-aq^{6k})q^{\binom{3k}{2}}T(k)}{(1-aq^{6k+\beta})(1-aq^{6k+\gamma})}\\
&\qquad-\sum_{k\geq 0}\begin{bmatrix} n\\ 3k+1\end{bmatrix}\frac{(-1)^{3k}(1-aq^{6k+1+\beta})}{(aq^{n+\alpha};q^3)_k(aq^{n+\beta},aq^{n+\gamma};q^3)_{k+1}}\\
&\qquad\times\frac{-aq^{6k+\alpha+1}(1-aq^{6k})(1-q^{3k+1})(1-q^{\beta-\alpha-1})}{(1-aq^{6k+\beta})(1-aq^{6k+1+\beta})(1-aq^{6k+\alpha+1})}\,q^{\binom{3k}{2}}T(k)\\
&\qquad+\sum_{k\geq 0}\begin{bmatrix} n\\ 3k+2\end{bmatrix}\frac{(-1)^{3k}(1-aq^{6k+2+\alpha})}{(aq^{n+\alpha},aq^{n+\beta},aq^{n+\gamma};q^3)_{k+1}}\\
&\qquad\times\frac{-aq^{6k+\alpha+1}(1-aq^{6k})(1-q^{3k+1})(1-q^{3k+2})}{(1-aq^{6k+\alpha+1})(1-aq^{6k+2+\alpha})}\,q^{\binom{3k}{2}}T(k).
\end{aligned}
\tag{4.3}
$$

By using Corollary 3, we get the terminating summation formulas

$$
\begin{aligned}
&\sum_{k=0}^{3n}\frac{(q^{-3n},aq^{3n-2};q)_k q^k}{(q;q)_k}\,\frac{(aq^{2+\alpha};q^2)_k}{(aq^{\alpha};q^2)_k}\,\frac{\omega(k)}{(aq;q)_k}\\
&\quad=\frac{(1-aq^{6n})}{(1-aq^{\alpha})(1-aq^{6n+\beta})(1-aq^{6n+\gamma})}\,\frac{T(n)}{(aq;q)_{3n-3}},
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{k=0}^{3n+1}\frac{(q^{-3n-1},aq^{3n-2},aq^{3n+1+\gamma};q)_k q^k}{(q,aq^{3n+\gamma};q)_k}\,\frac{(aq^{2+\alpha};q^2)_k}{(aq^{\alpha};q^2)_k}\,\frac{\omega(k)}{(aq;q)_k}\\
&\quad=\frac{(1-aq^{6n})(1-q^{3n+1})(1-q^{1+\alpha-\beta})}{(1-aq^{3n+\gamma})(1-aq^{\alpha})(1-aq^{6n+\beta})(1-aq^{6n+1+\beta})(1-aq^{6n+\alpha+1})}\\
&\quad\times\frac{aq^{3n+\beta}T(n)}{(aq;q)_{3n-3}},
\end{aligned}
$$

$$\sum_{k=0}^{3n+2} \frac{(q^{-3n-2}, aq^{3n-2}, aq^{3n+1+\beta}; q)_k q^k}{(q, aq^{3n+\beta}; q)_k} \frac{(aq^{3n+1+\gamma}; q)_k}{(aq^{3n+\gamma}; q)_k}$$
$$\times \frac{(aq^{2+\alpha}; q^2)_k}{(aq^\alpha; q^2)_k} \frac{\omega(k)}{(aq; q)_k}$$
$$= \frac{-aq^\alpha(1-aq^{6n})(1-q^{3n+1})(1-q^{3n+2})}{(1-aq^{3n+\beta})(1-aq^{3n+\gamma})(1-aq^\alpha)(1-aq^{6n+\alpha+1})(1-aq^{6n+2+\alpha})}$$
$$\times \frac{T(n)}{(aq; q)_{3n-3}},$$

which can be written in the equivalent form

(4.4)
$${}_7\phi_6 \left[\begin{matrix} q^{-3n}, aq^{3n-2}, q\sqrt{aq^\alpha}, -q\sqrt{aq^\alpha}, a^{1/3}, \omega a^{1/3}, \omega^2 a^{1/3} \\ \sqrt{aq^\alpha}, -\sqrt{aq^\alpha}, \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq} \end{matrix} ; q, q \right]$$
$$= \frac{(1-aq^{6n})}{(1-aq^\alpha)(1-aq^{6n+\beta})(1-aq^{6n+\gamma})} \frac{a^n (q, q^2, a; q^3)_n}{(a; q)_{3n-2}},$$

