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A NOTE ON SCHUR-CONVEXITY OF EXTENDED MEAN VALUES

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ABSTRACT. In this article, the Schur-convexity of the extended mean values is proved. Consequently, an inequality between the logarithmic mean values and the identric (exponential) mean values is deduced.

1. Introduction. It is well known that, in 1975, the extended mean values $E(r,s;x,y)$ were defined in [**21**] by Stolarsky as follows

(1) $$E(r,s;x,y)=\left[\frac{r}{s}\cdot\frac{y^s-x^s}{y^r-x^r}\right]^{1/(s-r)}, \quad rs(r-s)(x-y)\neq 0;$$

(2) $$E(r,0;x,y)=\left[\frac{1}{r}\cdot\frac{y^r-x^r}{\ln y-\ln x}\right]^{1/r}, \quad r(x-y)\neq 0;$$

(3) $$E(r,r;x,y)=\frac{1}{e^{1/r}}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r-y^r)}, \quad r(x-y)\neq 0;$$

(4) $$E(0,0;x,y)=\sqrt{xy}, \quad x\neq y; \qquad E(r,s;x,x)=x, \quad x=y;$$

where x, $y>0$ and r, $s\in\mathbf{R}$.

For x, $y>0$ and $t\in\mathbf{R}$, let us define a function g by

(5) $$g(t)=g(t;x,y)=\begin{cases}(y^t-x^t)/t, & t\neq 0;\\ \ln y-\ln x, & t=0.\end{cases}$$

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It is easy to see that g can be expressed in integral form as

$$g(t;x,y)=\int_x^y u^{t-1}\,du, \tag{6}$$

and

$$g^{(n)}(t)=\int_x^y (\ln u)^n u^{t-1}\,du. \tag{7}$$

Recently, a new expression for the ith order derivative of $g(t;x,y)$ with respect to variable t was obtained by the author as follows

$$(-1)^i g^{(i)}(t)=\frac{\Gamma(i+1,-t\ln y)-\Gamma(i+1,-t\ln x)}{t^{i+1}}, \tag{8}$$

where i is a nonnegative integer and Γ denotes the incomplete gamma function defined for $\operatorname{Re} z>0$ by

$$\Gamma(z,x)=\int_x^\infty t^{z-1}e^{-t}\,dt. \tag{9}$$

Therefore, the extended mean values $E(r,s;x,y)$ were represented in terms of g in [**2, 7, 10, 19**] by

$$E(r,s;x,y)=\begin{cases}\left(\dfrac{g(s;x,y)}{g(r;x,y)}\right)^{1/(s-r)} & (r-s)(x-y)\neq 0;\\[2ex] \exp\left(\dfrac{(\partial g(r;x,y)/\partial r)}{g(r;x,y)}\right) & r=s,\ x-y\neq 0\end{cases} \tag{10}$$

and

$$\ln E(r,s;x,y)=\begin{cases}\dfrac{1}{(s-r)}\displaystyle\int_r^s \frac{(\partial g(t;x,y)/\partial t)}{g(t;x,y)}\,dt & (r-s)(x-y)\neq 0;\\[2ex] \dfrac{(\partial g(r;x,y)/\partial r)}{g(r;x,y)} & r=s,\ x-y\neq 0.\end{cases} \tag{11}$$

In 1978, Leach and Sholander [**3**] showed that $E(r,s;x,y)$ are increasing with both r and s, or with both x and y. Later, the monotonicities of E were researched by the author and others in [**2, 12**–**16**] and [**19, 20**] using different ideas and simpler approaches.

In 1983 and 1988, Leach and Sholander [**4**] and Páles [**5**], respectively, solved the problem of comparison of E; that is, they found necessary and sufficient conditions for the parameters r, s and u, v in order that $E(r, s; x, y) \leq E(u, v; x, y)$ be satisfied for all positive x and y.

The concepts of mean values have been generalized or extended by the author in [**7**–**9**] and [**11, 12**].

Recently, the author verified the logarithmic convexity of $E(r, s; x, y)$ with two parameters r and s as follows

Theorem A [**10**]. *For all fixed x, $y > 0$ and $s \in [0, +\infty)$ (or $r \in [0, +\infty)$, respectively), the extended mean values $E(r, s; x, y)$ are logarithmically concave in r (or in s, respectively) on $[0, +\infty)$; for all fixed $x, y > 0$ and $s \in (-\infty, 0]$ (or $r \in (-\infty, 0]$, respectively), the extended mean values $E(r, s; x, y)$ are logarithmically convex in r (or in s, respectively) on $(-\infty, 0]$.*

Definition 1 [**6**, pp. 75–76]. A function f with n arguments defined on I^n is Schur-convex on I^n if $f(x) \leq f(y)$ for each two n-tuples $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in I^n such that $x \prec y$ holds, where I is an interval with nonempty interior.

