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A STUDY OF THE LIPPMANN-SCHWINGER EQUATION AND SPECTRA FOR SOME UNBOUNDED QUANTUM POTENTIALS

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ABSTRACT. In this article we study the Modified Lippmann–Schwinger equation for certain model potentials V defined on $\mathbf{R}^3$, not of Rollnik class, and solutions to the equation in a weak sense. Further, we study the resolvent and the spectrum of the operator $H = -\Delta + cV$ in our model for nonzero constants c. In particular, we find that, for sufficiently small $c > 0$, H has no singular spectrum.

Introduction. This article involves the study of the integral operator

$$(A_\lambda \phi)(x) = \frac{1}{4\pi} \int_{\mathbf{R}^3} \frac{|V(x)|^{1/2} e^{i\lambda|x-y|} V(y)^{1/2}}{|x-y|} \, \phi(y,\kappa) d\, y, \tag{0.1}$$

for certain classes of real-valued functions V defined on $\mathbf{R}^3$ where A_λ operates on a Hilbert space of functions ϕ also defined on $\mathbf{R}^3$ and where λ is a complex parameter. Here V is regarded as the potential for a (three-dimensional) Schrödinger operator $H \stackrel{\text{def}}{=} H_o + V = -\Delta + V$. We study a norm by Friedrichs [**1**] to develop a class of potentials V for which A_λ is not a Hilbert-Schmidt operator for any real λ, yet is compact for all real λ.

We apply our study of the operators A_λ to the so-called modified Lippmann-Schwinger equation:

$$\begin{aligned} \psi(x,\kappa) &= |V(x)|^{1/2} e^{i\kappa\cdot x} \\ &\quad - \frac{1}{4\pi} \int_{\mathbf{R}^3} \frac{|V(x)|^{1/2} e^{i|\kappa||x-y|} V(y)^{1/2}}{|x-y|} \, \psi(y,\kappa) d\, y. \end{aligned} \tag{0.2}$$

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with $V^{1/2} \stackrel{\text{def}}{=} |V|^{1/2}(\operatorname{sgn} V)$. Equation (0.2) arises in the study of Møller (wave) operators and of continuum eigenfunction expansions of the operator $H_o + V$ on $L^2(\mathbf{R}^3)$ [**5, 6, 9, 15**]. It is known [**2, 3, 8, 11**] that, for $\kappa \in \mathbf{R}^3$, except possibly those of a set of Lebesgue measure 0, (0.2) has a unique solution $\psi(x,\kappa) \in L^2(\mathbf{R}^3)$ when $V \in L^1(\mathbf{R}^3)$ and satisfies

$$\int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{|V(x)||V(y)|}{|x-y|^2}\,dx\,dy < \infty. \tag{0.3}$$

So, to motivate our study of the operator (0.1), we provide a sketch of proof. If V satisfies (0.3), then $A_{|\kappa|} : L^2(\mathbf{R}^3) \to L^2(\mathbf{R}^3)$ is a bounded operator. Indeed, it is a Hilbert-Schmidt operator and is, hence, compact. After rearrangement, equation (0.2) can be written as

$$(I + A_{|\kappa|})\psi(x,\kappa) = |V(x)|^{1/2} e^{i\kappa\cdot x} \tag{0.4}$$

where I denotes the identity operator on $L^2(\mathbf{R}^3)$. The result then follows via the analytic Fredholm theorem, see Theorem VI.41 of [**11**].

The condition (0.3) on V is satisfied if $V \in L^1_{\text{loc}}(\mathbf{R}^3)$ and $V(x) = O(e^{-\alpha|x|})$ as $|x| \to \infty$ for some positive α [**2, 3**]. Moreover, by Sobolev's inequality, this condition is also satisfied if $V(x) \in L^1(\mathbf{R}^3) \cap L^{3/2}(\mathbf{R}^3)$ [**11**]. However, for some potentials, the operators $A_{|\kappa|}$ may not be of Hilbert-Schmidt class, yet may be bounded—even compact. Indeed, using estimates from [**1**], see also [**10**], we demonstrate the existence of locally bounded V for which the operator $A_{|\kappa|}$ is not Hilbert-Schmidt for any κ, yet is compact for all κ.

The outline of this article is as follows: In Section 1 we introduce modes of compactness for operators $A_{|\kappa|}$ and check known results for some simple, bounded potentials to motivate more complicated examples. In Section 2 we introduce certain potentials of unbounded essential range to be used throughout the rest of the article. The associated operators $A_{|\kappa|}$ are then shown to be compact but not Hilbert-Schmidt. Using this model, in Section 3 we demonstrate the existence of weak solutions of the Lippmann-Schwinger equation, and in Section 4 we study the spectrum of the Schrödinger equation.

1. Compactness of $A_{|\kappa|}$ for some bounded potentials. A measurable function $V(x)$ defined on $\mathbf{R}^3$ is of Rollnik class [**11, 12**] if

$$||V||^2_{\text{Rollnik}} \stackrel{\text{def}}{=} \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{|V(x)||V(y)|}{|x-y|^2}\,dx\,dy < \infty. \tag{1.1}$$

And, for $0 < \beta \leq 1$, using an operator norm from [**1**], we will say V is of class $\mathrm{cl}\,(2\beta)$ if

$$||V||_{2\beta}^2 \stackrel{\text{def}}{=} \sup_{z\in\mathbf{R}^3} \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{|V(x)|^{\beta}|V(y)||V(z)|^{1-\beta}}{|x-y|^{2\beta}|y-z|^{2-2\beta}}\,dx\,dy < \infty \tag{1.2}$$

with $\mathrm{cl}\,(2)$ being the Rollnik-class potentials. Such classes are motivated by norms from [**1**] which, for $0 < \beta \leq 1$, are given by

$$||T||_{2\beta}^2 \stackrel{\text{def}}{=} \sup_{z\in\mathbf{R}^3} \int |K(x,y)|^{2\beta}|K(y,z)|^{2-2\beta}\,dx\,dy, \tag{1.3}$$

for an integral operator T on $L^2(\mathbf{R}^3)$

$$T\phi(x) = \int_{\mathbf{R}^3} K(x,y)\phi(y)\,dy$$

with integral kernel K.

T will be said to be 2β-bounded if (1.3) is finite. Indeed, a measurable function V is of class $\mathrm{cl}\,(2\beta)$ if and only if the associated operator $A_{|\kappa|}$ is 2β-bounded: Note that $||T||_{\mathrm{HS}} = ||T||_2$ where $||\cdot||_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm. It follows from (20.14) of [**1**] that integrals (1.3) produce upper bounds on the $L^2(\mathbf{R}^3)$ operator norms of integral operators T since

$$||T|| \leq ||T||_{2\beta};$$

and, for the case $\beta = 1$, we have

$$||T|| \leq ||T||_{\mathrm{HS}}.$$

Furthermore, we denote by $||T||_{\mathrm{Hol}}$ the Holmgren norm of an integral operator T which is defined by

$$||T||_{\mathrm{Hol}} \stackrel{\text{def}}{=} \sup_{z\in\mathbf{R}^3} \int |K(x,z)|\,dx.$$

Finally, we will denote, for positive a, the quantities

$$[|T|]_a \stackrel{\text{def}}{=} \sup_{z\in\mathbf{R}^3} \int |K(x,z)|^a\,dx.$$

It is easy to show that

$$||T||_{2\beta}^2 \le [|T|]_{2\beta} \cdot [|T|]_{2-2\beta}. \tag{1.4}$$

For bounded potentials, $V \in B(\mathbf{R}^3)$, we find a range of p for which $V \in L^p(\mathbf{R}^3) \implies V \in \mathrm{cl}\,(2\beta)$ for some $0 < \beta < 1$:

Proposition 1. *Suppose $V \in L^p(\mathbf{R}^3) \cap B(\mathbf{R}^3)$ for some $p < (3/16)(1+\sqrt{33}) \approx 1.2646$. Then, there exist positive numbers, $\beta_1(p)$ and $\beta_2(p)$, such that $\beta_1(p) < \beta_2(p)$ and that $V \in \mathrm{cl}\,(2\beta)$ whenever $\beta_1(p) < \beta < \beta_2(p)$.*

Proof. Since $V \in B(\mathbf{R}^3)$, we need only to appraise

$$\sup_{z\in\mathbf{R}^3} \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{|V(x)|^\beta |V(y)|}{|x-y|^{2\beta}|y-z|^{2-2\beta}}\,dx\,dy$$

as we determine p and q for which $V \in L^p(\mathbf{R}^3) \cap L^q(\mathbf{R}^3)$ implies $V \in \mathrm{cl}\,(2\beta)$.