(4.5)
$${}_8\phi_7 \left[\begin{matrix} q^{-3n-1}, aq^{3n-2}, aq^{3n+\gamma+1}, q\sqrt{aq^\alpha}, -q\sqrt{aq^\alpha}, a^{1/3}, \omega\, a^{1/3}, \omega^2 a^{1/3} \\ aq^{3n+\gamma}, \sqrt{aq^\alpha}, -\sqrt{aq^\alpha}, \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq} \end{matrix} ; q, q \right]$$
$$= \frac{a^{n+1} q^{3n+\beta}(1-aq^{6n})(1-q^{3n+1})(1-q^{1+\alpha-\beta})}{(1-aq^{3n+\gamma})(1-aq^\alpha)(1-aq^{6n+\beta})(1-aq^{6n+1+\beta})(1-aq^{6n+\alpha+1})}$$
$$\times \frac{(q, q^2, a; q^3)_n}{(a; q)_{3n-2}},$$

(4.6)
$${}_9\phi_8 \left[\begin{matrix} q^{-3n-2}, aq^{3n-2}, aq^{3n+\beta+1}, aq^{3n+\gamma+1}, q\sqrt{aq^\alpha}, -q\sqrt{aq^\alpha}, a^{1/3}, \omega\, a^{1/3}, \omega^2 a^{1/3} \\ aq^{3n+\beta}, aq^{3n+\gamma}, \sqrt{aq^\alpha}, -\sqrt{aq^\alpha}, \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq} \end{matrix} ; q, q \right]$$
$$= \frac{-a^{n+1} q^\alpha (1-aq^{6n})(1-q^{3n+1})(1-q^{3n+2})}{(1-aq^{3n+\beta})(1-aq^{3n+\gamma})(1-aq^\alpha)(1-aq^{6n+\alpha+1})(1-aq^{6n+2+\alpha})}$$
$$\times \frac{(q, q^2, a; q^3)_n}{(a; q)_{3n-2}}.$$

Similarly, by setting $c = fa$ in (4.1) and replacing a, b, d and f by $aq^{6k+\beta}$, $1/aq^{6k+\alpha}$, $aq^{n+3k+\alpha}$ and $1/a^2q^{12k+\alpha+\beta+1}$, respectively, we yield

$$\begin{aligned}
q^n &= \frac{q^{3k}(1-aq^{n+3k+\alpha})(1-aq^{n+3k+\beta})}{(1-aq^{6k+\alpha})(1-aq^{6k+\beta})} \\
&\quad - \frac{q^{3k}(1-a^2q^{12k+\beta+\alpha+1})(1-q^{n-3k})(1-aq^{n+3k+\alpha})}{(1-aq^{6k+\alpha})(1-aq^{6k+\beta})(1-aq^{6k+\alpha+1})} \\
&\quad + \frac{aq^{9k+\alpha+1}(1-q^{n-3k})(1-q^{n-3k-1})}{(1-aq^{6k+\alpha})(1-aq^{6k+\alpha+1})}.
\end{aligned} \tag{4.7}$$