The relationship of majorization $x \prec y$ means that

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \qquad \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}, \tag{12}$$

where $1 \leq k \leq n-1$, $x_{[i]}$ denotes the ith largest component in x.

A function f is Schur-concave if and only if $-f$ is Schur-convex.

In this article, our main purpose is to prove the Schur-convexity of the extended mean values $E(r, s; x, y)$ with (r, s), and then we obtain the following

Theorem 1. *For fixed (x, y) with $x > 0$, $y > 0$ and $x \neq y$, the extended mean values $E(r, s; x, y)$ are Schur-concave on $\mathbf{R}_+^2$ and Schur-convex on $\mathbf{R}_-^2$ with (r, s), where $\mathbf{R}_+^2$ and $\mathbf{R}_-^2$ denote $[0, +\infty) \times [0, +\infty)$ and $(-\infty, 0] \times (-\infty, 0]$, the first and third quadrants, respectively.*

Considering $(r_1, s_1) = (0, 2r)$ and $(r_2, s_2) = (r, r)$ for $r \neq 0$, as a direct consequence of Theorem 1, we obtain an inequality between the logarithmic mean values (2) and the identric (exponential) mean values (3) as follows

Corollary 1. *Let $x, y > 0$ and $x \neq y$. Then, for $r > 0$, we have*

$$\left[\frac{1}{2r}\cdot\frac{y^{2r}-x^{2r}}{\ln y-\ln x}\right]^{1/(2r)} \leq \frac{1}{e^{1/r}}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r-y^r)}. \tag{13}$$

For $r < 0$, inequality (13) *reverses.*

2. Lemmae. In order to prove Theorem 1, we need the following lemmae.

Lemma 1 [**1**]. *Let f be a continuous function on I. Then the arithmetic mean of function f* (*or the integral arithmetic mean*),

$$\phi(u, v) = \begin{cases} \dfrac{1}{(v-u)}\displaystyle\int_u^v f(t)\,dt & u \neq v, \\ f(u) & u = v, \end{cases} \tag{14}$$

is Schur-convex (*Schur-concave*) *on I^2 if and only if f is convex* (*concave*) *on I.*

By formula (11) and Lemma 1, it is easy to see that, to prove the Schur-convexity of the extended mean values $E(r, s; x, y)$ with (r, s), it suffices to verify the convexity of function

$$\frac{g'(t)}{g(t)} \triangleq \frac{g'_t(t;x,y)}{g(t;x,y)} \triangleq \frac{\partial g(t;x,y)}{\partial t}\cdot\frac{1}{g(t;x,y)} \tag{15}$$

with respect to t, where $g(t) = g(t; x, y)$ is defined by (5) or (6).

Straightforward computation results in

$$\left(\frac{g'(t)}{g(t)}\right)' = \frac{g''(t)g(t)-[g'(t)]^2}{g^2(t)}, \tag{16}$$

$$\left(\frac{g'(t)}{g(t)}\right)'' = \frac{g^2(t)g'''(t)-3g(t)g'(t)g''(t)+2[g'(t)]^3}{g^3(t)}. \tag{17}$$

Lemma 2 [**10**]. *If $y > x = 1$, then, for $t \geq 0$,*

$$g^2(t;1,y)g_t'''(t;1,y) - 3g(t;1,y)g_t'(t;1,y)g_t''(t;1,y) + 2[g_t'(t;1,y)]^3 \leq 0. \tag{18}$$

Lemma 3. *If $y > x = 1$, then, for $t \geq 0$, the function $g'(t)/g(t)$ is concave.*

Proof. This follows from using a combination of formulae (15), (16) and (17) with Lemma 2 easily. □

3. Proof of Theorem 1. It is evident that $E(r,s;x,y)$ is symmetric with (r,s) since we have $E(r,s;x,y) = E(s,r;x,y)$.

Combining Lemma 2 with equality (17) shows that the function $g_t'(t;1,y)/g(t;1,y)$ is concave on $[0,+\infty)$ with t for $y > x = 1$. Therefore, from Lemma 1, it follows that the extended mean values $E(r,s;1,y)$ are Schur-concave with (r,s) on $[0,+\infty) \times [0,+\infty)$ for $y > x = 1$.

By standard arguments, we obtain

$$E(r,s;x,y) = xE(r,s;1,(y/x)), \tag{19}$$

$$E(-r,-s;x,y) = \frac{xy}{E(r,s;x,y)}. \tag{20}$$

Hence, for fixed x and y, the extended mean values $E(r,s;x,y)$ are Schur-concave on $[0,+\infty) \times [0,+\infty)$ and Schur-convex on $(-\infty,0] \times (-\infty,0]$ with (r,s). The proof of Theorem 1 is complete.

Remark. Recently, the Schur-convexities with (x,y) of the extended mean values $E(r,s;x,y)$ were obtained, see [**13, 17**].

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