Write

$$\int \frac{|V(x)|^\beta}{|x-y|^{2\beta}}\,dx = \int_{|x-y|<1} \frac{|V(x)|^\beta}{|x-y|^{2\beta}}\,dx + \int_{|x-y|\ge 1} \frac{|V(x)|^\beta}{|x-y|^{2\beta}}\,dx.$$

We have

$$\int_{|x-y|<1} \frac{|V(x)|^\beta}{|x-y|^{2\beta}}\,dx \le \sup_{x\in\mathbf{R}^3}\left[|V(x)|^\beta \int_{|x-y|<1} |x-y|^{-2\beta}\,dx\right]$$

and, using Hölder's inequality, we have for appropriate $\beta < p < \infty$,

$$\begin{aligned}&\int_{|x-y|\ge 1} \frac{|V(x)|^\beta}{|x-y|^{2\beta}}\,dx \\ &\le \left[\int_{|x-y|\ge 1} |V(x)|^p\,dx\right]^{\beta/p}\left[\int_{|x-y|\ge 1} |x-y|^{(-2\beta p)/(p-\beta)}\,dx\right]^{1-\beta/p}.\end{aligned}$$

Likewise,

$$\int \frac{|V(y)|}{|y-z|^{2-2\beta}}\,dy = \int_{|y-z|<1} \frac{|V(y)|}{|y-z|^{2-2\beta}}\,dy + \int_{|y-z|\ge 1} \frac{|V(y)|}{|y-z|^{2-2\beta}}\,dy$$

with

$$\int_{|y-z|<1} \frac{|V(y)|}{|y-z|^{2-2\beta}}\,dy \le \sup_{y\in\mathbf{R}^3}\left[|V(y)|\int_{|y-z|<1}|y-z|^{2\beta-2}\,dy\right]; \tag{1.7}$$

and, for appropriate $q > 1$,

$$\int_{|y-z|\ge 1} \frac{|V(y)|}{|y-z|^{2-2\beta}}\,dy \le \left[\int_{|y-z|\ge 1}|V(y)|^q\,dy\right]^{1/q} \times \left[\int_{|y-z|\ge 1}|y-z|^{(2\beta-2)q/q-1}\,dy\right]^{1-1/q}.$$

The convergence of integral (1.8) for all $0 < \beta < 1$ is clear when $V \in L^q(\mathbf{R}^3)$ for $0 < q \le 1$.

We now determine for which p and β are the quantities (1.5)–(1.8) finite when $V \in L^p(\mathbf{R}^3) \cap B(\mathbf{R}^3)$. For any positive R,

$$\int_{|x-y|<R} |x-y|^{-r}\,dx < \infty, \tag{1.9}$$

for $r < 3$, and

$$\int_{|x-y|\ge R} |x-y|^{-r}\,dx < \infty \tag{1.10}$$

for $r > 3$. Now, $2\beta < 2$ and $2-2\beta < 2$ for $0 < \beta < 1$ so that (1.5) and (1.7) are finite for any p and q, respectively. Moreover, from (1.9) and (1.10), the quantities (1.6) and (1.8) are both finite provided that

$$\frac{2\beta p}{p-\beta} > 3 \tag{1.11}$$

for $p > \beta$ and

$$\frac{(2-2\beta)q}{q-1} > 3 \tag{1.12}$$

for $q > 1$.

Now, for fixed p and q, we determine the range of β for which $V \in \mathrm{cl}\,(2\beta)$. Simultaneous inequalities (1.11) and (1.12) give

$$\frac{3p}{2p+3} < \beta < \max_q \left[\min\left\{ \frac{3-q}{2q}, q \right\} \right], \tag{1.13}$$

where the maximum is taken over those q for which $V \in L^q(\mathbf{R}^3)$; namely, those q such that $q \geq p$. So, the statement of the proposition then holds for

$$\beta_1(p) \stackrel{\mathrm{def}}{=} \frac{3p}{2p+3}$$

and

$$\beta_2(p) \stackrel{\mathrm{def}}{=} \begin{cases} 1 & : 0 < p \leq 1 \\ \dfrac{3-p}{2p} & : 1 < p < \left(\dfrac{3}{16}\right)(1+\sqrt{33}). \end{cases} \qquad \square$$

Corollary 1. *Given $p \leq 1$, $L^p(\mathbf{R}^3) \cap B(\mathbf{R}^3) \subset \mathrm{cl}\,(2\beta)$ for every $\beta_1(p) < \beta \leq 1$.*

Proof. We have only to show that $V \in \mathrm{cl}\,(2)$ which follows from Sobolev's inequality. $\square$

Remark 1.14. We note that, for $V \in L^\infty(\mathbf{R}^3)$, an estimate on Riesz potentials [**13**] shows that $A_{|\kappa|}$ is bounded as an operator from $L^p(\mathbf{R}^3)$ to $L^q(\mathbf{R}^3)$ for $1/q = 1/p - 2/3$ where $1 < p < 3/2$.

To investigate the compactness of the operators $A_{|\kappa|}$ for possibly unbounded V, we will use the following

Lemma 1. *Suppose that $V \in \mathrm{cl}\,(2\beta) \cap L^2_{\mathrm{loc}}(\mathbf{R}^3)$ for some $0 < \beta < 1$. Then, the associated operator $A_{|\kappa|}$ is compact.*

Proof. Let $0 \leq g_R(x) \leq 1$ be defined by

$$g_R(x) = \begin{cases} 1 & : |x| \leq R \\ 0 & : |x| > R. \end{cases} \tag{1.15}$$

Then, for each $R > 0$ we define the operators $A_{|\kappa|,R}$ by

$$4\pi A_{|\kappa|,R}\phi(x) = \int \frac{e^{i|\kappa||x-y|}|V(x)|^{1/2}g_R(x)V(y)^{1/2}}{|x-y|}\, g_R(x-y)\phi(y)\,dy.$$

Now, using the changes of variables $u = y - x$ and $r = |u|$, we obtain

$$\begin{aligned}\iint \frac{|V(x)g_R^2(x)||V(y)|}{|x-y|^2}\, g_R^2(x-y)\,dy\,dx \\ &\leq \int_{|x|\leq R}\int_{|u|\leq R} \frac{|V(x)||V(u+x)|}{|u|^2}\,du\,dx \\ &\leq \int_{|x|\leq R}\int_{S^2}\int_0^R |V(x)||V(x+r\omega)|\,dr\,d\omega\,dx.\end{aligned}$$

Since

$$|V(x)||V(x+r\omega)| \leq \frac{(V(x))^2 + (V(x+r\omega))^2}{2},$$

we have, for each $r \in [0, R]$ and for each $\omega \in S^2$,

$$\begin{aligned}\int_{|x|\leq R} |V(x)||V(x+r\omega)|\,dx &\leq 1/2\int_{|x|\leq R} |V(x)|^2 + |V(x+r\omega)|^2\,dx \\ &\leq \int_{|x|\leq 2R} |V(x)|^2\,dx.\end{aligned}$$