Using this factor-separation in (3.1), we obtain the summation formula

$$\begin{aligned}
&\frac{q^n\omega(n)}{(1-aq^{n+\alpha})(1-aq^{n+\beta})(1-aq^{n+\gamma})} \\
&= \sum_{k\geq 0} \begin{bmatrix} n \\ 3k \end{bmatrix} \frac{(-1)^{3k}(1-aq^{6k+\gamma})}{(aq^{n+\alpha}, aq^{n+\beta}; q^3)_k (aq^{n+\gamma}; q^3)_{k+1}} \\
&\quad \times \frac{(1-aq^{6k})q^{\binom{3k}{2}+3k}T(k)}{(1-aq^{6k+\alpha})(1-aq^{6k+\beta})(1-aq^{6k+\gamma})} \\
&\quad - \sum_{k\geq 0} \begin{bmatrix} n \\ 3k+1 \end{bmatrix} \frac{(-1)^{3k}(1-aq^{6k+1+\beta})}{(aq^{n+\alpha}; q^3)_k (aq^{n+\beta}, aq^{n+\gamma}; q^3)_{k+1}} \\
&\quad \times \frac{(1-aq^{6k})(1-q^{3k+1})(1-a^2q^{12k+\beta+\alpha+1})}{(1-aq^{6k+\alpha})(1-aq^{6k+\beta})(1-aq^{6k+1+\beta})(1-aq^{6k+\alpha+1})} \\
&\quad \times q^{\binom{3k}{2}+3k}T(k) + \sum_{k\geq 0} \begin{bmatrix} n \\ 3k+2 \end{bmatrix} \frac{(-1)^{3k}(1-aq^{6k+\alpha+2})}{(aq^{n+\alpha}, aq^{n+\beta}, aq^{n+\gamma}; q^3)_{k+1}} \\
&\quad \times \frac{aq^{9k+\alpha+1}(1-aq^{6k})(1-q^{3k+1})(1-q^{3k+2})}{(1-aq^{6k+\alpha})(1-aq^{6k+\alpha+1})(1-aq^{6k+\alpha+2})}\, q^{\binom{3k}{2}}T(k).
\end{aligned} \tag{4.8}$$

By using (2.14), (2.15) and (2.16), we get the following three terminating summation formulas

$$\sum_{k=0}^{3n} \frac{(q^{-3n}, aq^{3n-2}; q)_k}{(q;q)_k} \frac{q^{2k}\omega(k)}{(aq;q)_k}$$
$$= \frac{(1-aq^{6n})}{(1-aq^{6n+\alpha})(1-aq^{6n+\beta})(1-aq^{6n+\gamma})} \frac{q^{3n}T(n)}{(aq;q)_{3n-3}}$$
$$= \frac{q^{3n}T(n)}{(1-aq^{6n-1})(1-aq^{6n-2})(aq;q)_{3n-3}},$$

$$\times \sum_{k=0}^{3n+1} \frac{(q^{-3n-1}, aq^{3n-2}; q)_k}{(q;q)_k} \frac{(aq^{3n+1+\gamma}; q)_k}{(aq^{3n+\gamma}; q)_k} \frac{q^{2k}\omega(k)}{(aq;q)_k}$$
$$= \frac{(1-aq^{6n})(1-q^{3n+1})(1-a^2q^{12n+\beta+\alpha+1})}{(1-aq^{3n+\gamma})(1-aq^{6n+\alpha})(1-aq^{6n+\beta})(1-aq^{6n+1+\beta})(1-aq^{6n+\alpha+1})}$$
$$\times \frac{T(n)}{(aq;q)_{3n-3}},$$

$$\times \sum_{k=0}^{3n+2} \frac{(q^{-3n-2}, aq^{3n-2}; q)_k}{(q;q)_k} \frac{(aq^{3n+1+\beta}; q)_k}{(aq^{3n+\beta}; q)_k} \frac{(aq^{3n+1+\gamma}; q)_k}{(aq^{3n+\gamma}; q)_k} \frac{q^{2k}\omega(k)}{(aq;q)_k}$$
$$= \frac{(1-aq^{6n})(1-q^{3n+1})(1-q^{3n+2})}{(1-aq^{3n+\beta})(1-aq^{3n+\gamma})(1-aq^{6n+\alpha})(1-aq^{6n+\alpha+1})(1-aq^{6n+\alpha+2})}$$
$$\times \frac{aq^{3n+\alpha}T(n)}{(aq;q)_{3n-3}},$$

which also can be written in the equivalent form

$$(4.9) \quad {}_5\phi_4 \left[\begin{matrix} q^{-3n}, aq^{3n-2}, a^{1/3}, \omega a^{1/3}, \omega^2 a^{1/3} \\ \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq} \end{matrix} ; q, q^2 \right]$$
$$= \frac{a^n q^{3n}(q, q^2, a; q^3)_n}{(1-aq^{6n-1})(1-aq^{6n-2})(a;q)_{3n-2}},$$

(4.10)

$$
{}_6\phi_5\left[\begin{matrix} q^{-3n-1}, aq^{3n-2}, aq^{3n+\gamma+1}, a^{1/3}, \omega a^{1/3}, \omega^2 a^{1/3} \\ aq^{3n+\gamma}, \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq} \end{matrix}; q, q^2\right]
$$

$$
= \frac{(1-aq^{6n})(1-q^{3n+1})(1-a^2q^{12n+\beta+\alpha+1})}{(1-aq^{3n+\gamma})(1-aq^{6n+\alpha})(1-aq^{6n+\alpha+1})(1-aq^{6n+\beta})(1-aq^{6n+1+\beta})} \times \frac{a^n(q, q^2, a; q^3)_n}{(a;q)_{3n-2}},
$$