So, by the Fubini-Tonelli Theorem, for all $R > 0$,

$$||A_{|\kappa|,R}||_{\text{HS}} \leq \sqrt{R}\bigg(\int_{|x|\leq 2R} |V(x)|^2\,dx\bigg)^{1/2}.$$

Therefore, for such R, $A_{|\kappa|,R}$ is of Hilbert-Schmidt class and is, hence, compact. Clearly, $||A_{|\kappa|} - A_{|\kappa|,R}||_{2\beta} \leq ||A_{|\kappa|}||_{2\beta}$ so that by the Lebesgue dominated convergence theorem,

$$\lim_{R\to\infty} ||A_{|\kappa|} - A_{|\kappa|,R}||_{2\beta} = 0$$

and, hence, in the $L^2(\mathbf{R}^3)$ operator norm,

$$\lim_{R\to\infty} ||A_{|\kappa|} - A_{|\kappa|,R}|| = 0.$$

This shows that $A_{|\kappa|}$ is the operator-norm limit of compact operators and is therefore compact. $\square$

We now provide a necessary condition for bounded, central potentials to be of class $\mathrm{cl}\,(2\beta)$: V is said to be a central potential if there is a function $\mathcal{V}$, defined on $\mathbf{R}^+$, such that $V(x) = \mathcal{V}(|x|)$. For $r = |x|$ we state the following

Proposition 2. *A bounded, central potential* $V \in \mathrm{cl}\,(2\beta)$ *only if the associated function* $\mathcal{V}$ *satisfies*

$$\mathcal{V}(r) \in L^1(\mathbf{R}^+;\, dr) \cap L^\beta(\mathbf{R}^+; r^{2-2\beta}\, dr).$$

Proof. For each z, we use the Fubini-Tonelli theorem and a change of coordinates to obtain

$$\begin{aligned} ||V||_{2\beta}^2 &\geq \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|V(x)|^\beta |V(y)| |V(z)|^{1-\beta}}{|x-y|^{2\beta} |y-z|^{2-2\beta}}\, dx\, dy \\ &\geq 4\pi |V(z)|^{1-\beta} \int_0^\infty \frac{|\mathcal{V}(r)|^\beta r^2}{(R+r)^{2\beta}}\, dr \\ &\quad \times \int_{|y|<R} \frac{|V(y)|}{|y-z|^{2-2\beta}}\, dy. \end{aligned} \tag{1.16}$$

Likewise, choosing R so large that

$$\int_{|x|<R} |V(x)|^\beta\, dx \overset{\mathrm{def}}{=} \delta > 0,$$

we have

$$\begin{aligned} ||V||_{2\beta}^2 &\geq 4\pi \int_{\mathbf{R}^3} \frac{|V(z)|^{1-\beta} |V(y)|\, dy}{(R+|y|)^{2\beta} (|z|+|y|)^{2-2\beta}} \int_{|x|<R} |V(x)|^\beta\, dx \\ &= 4\pi |V(z)|^{1-\beta} \delta \int_0^\infty \frac{\mathcal{V}(r) r^2}{(R+r)^{2\beta} (|z|+r)^{2-2\beta}}\, dr \end{aligned} \tag{1.17}$$

Choosing z, not a root of V, and sufficiently large R, it is clear that, since $\mathcal{V}$ is bounded, (1.16) is finite only if $\mathcal{V}(r) \in L^\beta(\mathbf{R}^+; r^{2-2\beta}\, dr)$ and that (1.17) is finite only if $\mathcal{V}(r) \in L^1(\mathbf{R}^+;\, dr)$. $\square$

Our object, which we postpone until the next section, will be to construct cl (2β)-class potentials which are not Rollnik-class. To motivate those constructions, we first consider operators $A_{|k|}$ for certain bounded potentials given by

$$V_\gamma(x) \stackrel{\text{def}}{=} (1+|x|)^{-\gamma}.$$

(Such potentials are well-studied in, for instance, the study of Møller operators [**4, 6, 7, 17**].)

Proposition 3. *$V_\gamma(x) \in \mathrm{cl}\,(2\beta)$ if and only if $\gamma > (3/\beta) - 2$. Hence, $V_\gamma \in \mathrm{cl}\,(2\beta)$ for each $3/(\gamma+2) \le \beta \le 1$.*

Proof. By Proposition 2, $V_\gamma \in \mathrm{cl}\,(2\beta)$ only if

$$\gamma > \max\left\{\frac{3}{\beta} - 2, 1\right\}.$$

Yet, $(3/\beta) - 2 \ge 1$ for any $0 < \beta \le 1$ and, hence, $\gamma > (3/\beta) - 2$. Conversely, $V \in L^p(\mathbf{R}^3)$ if and only if $p > 3/\gamma$; and, by Proposition 1 we have $V_\gamma \in \mathrm{cl}\,(2\beta)$ for $p < [3\beta/(3-2\beta)]$. The combined inequalities give $\gamma > (3/\beta) - 2$, and the proof is complete. $\square$

Remark 1.18. The associated operator $A_{|\kappa|}$ is already known to be compact, indeed Hilbert-Schmidt, for $V_\gamma(x) = \mathcal{V}(|x|)$ as in Proposition 3 via Sobolev's inequality for $\gamma > 2$.

Remark 1.19. It follows immediately from Proposition 3 that the associated operator $A_{|\kappa|}$ is bounded for $\gamma > 1$ which is already known [**4**].

2. Compactness of $A_{|\kappa|}$ for some unbounded potentials. We now introduce a class of potentials which admits functions which do not decay as $|x| \to \infty$ to construct potentials of various classes cl (2β). Consider functions that are supported on $\cup_{k=1}^{\infty} E_k$ for Lebesgue measurable sets E_k satisfying the following properties:

(i) The sets E_k are disjoint, and for $k \ne l$ the distance $d(E_k, E_l)$ between sets E_k and E_l satisfies

$$c_1|k-l| \le d(E_k, E_l) \le c_2|k-l|$$

for some positive constants c_1 and c_2, independent of k and l.

(ii) There are positive constants C_1, C_2 and b such that for every k the Lebesgue measure $\mu(E_k)$ of E_k satisfies

$$C_1 k^{-b} \le \mu(E_k) \le C_2 k^{-b}.$$

(iii) For every $1/2 \le \beta' < 1$, there is a positive constant $C_{\beta'}$, depending only on β', such that, for every k,

$$\int_{E_k} |x-y|^{-2\beta'}\, dx \le C_{\beta'} \mu(E_k)$$

uniformly for $y \in E_k$.

(iv) There is a positive constant D such that, for every k, the diameter, $\operatorname{diam}(E_k)$, of E_k satisfies $\operatorname{diam}(E_k) \le D$.

For fixed $b > 0$, the collection of sets

$$E_k = \left\{ (x_1, x_2, x_3) : \sqrt{x_2^2 + x_3^2} < 1,\ k < x_1 < k + \frac{1}{2k^b} \right\}$$

and

$$\tilde{E}_k = \left\{ (x_1, x_2, x_3) : \sqrt{x_2^2 + x_3^2} < \frac{1}{2},\ k + \frac{1}{8k^b} < x_1 < k + \frac{1}{4k^b} \right\}$$

for $k = 1, 2, 3, \ldots$ satisfy criteria (i)–(iv) and the property that $\tilde{E}_k \subset E_k$ for all k.

We now construct model potentials which are not Rollnik-class yet are 2β-class and, in fact, $C^\infty(\mathbf{R}^3)$-class. Let $\chi_1(x)$ be a nonnegative, $C^\infty(\mathbf{R}^3)$-class function such that $\chi_1(x) = 1$ for all $x \in \tilde{E}_1$ and $\operatorname{supp} \chi_1 \subset E_1$. Then, define for $k = 1, 2, 3, \ldots$

$$\chi_k(x) \stackrel{\text{def}}{=} \chi_1\big((x_1 - k)k^b + 1, x_2, x_3\big)$$

and

$$V_{a,b}(x) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \chi_k(x) k^a.$$

We note that $\operatorname{supp} V_{a,b} \subset \cup_{k=1}^{\infty} E_k$ and that, for each k, $\operatorname{supp} \chi_k \subset E_k$ and $V_{a,b}(x) = k^a$ for all $x \in \tilde{E}_k$.

Given $0 < \beta < 1$, we determine parameters a and b for which $V_{a,b}$ are of class $\mathrm{cl}\,(2\beta)$. First, we introduce some notation: Given two functions, f and g, the expression $f \lesssim g$ means that there is a positive constant c such that $|f| \le c|g|$ uniformly on the domain of both f and g and $f \asymp g$ means that both $f \lesssim g$ and $g \lesssim f$ hold. We are now ready to state

Proposition 4. *Given $0 < \beta < 1$ and $0 < \alpha < 2\beta - 1$, we have the following estimates for $l \in \mathbf{Z}^+$:*

$$\sum_{\substack{k>0\\k\neq l}} \frac{k^\alpha}{|k-l|^{2\beta}} \lesssim \begin{cases} l^{\alpha-2\beta+1} & : 0 < \beta < \dfrac{1}{2} \\ l^\alpha \ln l & : \beta = \dfrac{1}{2} \\ l^\alpha & : \dfrac{1}{2} < \beta < 1 \end{cases} \qquad (l \longrightarrow \infty).$$

We note that these sums diverge for every l when $\alpha \ge 2\beta - 1$.

Proof. From standard sum and integral estimates along with a change of variables, we find, for $0 < \beta < 1$,

$$\begin{aligned} \sum_{\substack{k>0\\k\neq l}} \frac{k^\alpha}{|k-l|^{2\beta}} &\lesssim \int_1^\infty \frac{(t+l)^\alpha}{|t|^{2\beta}}\,dt \\ &= l^{\alpha+1-2\beta} \int_{1/l}^\infty \frac{(w+1)^\alpha}{w^{2\beta}}\,dw \\ &\lesssim l^{\alpha+1-2\beta}\left[\int_{1/l}^1 w^{-2\beta}\,dw + \int_1^\infty w^{\alpha-2\beta}\,dw\right]. \end{aligned} \tag{2.1}$$

The second integral of (2.1) is finite for $\alpha < 2\beta - 1$ while

$$\int_{1/l}^1 w^{-2\beta}\,dw \lesssim \begin{cases} 1 & : 0 < \beta < \dfrac{1}{2} \\ \ln l & : \beta = \dfrac{1}{2} \\ l^{2\beta-1} & : \dfrac{1}{2} < \beta < 1 \end{cases} \qquad (l \longrightarrow \infty).$$

The combined estimates prove the claim. □

Write

$$|V_{a,b}(x)|^{\beta} = \sum_{k=1}^{\infty} (\chi_k(x))^{\beta} k^{a\beta}.$$

Now, supposing that $z \in E_l$ for some l (for otherwise $V_{a,b}(z) = 0$), for some constant $\delta > 0$, depending only on β, and for $k \neq l$,

$$\begin{aligned} \int \big(\chi_k(x)\big)^{\beta} \frac{k^{a\beta}}{|x-z|^{2\beta}}\, dx &\leq \delta \cdot \frac{k^{a\beta}\mu(E_k)}{|k-l|^{2\beta}} \\ &\lesssim \frac{k^{a\beta-b}}{|k-l|^{2\beta}}; \end{aligned} \tag{2.2}$$

and, for $k = l$,

$$\begin{aligned} \int \big(\chi_l(x)\big)^{\beta} \frac{l^{a\beta}}{|x-z|^{2\beta}}\, dx &\leq \delta \cdot l^{a\beta}\mu(E_l) \\ &\lesssim l^{a\beta-b}. \end{aligned} \tag{2.3}$$

Let us set $A_{|\kappa|}$ as the associated operator (0.1) with $V = V_{a,b}$. We now apply estimates (2.2) and (2.3) along with Proposition 4 to estimate $[|A_{|\kappa|}|]_{\beta}$ and $[|A_{|\kappa|}|]_{1-\beta}$ for $0 < \beta \leq 1/2$, thereby making estimates for $1/2 < \beta < 1$ immediate.

For some positive constant $\tilde{\delta}$, depending only on β, we have the following estimates uniform for $z \in \cup_{l=1}^{\infty} E_l$: For $0 < \beta < 1/2$ and $a\beta - b - 2\beta < -1$,

$$\begin{aligned} \int \frac{V_{a,b}^{\beta}(x)V_{a,b}^{\beta}(z)}{|x-z|^{2\beta}}\, dx &\leq \tilde{\delta}\Bigg[l^{a\beta-b} \cdot l^{a\beta} + \sum_{\substack{k>0 \\ k\neq l}} \frac{k^{a\beta-b}}{|k-l|^{2\beta}}\, l^{a\beta}\Bigg] \\ &\lesssim l^{2a\beta-b} + l^{1+a\beta-b-2\beta} \cdot l^{a\beta} \\ &\lesssim l^{2a\beta-b+1-2\beta}; \end{aligned}$$

for $a(1-\beta) - b < -1$ (noting that $1/2 < 1-\beta < 1$)

$$\begin{aligned} \int \frac{V_{a,b}^{1-\beta}(x)V_{a,b}^{1-\beta}(z)}{|x-z|^{2-2\beta}}\, dx &\leq \tilde{\delta}\Bigg[l^{a(1-\beta)-b} \cdot l^{a(1-\beta)} + \sum_{\substack{k>0 \\ k\neq l}} \frac{k^{a(1-\beta)-b}}{|k-l|^{2(1-\beta)}}\, l^{a(1-\beta)}\Bigg] \\ &\lesssim l^{2a(1-\beta)-b} + l^{a(1-\beta)-b} \cdot l^{a(1-\beta)} \\ &\lesssim l^{2a(1-\beta)-b}; \end{aligned}$$

finally, for $\beta = 1/2$ and $a/2 - b < -1$,

$$\int \frac{V_{a,b}^{1/2}(x)V_{a,b}^{1/2}(z)}{|x-z|}\,dx \le \tilde{\delta}\Big[l^{a/2-b}\cdot l^{a/2} + \sum_{\substack{k>0\\k\neq l}} \frac{k^{a/2-b}}{|k-l|}\,l^{a/2}\Big]$$
$$\lesssim l^{a-b} + l^{a/2-b}\cdot l^{a/2}\cdot \ln l$$
$$\lesssim l^{a-b}(1+\ln l).$$

Since these estimates provide a finite supremum for $l \in \mathbf{Z}^+$, we have that for $0 < \beta < 1/2$ the quantities $[|A_{|\kappa|}|]_{1-\beta}$ and $[|A_{|\kappa|}|]_{\beta}$ are both finite if $2a\beta - b + 1 - 2\beta \le 0$ and $2a(1-\beta) - b \le 0$ and that $[|A_{|\kappa|}|]_{1/2}$ is finite if $a - b < 0$.