(4.11)

$$
{}_7\phi_6\left[\begin{matrix} q^{-3n-2}, aq^{3n-2}, aq^{3n+\beta+1}, aq^{3n+\gamma+1}, a^{1/3}, \omega a^{1/3}, \omega^2 a^{1/3} \\ aq^{3n+\beta}, aq^{3n+\gamma}, \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq} \end{matrix}; q, q^2\right]
$$

$$
= \frac{(1-aq^{6n})(1-q^{3n+1})(1-q^{3n+2})}{(1-aq^{3n+\beta})(1-aq^{3n+\gamma})(1-aq^{6n+\alpha})(1-aq^{6n+\alpha+1})(1-aq^{6n+\alpha+2})} \times \frac{a^{n+1}q^{3n+\alpha}(q, q^2, a; q^3)_n}{(a;q)_{3n-2}}.
$$

5. Applications of Slater's identities. In her paper [**10**], Slater obtained the following terminating formula:

(5.1)

$$
\frac{1-q^2}{(q;q)_n(q^2;q)_{n+1}} + \sum_{k=1}^{[n/3]}\left\{\frac{q^{6k^2-k}}{(q;q)_{n-3k+1}(q^2;q)_{n+3k-1}} - \frac{q^{6k^2-7k+2}}{(q;q)_{n-3k+2}(q^2;q)_{n+3k-2}} + \frac{q^{6k^2+k}}{(q;q)_{n-3k}(q^2;q)_{n+3k}} - \frac{q^{6k^2+7k+2}}{(q;q)_{n-3k-1}(q^2;q)_{n+3k+1}}\right\} = \frac{1}{(q^2;q)_{2n}},
$$

(5.2)

$$
\frac{q^n(1-q^2)}{(q;q)_n(q^2;q)_{n+1}} + \sum_{k=1}^{[n/3]}\left\{\frac{q^{6k^2-4k}}{(q;q)_{n-3k+1}(q^2;q)_{n+3k-1}} - \frac{q^{6k^2-4k}}{(q;q)_{n-3k+2}(q^2;q)_{n+3k-2}} + \frac{q^{6k^2+4k}}{(q;q)_{n-3k}(q^2;q)_{n+3k}} - \frac{q^{6k^2+4k}}{(q;q)_{n-3k-1}(q^2;q)_{n+3k+1}}\right\} = \frac{q^n}{(q^2;q)_{2n}},
$$

(5.3)
$$\frac{1-q^2}{(q;q)_n(q^2;q)_{n+1}} + \sum_{k=1}^{[n/3]}\left\{\frac{q^{3k^2+k}}{(q;q)_{n-3k+1}(q^2;q)_{n+3k-1}} - \frac{q^{3k^2-5k+2}}{(q;q)_{n-3k+2}(q^2;q)_{n+3k-2}} + \frac{q^{3k^2-k}}{(q;q)_{n-3k}(q^2;q)_{n+3k}} - \frac{q^{3k^2+5k+2}}{(q;q)_{n-3k-1}(q^2;q)_{n+3k+1}}\right\} = \frac{q^{n^2}}{(q^2;q)_{2n}},$$

(5.4)
$$\frac{q^n(1-q^2)}{(q;q)_n(q^2;q)_{n+1}} + \sum_{k=1}^{[n/3]}\left\{\frac{q^{3k^2-2k}}{(q;q)_{n-3k+1}(q^2;q)_{n+3k-1}} - \frac{q^{3k^2-2k}}{(q;q)_{n-3k+2}(q^2;q)_{n+3k-2}} + \frac{q^{3k^2+2k}}{(q;q)_{n-3k}(q^2;q)_{n+3k}} - \frac{q^{3k^2+2k}}{(q;q)_{n-3k-1}(q^2;q)_{n+3k+1}}\right\} = \frac{q^{n^2+n}}{(q^2;q)_{2n}}.$$

By using Corollary 2 in these formulae we can obtain some new terminating formulae. To simplify our proceedings we consider the special case

$$\alpha=-1, \quad \beta=-2, \quad \gamma=0, \quad \text{and} \quad a=q$$

of Corollary 2, we can get the following result:

Corollary 4. *The solutions of the system of equations*

$$\begin{aligned}\Omega(n) &= \sum_{k\geq 0}\begin{bmatrix} n \\ 3k\end{bmatrix}\frac{(-1)^{3k}(1-q^{6k+1})}{(q^{n+1};q)_{3k+1}}\,f(k) \\ &- \sum_{k\geq 0}\begin{bmatrix} n \\ 1+3k\end{bmatrix}\frac{(-1)^{3k}(1-q^{6k+3})}{(q^{n+1};q)_{3k+2}}\,g(k) \\ &+ \sum_{k\geq 0}\begin{bmatrix} n \\ 2+3k\end{bmatrix}\frac{(-1)^{3k}(1-q^{6k+5})}{(q^{n+1};q^3)_{3k+3}}\,h(k)\end{aligned} \tag{5.5}$$

are given by the following summations

$$f(n) = (q;q)_{3n} q^{\binom{3n}{2}} \sum_{k=0}^{3n} \frac{(q^{-3n}, q^{3n+1};q)_k q^k}{(q,q;q)_k} \Omega(k), \tag{5.6}$$

$$g(n) = (q;q)_{3n+1} q^{\binom{3n+1}{2}} \sum_{k=0}^{3n+1} \frac{(q^{-3n-1}, q^{3n+2};q)_k q^k}{(q,q;q)_k} \Omega(k), \tag{5.7}$$

$$h(n) = (q;q)_{3n+2} q^{\binom{2+3n}{2}} \sum_{k=0}^{3n+2} \frac{(q^{-3n-2}, q^{3n+3};q)_k q^k}{(q,q;q)_k} \Omega(k). \tag{5.8}$$

Note that formula (5.1) can be rewritten as

$$\begin{aligned}
&\frac{(q;q)_n}{(q^{1+n};q)_{n+1}} \\
&= \sum_{k\geq 0} \begin{bmatrix} n \\ 3k \end{bmatrix} \frac{(-1)^{3k}(1-q^{6k+1})}{(q^{n+1};q)_{3k+1}} \frac{(-1)^{3k} q^{6k^2+k}(q;q)_{3k}}{(1-q^{6k+1})} \\
&\quad - \sum_{k\geq 0} \begin{bmatrix} n \\ 1+3k \end{bmatrix} \frac{(-1)^{3k}(1-q^{6k+3})}{(q^{n+1};q)_{3k+2}} \\
&\quad \times \frac{(-1)^{3k} q^{6k^2+5k+1}(1+q^{2k})(q;q)_{3k+1}}{(1-q^{6k+3})} \\
&\quad + \sum_{k\geq 0} \begin{bmatrix} n \\ 2+3k \end{bmatrix} \frac{(-1)^{3k}(1-q^{6k+5})}{(q^{n+1};q^3)_{3k+3}} \frac{(-1)^{3k} q^{6k^2+11k+5}(q;q)_{3k+2}}{(1-q^{6k+5})}.
\end{aligned}$$

By using Corollary 4, we can get

$$\sum_{k=0}^{3n} \frac{(q^{-3n}, q^{3n+1};q)_k q^k}{(q;q)_{2k+1}} = \frac{(-1)^{3n} q^{n(3n+5)/2}}{1-q^{6n+1}}, \tag{5.9}$$

$$\sum_{k=0}^{3n+1} \frac{(q^{-3n-1}, q^{3n+2};q)_k q^k}{(q;q)_{2k+1}} = \frac{(-1)^{3n} q^{(3n+1)(n+2)/2}(1+q^{2n+1})}{1-q^{6n+3}}, \tag{5.10}$$

$$\sum_{k=0}^{3n+2} \frac{(q^{-3n-2}, q^{3n+3};q)_k q^k}{(q;q)_{2k+1}} = \frac{(-1)^{3n} q^{(3n^2+13n+8)/2}}{1-q^{6n+5}}. \tag{5.11}$$