We are now ready to prove

Theorem 1. *Given $0 < \beta < 1$, there are functions of the form $V_{a,b}$ which are $\mathrm{cl}\,(2\beta)$-class, but not Rollnik-class. Indeed, for each such β, numbers $a \ge 0$ and $b > 0$ may be chosen so that the associated operator $A_{|\kappa|}$ is compact but not Hilbert-Schmidt.*

Proof. First, we will show that, given any $a \ge 0$ and $b > 0$ for which $a - b > -1/2$, the function $V_{a,b}$ is not of Rollnik class. We note that, since $\tilde{E}_k \subset E_k$ for each k, given D as in property (iv) and $y \in \tilde{E}_k$,

$$E_k \subset \{u + y : |u| \le D\}.$$

So,

$$\iint \frac{|V_{a,b}(x)||V_{a,b}(y)|}{|x-y|^2}\,dx\,dy$$
$$= \iint \frac{|V_{a,b}(u+y)||V_{a,b}(y)|}{|u|^2}\,du\,dy$$
$$\ge \iint_{|u|\le D} \frac{|V_{a,b}(u+y)||V_{a,b}(y)|}{|u|^2}\,du\,dy$$
$$\ge \iint_{|u|\le D} \frac{\sum_{k,l\ge 1}\chi_k(u+y)\chi_l(y)(kl)^a}{D^2}\,du\,dy$$
$$\ge \iint \frac{\sum_{k\ge 1}\chi_k(u+y)\chi_k(y)(k)^{2a}}{D^2}\,du\,dy$$

$$\geq \frac{1}{D^2}\sum_{k=1}^{\infty}\big(\mu(\tilde{E}_k)\big)^2 k^{2a}$$
$$\geq \left(\frac{\pi}{32D}\right)^2 \sum_{k=1}^{\infty} k^{2(a-b)}.$$

Now, to find non-Rollnik potentials $V_{a,b}$ for which Lemma 1 applies, we seek nonnegative numbers a and b which satisfy the following simultaneous inequalities:

$$2a\beta + 1 - 2\beta < b \tag{2.4}$$
$$2a(1-\beta) < b \tag{2.5}$$
$$a + \frac{1}{2} > b \tag{2.6}$$

for $0 < \beta \leq 1$. The lefthand sides (LHS) of inequalities (2.4)–(2.6) compare as follows:

LHS (2.4) < LHS (2.6) when $a > [(1/2 - 2\beta)/(1-2\beta)]$ for $0 < \beta \leq 1/4$, when $a \geq 0$ for $1/4 < \beta < 1/2$, and when $a < [(2\beta - 1/2)/(2\beta - 1)]$ for $1/2 < \beta < 1$.

LHS (2.5) < LHS (2.6) when $0 < a < [(1/2)/(1-2\beta)]$ for $0 < \beta < 1/2$ and when $a \geq 0$ for $1/2 \leq \beta < 1$.

So, given $0 < \beta < 1$, let b satisfy

$$\max\{\text{LHS}\,(2.4),\, \text{LHS}\,(2.5)\} < b < \text{LHS}\,(2.6)$$

for which in the following cases consistent solutions exist:

i) $[(1/2 - 2\beta)/(1-2\beta)] < a < [(1/2)/(1-2\beta)]$ for $0 < \beta \leq 1/4$;

ii) $a \geq 0$ for $1/4 \leq \beta < 1/2$;

iii) $a > 0$ for $\beta = 1/2$;

iv) and, $0 \leq a < [(2\beta - 1/2)/(2\beta - 1)]$ for $1/2 \leq \beta < 1$.

Remark 2.7. We note that, for a and b as above, $V_{a,b} \notin L^1(\mathbf{R}^3)$.

Remark 2.8. In case iii) above, the associated operator $A_{|\kappa|}$ is bounded in Holmgren norm but not in Hilbert-Schmidt norm.

3. Weak solutions to the Lippmann-Schwinger equation. In this section we will analyze solutions to equation (0.2) in an abstract sense, vis-à-vis [**5**], for a general subclass of $\mathrm{cl}\,(2\beta)$-class potentials. We proceed using the following result, whose proof closely follows part II of the proof of Theorem XI.41 from [**11**]:

Theorem 2. *Given* $V \in \mathrm{cl}\,(2\beta)$ *for some* $0 < \beta \leq 1$*, the operator* $A_{|\kappa|} + I$ *is invertible on* $L^2(\mathbf{R}^3)$ *for all* $|\kappa|$ *except, perhaps, for* $|\kappa|^2 \in \mathcal{E}$*, where* $\mathcal{E}$ *is a certain set of Lebesgue measure zero.*

Proof. We consider A_λ for complex λ. From the estimate (20.8) of [**1**], we find

$$
\begin{aligned}
&(4\pi)^2||A_\lambda||^2 \\
&\leq \sup_{z\in\mathbf{R}^3} \int_{\mathbf{R}^6} \frac{|V(x)|^\beta e^{-2\beta \operatorname{Im}\lambda|x-y|}\,|V(y)||V(z)|^{1-\beta} e^{(-2+2\beta)\operatorname{Im}\lambda|y-z|}}{|x-y|^{2\beta}|y-z|^{2-2\beta}}\,dx\,dy \\
&\leq ||V||_{2\beta}^2.
\end{aligned}
$$

So, by Fubini's theorem and Morera's theorem, A_λ is an analytic, operator-valued function defined on the upper half-plane, $\operatorname{Im}\lambda > 0$. (See the first two paragraphs of Section 4 in [**2**] for details.) Furthermore, since

$$||A_{\lambda_1} - A_{\lambda_2}||_{2\beta} \leq 2||A_0||_{2\beta}$$

for any real numbers λ_1 and λ_2, we have by the Lebesgue dominated convergence theorem and the mean value theorem that $||A_\lambda||_{2\beta}$ is continuous for λ on the real axis, $\operatorname{Im}\lambda = 0$, and, hence, so is $||A_\lambda||$. Similarly, one can show that

$$\lim_{\operatorname{Im}\lambda\to+\infty} ||A_\lambda|| = 0$$

where the limit is independent of $\operatorname{Re}\lambda$. Therefore, there is a positive number γ_o for which $(A_\lambda + I)^{-1}$ is analytic whenever $\operatorname{Im}\lambda > \gamma_o$. Now, the statement of the theorem follows from a version of the analytic Fredholm theorem (see Proposition of page 101 in [**11**] and the two paragraphs which follow) whereby the exceptional set $\mathcal{E} \subset \mathbf{R}$ is closed and of Lebesgue measure 0. $\square$

Remark 3.1. As in [**11**], we likewise note that, by the Riemann-Lebesgue lemma, the set $\mathcal{E}$ is bounded.

Remark 3.2. Given a potential of the form cV where $V \in \mathrm{cl}\,(2\beta)$ for some $0 < \beta \leq 1$ and $c > 0$ is sufficiently small, $||A_{|\kappa|}||$ can be made so small that $(A_{|\kappa|} + I)^{-1}$ exists for all $|\kappa|$; in which case, $\mathcal{E}$ is empty.

In the next theorem we consider, for certain measure spaces, solutions to equation (0.2) as weak limits. For $\kappa^2 \notin \mathcal{E}$, define for $m = 1, 2, \ldots$ the bounded operators $G_{|\kappa|,m} \stackrel{\text{def}}{=} (A_{|\kappa|} + I)^{-1} g_m$ on $L^2(\mathbf{R}^3)$ for functions $g_{_R}$ as in (1.15). Suppose $V^{1/2} e^{i\kappa \cdot x} \in \mathfrak{X}^*$, the dual space of a closed subspace $\mathfrak{X}$ of $L^2(\mathbf{R}^3)$, and let $\mathfrak{Y} \stackrel{\text{def}}{=} (A_{|\kappa|} + I)(\mathfrak{X})$ (which, since $A_{|\kappa|}$ is compact, is also a closed subspace of $L^2(\mathbf{R}^3)$). We construct weak solutions to (0.2) in the sense that $G^*_{|\kappa|,m}(V^{1/2} e^{i\kappa \cdot x})$ converges almost everywhere to a function $g \in \mathfrak{Y}^*$. Indeed, we state

Theorem 3. *For all $|\kappa|^2 \notin \mathcal{E}$, the sequence of operators $G^*_{|\kappa|,m}$ for $m = 1, 2, \ldots,$ converges in the weak-$*$ sense to an operator*

$$G^* : \mathfrak{X}^* \rightarrow \mathfrak{Y}^*.$$

In particular, $G^*(e^{i\kappa \cdot (\cdot)} V^{1/2}) \in \mathfrak{Y}^*$.

Proof. Choose functions $w \in \mathfrak{Y}$ and $v \in \mathfrak{X}^*$. Then, $g \stackrel{\text{def}}{=} (A_{|\kappa|} + I)^{-1}(w) \in \mathfrak{X}$ and, therefore, for each m

$$\begin{aligned} \int_{\mathbf{R}^3} G^*_{|\kappa|,m}(v)(x) w(x)\, dx &= \int_{\mathbf{R}^3} v(x) g_m(x) (A_{|\kappa|} + I)^{-1}(w)(x)\, dx \\ &= \int_{\mathbf{R}^3} v(x) g_m(x) g(x)\, dx. \end{aligned} \tag{3.3}$$

The result now follows by the Lebesgue dominated convergence theorem. □

Before we state the next result, we make the following definitions. We will denote by $\mathcal{C}_{\eta,\delta}$ the open cone given by

$$\mathcal{C}_{\eta,\delta} \stackrel{\text{def}}{=} \left\{ x : \frac{x \cdot \eta}{|x|} > \delta \right\}$$

for some $-1 < \delta < 1$ and for some unit vector $\eta \in \mathbf{R}^3$. Given δ, a function $\phi(x)$ will be said to be rapidly decreasing on the cone $\mathcal{C}_{\eta,\delta}$ if, for every positive integer j,

$$\lim_{|x|\to\infty} \sup_{(x\cdot\eta)/|x|>\delta} |x|^j |\phi(x)| = 0$$

and the expression $f \sim h$ on $\mathcal{C}_{\eta,\delta}$ will mean that the difference $f - h$ is rapidly decreasing on $\mathcal{C}_{\eta,\delta}$. Finally, a function f is said to be polynomially bounded if $f(x) \lesssim (1+|x|)^\alpha$ for some $\alpha > 0$.

In the context of Theorem 3, we find asymptotic relationships between certain functions $g(x)$ and the associated functions $w(x)$ for large $r \overset{\text{def}}{=} |x|$. Defining $F \overset{\text{def}}{=} V^{1/2} g$, for $|\kappa|^2 \notin \mathcal{E} \bigcup \{0\}$ we state

Theorem 4. *Suppose that $V(x) \in C^\infty(\mathbf{R}^3)$ is polynomially bounded and that F as above is supported in the complement of a cone $\mathcal{C}_{\eta,\delta}$ where, for some $\gamma > 3$, F satisfies*

$$\left| \frac{d^j}{d\,r^j} F \right| \lesssim (1+r^2)^{-(\gamma+j)/2}$$

on $\mathbf{R}^3$ for each $j = 0, 1, 2, \ldots$. Then, $w(x) \sim g(x)$ on $\mathcal{C}_{-\eta,\delta'}$ for any δ' such that $\delta < \delta' < 1$.

Proof. For a given cone $\mathcal{C}_{\eta,\delta}$, we will show that $w(x) = g(x) + \phi(x)$ where $\phi(x) = A_{|\kappa|}(g)(x)$ is rapidly decreasing on $\mathcal{C}_{-\eta,\delta'}$. To this end, it suffices to show that, for $|\kappa|^2 \notin \mathcal{E} \cup \{0\}$,

$$T_{|\kappa|}(g)(x) \overset{\text{def}}{=} \int_{\mathbf{R}^3} \frac{e^{i|\kappa||x-y|}}{|x-y|} F(y)\, dy$$

is rapidly decreasing on $\mathcal{C}_{-\eta,\delta'}$. For $\omega \overset{\text{def}}{=} (x/r)$, $x \neq 0$, fixed, we introduce the variable $u \overset{\text{def}}{=} (y/r) - \omega$, and we define $s \overset{\text{def}}{=} |u|$ and $\nu \overset{\text{def}}{=} (u/s)$

to write

$$\begin{aligned}
T_{|\kappa|}(g)(x) &= T_{|\kappa|}(g)(r\omega) \\
&= \int_{\mathbf{R}^3} \frac{e^{ir|\kappa||\omega-(y/r)|}}{r|\omega-(y/r)|}\, F(y)\, dy \\
&= \int \frac{e^{ir|\kappa||u|}}{|u|}\, F\big(r(\omega+u)\big) r^2\, du \\
&= r^2 \int_{S^2\setminus \mathcal{C}_{\eta,\delta}} \int_0^\infty e^{ir|\kappa|s} sF\big(r(\omega+s\nu)\big)\, ds\, d\Omega(\nu).
\end{aligned}$$

Now, by the Lebesgue dominated convergence theorem and the Fubini-Tonelli theorem, it suffices to show that

$$\int_0^\infty e^{ir|\kappa|s} sF(r(\omega+s\nu))\, ds \tag{3.4}$$

rapidly decreases, as $r \to \infty$, uniformly in ν. Supposing $r \geq 1$, it follows by induction and the chain rule that, for each $j = 0, 1, 2, \ldots$ with $\mathfrak{d} \stackrel{\text{def}}{=} \delta' - \delta$,

$$\begin{aligned}
\frac{d^j}{d\, s^j}\,[sF(r(\omega+s\nu))] &\lesssim \frac{r^j(s^{j+2}+1)}{(1+r^2(s^2+2s\omega\cdot\nu+1))^{(\gamma+j)/2}} \\
&\lesssim \frac{r^j(s^{j+2}+1)}{(1+r^2(s-1)^2+2\mathfrak{d}r^2 s)^{(\gamma+j)/2}} \\
&\lesssim \begin{cases} [(r^j s^{j+2})/((rs)^{\gamma+j})] & : s \geq 2 \\ [r^j/((1+r^2 s))^{(\gamma+j)/2}] & : 0 \leq s < 2 \end{cases} \\
&\lesssim (s+1)^{2-\gamma}
\end{aligned} \tag{3.5}$$

uniformly for $r \geq 1$. In (3.5) we use that $\omega \cdot \nu \geq 1 - d$ for $\omega \in \mathcal{C}_{-\eta,\delta}$ and $s\nu \in \operatorname{supp} F$. Therefore,

$$\frac{d^j}{d\, s^j}\,[sF(r(\omega+s\nu))] \in L^1(\mathbf{R},\, ds)$$

for each j and, hence, it follows from the Riemann-Lebesgue lemma that the integral (3.4) indeed rapidly decreases, as $r \to \infty$; so the result is now immediate. □

We apply this result to some non-Rollnik, cl (2β)-class potentials: We state

Corollary 2. *For $V = V_{a,b}$ as in Theorem* 1, *the conclusion of Theorem* 1 *holds for any $g \in \mathcal{S}(\mathbf{R}^3)$.*

Proof. We need only to show that all derivatives of $V_{a,b}$ are polynomially bounded. Using the chain rule,

$$\frac{\partial^j}{\partial\, x_1^j}\,\chi_k(x) = k^{bj}\left(\frac{\partial^j}{\partial\, x_1^j}\,\chi_1\right)((x_1+1)k^b - k, x_2, x_3)$$

and

$$\frac{\partial^j}{\partial\, x_l^j}\,\chi_k(x) = \left(\frac{\partial^j}{\partial\, x_l^j}\,\chi_1\right)((x_1+1)k^b - k, x_2, x_3)$$

for $l = 2, 3$ so that for any 3-index variable $\alpha = (\alpha_1, \alpha_2, \alpha_3)$,

$$|\partial_x^\alpha \chi_k(x)| \le k^{\alpha_1 b} |\partial_x^\alpha \chi_k(x)|.$$