Similarly, by using (5.2), (5.3) and (5.4), we can obtain

$$\sum_{k=0}^{3n} \frac{(q^{-3n}, q^{3n+1}; q)_k q^{2k}}{(q;q)_{2k+1}} = \frac{(-1)^{3n} q^{n(3n+11)/2}}{1-q^{6n+1}}, \tag{5.12}$$

$$\sum_{k=0}^{3n+1} \frac{(q^{-3n-1}, q^{3n+2}; q)_k q^{2k}}{(q;q)_{2k+1}} = \frac{(-1)^{3n} q^{n(3n+5)/2}(1+q^{4n+2})}{1-q^{6n+3}}, \tag{5.13}$$

$$\sum_{k=0}^{3n+2} \frac{(q^{-3n-2}, q^{3n+3}; q)_k q^{2k}}{(q;q)_{2k+1}} = \frac{(-1)^{3n} q^{(3n+1)(n+2)/2}}{1-q^{6n+5}}, \tag{5.14}$$

$$\sum_{k=0}^{3n} \frac{(q^{-3n}, q^{3n+1}; q)_k q^{k^2+k}}{(q;q)_{2k+1}} = \frac{(-1)^{3n} q^{-n(3n-1)/2}}{1-q^{6n+1}}, \tag{5.15}$$

$$\sum_{k=0}^{3n+1} \frac{(q^{-3n-1}, q^{3n+2}; q)_k q^{k^2+k}}{(q;q)_{2k+1}} = \frac{(-1)^{3n} q^{-n(3n+1)/2}(1+q^{4n+2})}{1-q^{6n+3}}, \tag{5.16}$$

$$\sum_{k=0}^{3n+2} \frac{(q^{-3n-2}, q^{3n+3}; q)_k q^{k^2+k}}{(q;q)_{2k+1}} = \frac{(-1)^{3n} q^{(-3n^2+5n+6)/2}}{1-q^{6n+5}}, \tag{5.17}$$

$$\sum_{k=0}^{3n} \frac{(q^{-3n}, q^{3n+1}; q)_k q^{k^2+2k}}{(q;q)_{2k+1}} = \frac{(-1)^{3n} q^{-n(3n-7)/2}}{1-q^{6n+1}}, \tag{5.18}$$

$$\sum_{k=0}^{3n+1} \frac{(q^{-3n-1}, q^{3n+2}; q)_k q^{k^2+2k}}{(q;q)_{2k+1}} = \frac{(-1)^{3n} q^{-n(3n-1)/2}(1+q^{2n+1})}{1-q^{6n+3}}, \tag{5.19}$$

$$\sum_{k=0}^{3n+2} \frac{(q^{-3n-2}, q^{3n+3}; q)_k q^{k^2+2k}}{(q;q)_{2k+1}} = \frac{(-1)^{3n} q^{-n(3n+1)/2}}{1-q^{6n+5}}. \tag{5.20}$$

6. q-multiplicate inverse series relations. Before we present the more general case of inverse series relations, we also need to define parameters $\{\alpha_1, \alpha_2, \dots, \alpha_l\}$

$$\alpha_i \in [1, l], \quad i = 1, 2, \dots, l \quad \text{and} \quad \alpha_i \neq \alpha_j \quad \text{if} \quad i \neq j.$$

By using a similar method, finite q-differences, with q-triplicate inverse series relations, we can establish the q-multiplicate inverse series relations as follows

Theorem 5. *With ϕ_i-polynomials defined by* (2.1), *the system of equations*

$$\Omega(n) = \sum_{m=0}^{l-1}\left\{\sum_{k\geq 0}\begin{bmatrix} n \\ lk+m\end{bmatrix}\frac{(-1)^{lk+m}(A_{l-m-1,k}+q^{lk+m}B_{l-m-1,k})\Xi_m(k)}{\prod_{i=0}^{l-m-2}\phi_i(q^n;k)\prod_{i=l-m-1}^{l-1}\phi_i(q^n;k+1)}\right\} \tag{6.1}$$

is equivalent to the system of equations

$$\Xi_m(n) = \sum_{k=0}^{ln+m}(-1)^k\begin{bmatrix} ln+m \\ k\end{bmatrix}q^{\binom{ln+m-k}{2}}\prod_{i=0}^{l-m-1}\phi_i(q^k;n) \times\prod_{i=l-m}^{l-1}\phi_i(q^k;n+1)\Omega(k), \tag{6.2}$$

where $m = 0, 1, 2, \ldots, l-1$.

This is the q-analog of Chu's multiplicate inverse series relations [**2**, Appendix B]. We will give some applications to hypergeometric identities in forthcoming work.

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