Now, for $C = \sup_{x \in E_1} |\partial_x^\alpha \chi_1(x)|$, and for h_k denoting the characteristic function of the set E_k, we have

$$
\begin{aligned}
|\partial_x^\alpha V_{a,b}(x)| &= \left|\sum_{k=1}^{\infty} k^a \partial_x^\alpha \chi_k(x)\right| \\
&\le \sum_{k=1}^{\infty} k^a |\partial_x^\alpha \chi_k(x)| \\
&= \sum_{k=1}^{\infty} k^{\alpha_1 b + a} \big|(\partial_x^\alpha \chi_1)((x_1+1)k^b - k, x_2, x_3)\big| \\
&\le \sum_{k=1}^{\infty} k^{\alpha_1 b + a} C h_k(x)
\end{aligned}
\tag{3.6}
$$

so that

$$\partial_x^\alpha V_{a,b}(x) \lesssim (1+|x|)^{b|\alpha|+a}. \qquad \square$$

4. Resolvent and spectrum of H. We now consider operators of the form $H = H_o + cV_{a,b}$ with real, nonzero (coupling) constants c and potential $V_{a,b}$ as above. With fixed a and b, we estimate L^2 inner products of the form $(f, R(\lambda)g)$ for appropriate f and g where $R(\lambda)$ is the resolvent operator for H, given by $R(\lambda) \stackrel{\text{def}}{=} (H-\lambda)^{-1}$. We then apply these results in the study of the spectrum of H. First, we consider the operator B_λ, defined by

$$(B_\lambda f)(x) = V_{a,b}^{1/2}(x) \int_{\mathbf{R}^3} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} f(y)\, dy,$$

and seek a closed subspace $\tilde{\mathcal{H}}$ of $L^2(\mathbf{R}^3)$ for which the operator-valued function B_λ takes values in $\mathcal{L}(\tilde{\mathcal{H}}; L^2(\mathbf{R}^3))$. (We will take $\sqrt{\lambda}$ to have positive imaginary part for $\lambda \in \mathbf{C} \setminus [0,\infty)$.) In particular, we proceed to construct such a space of the form $\tilde{\mathcal{H}} = L^2(\mathbf{R}^3; d\nu)$ for a measure ν equivalent to Lebesgue measure. To this end, we find a class of functions $\phi \in L^2(\mathbf{R}^3)$ for which $\operatorname{supp}\phi = \mathbf{R}^3$ and $B_\lambda(\phi) \in L^2(\mathbf{R}^3)$. Indeed, we have

Proposition 5. *There are measurable functions ϕ which are positive throughout $\mathbf{R}^3$ for which the operator-valued function*

$$\lambda \longmapsto B_\lambda \circ \phi \tag{4.1}$$

takes values of Hilbert-Schmidt class for each $\lambda \in [0,\infty)$.

In (4.1), ϕ represents the operation of multiplication by the function ϕ.

Proof. Denote by S_r the set

$$S_r \stackrel{\text{def}}{=} \left\{ (y_1, y_2, y_3) | y_1 \geq 0, \sqrt{y_2^2 + y_3^2} \leq r \right\}$$

and write ϕ in the form $\phi^2(y) = \phi_1(y) + \phi_2(y)$ where $\operatorname{supp}\phi_1 \subset S_3$ and $\operatorname{supp}\phi_2 \subset \mathbf{R}^3 \setminus S_2$. Denote by D the set

$$D \stackrel{\text{def}}{=} \left\{ (y_1, y_2, y_3) | 0 \leq y_1 < 1, \sqrt{y_2^2 + y_3^2} \leq 3 \right\}$$

and for $k \in \mathbf{N}$ define $D_k \stackrel{\text{def}}{=} \{y - (k-1, 0, 0) \mid y \in D\}$. Let ϕ_1 be the function, positive-valued on S_3, given by

$$\phi_1(y) = \sum_{l=1}^{\infty} l^{-\alpha} \mathfrak{D}_l(y)$$

for $\alpha > a + 2$ where $\mathfrak{D}_l$ denotes the characteristic function of the set D_l. We compute according to a change of variables as before

$$\begin{aligned}\iint \frac{\chi_1(x)\mathfrak{D}_1(y)}{|x-y|^2}\, dy\, dx &\le \int 4\pi \int_0^{r_0} \chi_1(x)\, dr\, dx \\ &= 4\pi r_0 \mu(E_1)\end{aligned}$$

where $r_0 = \operatorname{diam}(D \cup E_1)$. For $k \ge 2$,

$$\begin{aligned}\iint \frac{\chi_k(x)\mathfrak{D}_1(y)}{|x-y|^2}\, dx\, dy &\le \iint \frac{\chi_k(x)\mathfrak{D}_1(y)}{(k-1)^2}\, dx\, dy \\ &= \frac{\mu(E_k)\mu(\mathfrak{D}_1)}{(k-1)^2} \\ &= \frac{9\pi\mu(E_k)}{(k-1)^2}.\end{aligned}$$

So,

$$\iint \frac{V_{a,b}(x)\mathfrak{D}_1(y)}{|x-y|^2}\, dx\, dy \le 4\pi r_0 \mu(E_1) + 9\pi \sum_{k=2}^{\infty} \frac{k^a \mu(E_k)}{(k-1)^2}$$

which is finite for $b - a > -1$.

Now, for $k \ge l$,

$$\chi_k(x + (l-1, 0, 0)) \le \chi_{k-l+1}(x)$$

so that, for $l \ge 2$,

$$\begin{aligned}&\iint \frac{\sum_{k \ge l} k^a \chi_k(x)\mathfrak{D}_l(y)}{|x-y|^2}\, dx\, dy \\ &\qquad = \iint \frac{\sum_{j=0}^{\infty} (j+l)^a \chi_{j+l}(x)\mathfrak{D}_l(y)}{|x-y|^2}\, dx\, dy \\ &\qquad \le \iint \frac{\sum_{j=0}^{\infty} l^a((j/l)+1)^a \chi_{j+1}(x)\mathfrak{D}_1(y)}{|x-y|^2}\, dx\, dy \\ &\qquad \le l^a \iint \frac{V_{a,b}(x)\mathfrak{D}_1(y)}{|x-y|^2}\, dx\, dy.\end{aligned}$$

For $k < l$,

$$\iint \frac{k^a \chi_k(x) \mathfrak{D}_l(y)}{|x-y|^2}\, dx\, dy \leq l^a \mu(E_k)\mu(\mathfrak{D}_1)/(1/4)$$

so that

$$\iint \frac{\sum_{j=1}^{l-1} \chi_j(x) \mathfrak{D}_l(y)}{|x-y|^2}\, dx\, dy \leq 4 l^{a+1} \mu(E_1)\mu(\mathfrak{D}_1). \tag{4.3}$$

Therefore, by (4.2) and (4.3) we have that, for some positive constant C independent of l,

$$\iint \frac{V_{a,b}(x) \mathfrak{D}_l(y)}{|x-y|^2}\, dx\, dy < C l^{a+1} \tag{4.4}$$

and, hence, the integral

$$\iint \frac{V_{a,b}(x) \phi_1(y)}{|x-y|^2}\, dx\, dy$$

converges.

Next, we consider functions ϕ_2 with the following properties:

$$\phi_2 \in L^1(\mathcal{C}_0)$$

where

$$\mathcal{C}_0 \stackrel{\text{def}}{=} \{(x_1,x_2,x_3) | x_1 < 0\} \bigcup \left\{(x_1,x_2,x_3) | x_1 \geq 0,\, 2 + x_1 \leq \sqrt{x_2^2 + x_3^2}\right\};$$

and

$$\phi_2(x) \leq e^{-(k+1)|x|^2}$$

on

$$\mathcal{C}_k \stackrel{\text{def}}{=} \left\{(x_1,x_2,x_3) | x_1 > 0,\, 2 + \frac{1}{k+1}\, x_1 < \sqrt{x_2^2 + x_3^2} \leq 2 + \frac{1}{k}\, x_1\right\}$$

for each $k \in \mathbf{N}$, respectively.

Since $d(E_k, \mathcal{C}_0) > k$, we have, for some positive constant C, independent of k,

$$\int_{\mathcal{C}_0}\int_{\mathbf{R}^3} \frac{\chi_k(x)\phi_2(y)}{|x-y|^2}\,dx\,dy \le \frac{1}{k^2}\,\mu(E_k)\int_{\mathcal{C}_0}\phi_2(y)\,dy \le Ck^{-b-2}$$

so that

$$\int_{\mathcal{C}_0}\int_{\mathbf{R}^3} \frac{V_{a,b}(x)\phi_2(y)}{|x-y|^2}\,dx\,dy \le \sum_{k=1}^{\infty} C/k^{b-a+2}$$

which is finite for $b-a>-1$.

For $x \in \operatorname{supp} V$ and $y \in \mathcal{C}_{k-1}$ for $k \ge 2$, we estimate $|x-y|$: It is not difficult to show that for $x=(x_1,1,0)$ and $y=(y_1, 2+(1/k)y_1, 0)$

$$\begin{aligned} |x-y|^2 &\ge \frac{k^2}{(k^2+1)^2}\left[\left(\frac{x_1}{k}+1\right)^2 + (x_1+k)^2\right] \\ &> \frac{1}{2}\,\frac{1}{k^2}\,[x_1+1]^2; \end{aligned} \tag{4.5}$$

and, by the symmetry of these sets about the positive x_1-axis, the same estimate (4.5) holds for all $y \in \mathcal{C}_{k-1}$ and $x \in \operatorname{supp} V$.

Now, for $y \in \mathcal{C}_{k-1}$, we have that $\phi_2(y) \le e^{-k(4+y_1^2)}$ and that

$$\int_{E_l} \frac{\chi_l(x)}{|x-y|^2}\,dx \le \int_{E_l} \frac{2k^2}{(1+x_1)^2}\,dx \le \frac{2k^2\mu(E_l)}{(l+1)^2}.$$

So, we compute, using cylindrical coordinates with $r^2 = y_2^2 + y_3^2$,

$$\begin{aligned} \iint \frac{\chi_l(x)\phi_2(y)}{|x-y|^2}\,dx\,dy &\le 2\pi \int_{E_l}\int_0^{\infty}\int_{2+(y_1/(k+1))}^{2+(y_1/k)} \frac{e^{-k(4-y_1^2)}}{(1+x_1)^2}\,r\,dr\,dy_1\,dx \\ &\le \frac{\pi\mu(E_l)e^{-4k}}{(l+1)^2} \\ &\quad\times \int_0^{\infty} k^2\left[\left(2+\frac{y_1}{k}\right)^2 - \left(2+\frac{y_1}{k+1}\right)^2\right] e^{-ky_1^2}\,dy_1 \\ &\le \frac{Ce^{-4k}}{l^{b+2}} \end{aligned} \tag{4.6}$$

for some positive constant C independent of l and k. Hence, for each l,

$$\iint_{\bigcup_{k=1}^{\infty} \mathcal{C}_k} \frac{\chi_l(x)\phi_2(y)}{|x-y|^2}\, dy\, dx \le \sum_{k=1}^{\infty} \iint_{\mathcal{C}_k} \frac{\chi_l(x)\phi_2(y)}{|x-y|^2}\, dx\, dy$$
$$\le \frac{C}{l^{b+2}} \sum_{k=1}^{\infty} e^{-4k} \le \frac{\tilde{C}}{l^{b+2}}$$

for $\tilde{C} = C/(e^4 - 1)$.

It follows that, for $b - a > -1$,

$$\iint_{\bigcup_{k=1}^{\infty} \mathcal{C}_k} \frac{V_{a,b}(x)\phi_2(y)}{|x-y|^2}\, dx\, dy$$

also converges, and we are done. □

Now, given ϕ as in Proposition 5, define the Hilbert space

$$\tilde{\mathcal{H}} \overset{\text{def}}{=} \left\{ f \in L^2(\mathbf{R}^3) : \frac{f}{\phi} \in L^2(\mathbf{R}^3) \right\}.$$

Since $\phi(x) \in L^2(\mathbf{R}^3)$, $\tilde{\mathcal{H}} \subset L^1(\mathbf{R}^3) \cap L^2(\mathbf{R}^3)$. Hence, functions $f \in \tilde{\mathcal{H}}$, are of Rollnik class and satisfy $|f|^{1/2} \in L^2(\mathbf{R}^3)$. Therefore, the operator $|f|^{1/2}(H_0 - \lambda)^{-1}|f|^{1/2}$ is of Hilbert-Schmidt class. This immediately gives

Proposition 6. *Given $f \in \tilde{\mathcal{H}}$, the function $\lambda \to (f, (H_0 - \lambda)^{-1} f)$ is uniformly bounded for $\lambda \in \mathbf{C} \setminus [0, \infty)$.*

Noting that $\tilde{\mathcal{H}}$ is dense in $L^2(\mathbf{R}^3)$, we apply the criteria of Theorem XIII.19 [**11**] to demonstrate the absence of singular spectrum, $\sigma_{\text{sing}}(H)$, of H with $V = cV_{a,b}$ for certain nonzero constants c.

Theorem 5. *Let $0 < s < t$ be chosen so that $[s, t] \cap \mathcal{E} = \varnothing$.*

a) *If $(I + A_{\sqrt{\lambda}})^{-1}$ is uniformly bounded for λ in a complex neighborhood containing $[s, t]$, then $\sigma_{\text{sing}}(H) \bigcap [s, t] = \varnothing$.*

b) *If c is chosen so that the integral operator $A_{|\kappa|}$ satisfies for some* $0 < \beta < 1$

$$||A_{|\kappa|}||_{2\beta} < 1$$

for some, hence for all, κ, *then* $\sigma_{\text{sing}}(H) = \varnothing$.

Proof. It suffices to show that $(f, R(\lambda)f)$ is uniformly bounded for $\operatorname{Re}\lambda \in [s,t]$ as such for $\operatorname{Im}\lambda > 0$. Choose $f \in \tilde{\mathcal{H}}$ and note that $B_{\bar{\lambda}} \circ \phi$ and $(I + A_{\sqrt{\lambda}})^{-1} \circ B_\lambda \circ \phi$ are each Hilbert-Schmidt (bounded) operators. For $\lambda \notin [0,\infty)$,

$$B_\lambda = V_{a,b}(H_0 - \lambda)^{-1}$$

so that, by using an identity from Section XI.6 [**11**], we obtain for $\lambda \notin \sigma(H)$

$$(H - \lambda)^{-1} = (H_o - \lambda)^{-1} - (B_{\bar{\lambda}})^* \circ [I + A_{\sqrt{\lambda}}]^{-1} \circ B_\lambda.$$

Therefore, for $\operatorname{Im}\lambda > 0$,

$$\begin{aligned}\big(f, (H - \lambda)^{-1} f\big) &- \big(f, (H_0 - \lambda)^{-1} f\big) \\ &= -\left(B_{\bar{\lambda}} \circ \phi\left(\frac{f}{\phi}\right), [I + A_{\sqrt{\lambda}}]^{-1} B_\lambda \circ \phi\left(\frac{f}{\phi}\right)\right) \\ &= -\left(B_{\bar{\lambda}} \circ \phi\left(\frac{f}{\phi}\right), [I + A_{\sqrt{\lambda}}]^{-1} B_\lambda \circ \phi\left(\frac{f}{\phi}\right)\right).\end{aligned}$$

With Proposition 5 in hand, the result of part a) follows since $B_\lambda \circ \phi$ and $B_{\bar{\lambda}} \circ \phi$ are each uniformly bounded in λ.

To prove part b), we note that $||A_{\sqrt{\lambda}}|| \le ||A_{\sqrt{\lambda}}||_{2\beta} < 1$, so that $(I + A_{\sqrt{\lambda}})^{-1}$ is uniformly bounded in λ. □

Remark 4.7. We note that the absence of singular spectra for our operators $H = H_o + cV_{a,b}$ may be shown simply by applying Stone's formula merely for a dense subspace of functions f. Yet, the method above produces an actual weighted Hilbert space on which $(f, R(\lambda)f)$ for $\operatorname{Im}\lambda > 0$ extends continuously to $[0,\infty) \setminus \mathcal{E}$.

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