ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 35, Number 5, 2005

SYMPLECTIC GEOMETRY FOR PAIRS OF SUBMANIFOLDS

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ABSTRACT. Darboux's classical theorem in symplectic geometry is generalized to pairs of transversal submanifolds.

1. Introduction. A smooth manifold V imbued with a closed, nondegenerate 2-form ω is called a symplectic manifold. The symplectic form ω gives the manifold a geometric structure (signed area, instead of length as in Riemannian geometry), and the closedness controls the topology of V. Symplectic manifolds play an important role in classical mechanics, geometrical optics, representation theory, and Kähler geometry. A variety of fundamental results in symplectic geometry provide for local characterizations of various geometric objects: symplectic manifolds, submanifolds, foliations, etc., the most fundamental and elementary of which is Darboux's theorem:

Theorem 1 (Darboux's theorem). *Every point of a symplectic manifold has local coordinates* (x_i, y_i), $i = 1, \dots, n$, *so that*

$$\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n.$$

We can conclude that, in stark contrast to Riemannian geometry, there are no local invariants other than dimension and that this dimension must be even. Another perspective on Darboux's theorem is this: Any two symplectic forms induce the same form on a point (the zero form) and so the intrinsic symplectic geometry of a point completely determines the symplectic geometry nearby.

In this paper we examine the extent to which the interior geometry of a pair of submanifolds determines its exterior geometry, a special

2000 AMS *Mathematics Subject Classification.* Primary 53Cxx, Secondary 53C15.
Received by the editors on August 1, 2001, and in revised form on January 28, 2003.

case of the much more difficult problem of understanding symplectic singularities. (For simplicity we will assume that all manifolds and submanifolds are closed.) Similar results for submanifolds were given by Weinstein and others, such as:

Definition 1. A *Lagrangian submanifold* of a $2n$-dimensional symplectic manifold (V, ω) is an embedding $f : M \to V$ of an n-dimensional submanifold so that $f^{\star}\omega = 0$ everywhere. We will also call the image $f(M)$ a Lagrangian submanifold.

Theorem 2 [**12**]. *Let V be a smooth $2n$-dimensional manifold with symplectic form ω, and suppose that M is a Lagrangian submanifold. Then there is a diffeomorphism ψ of a neighborhood of M with a neighborhood of any other Lagrangian embedding of M so that ψ preserves the symplectic structures.*

This remarkable result states that the interior geometry of a Lagrangian submanifold determines the geometry nearby. In addition to other results of this nature, some progress has been made for pairs of submanifolds. Melrose proved the following for transversal pairs of hypersurfaces (actually, glancing hypersurfaces which have additional restrictions which I won't go into) as part of a solution to the boundary value problem for glancing rays in the theory of billiards:

Theorem 3 [**10**]. *All glancing hypersurfaces in symplectic manifolds of a fixed dimension are locally equivalent.*

Our main result gives a condition for the global equivalence of transversal pairs of submanifolds.

Definition 2. Let V be a smooth manifold with two symplectic forms ω_0 and ω_1, and suppose M and N are transversal submanifolds. We will say that the forms are coincident if they induce the same forms on the submanifolds M and N, and if the forms agree at points lying in $M \cap N$.

Theorem 4 (Main theorem). *Let there be given a transversal pair of smooth closed submanifolds and two germs of coincident symplectic structures. If these symplectic forms can be deformed into one another inside the class of symplectic structures coincident with them, then the germs are symplectomorphic.*

2. Related results. The spirit of our main theorem is that the interior geometry does not give us enough information to determine the geometry nearby; that we must also know the exterior geometry along the intersection of the two submanifolds. The following straightforward example illustrates the problem:

Example. Let $M,\ N \subset \mathbf{R}^4$ be the 2-planes $\{(x_1, y_1, 0, 0)\}$ and $\{(0, 0, x_2, y_2)\}$, respectively. Let $\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ and $\omega_1 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_1 \wedge dx_2$. So ω_1 and ω_2 induce the same symplectic forms on M and N, but there can be no symplectomorphism between neighborhoods of $M \cup N$ as M and N are skew-orthogonal with respect to ω_0 but not with respect to ω_1.

Problem. This example brings up an interesting question: Let $\mathbf{C}^\star := \mathbf{C} \setminus \{0\}$. Are there then two symplectic forms on $\mathbf{C} \times \mathbf{C}$ where $\mathbf{C}^\star \times \mathbf{C}^\star$ is a symplectic submanifold for both forms so that no symplectomorphism between them can be extended to all of $\mathbf{C} \times \mathbf{C}$?

According to Melrose and Arnold [**2**], Riemannian geometry is a special case of the symplectic geometry of pairs of submanifolds, and a deeper understanding of all invariants associated to such pairs should prove interesting. For example let us start with a convex planar curve. The set of all tangent vectors of length 1 defines a hypersurface in $\mathbf{R}^4 \simeq T\mathbf{R}^2$ called the *surface of unit vectors*. The set of all vectors, of any length, based along the curve is also a hypersurface called the *surface of boundary vectors*. There is a natural flow defined on these surfaces (the characteristic flow) by the natural symplectic form on $\mathbf{R}^4$, and in fact the flow on the surface of unit vectors is the geodesic flow. The symplectic geometry of this pair has much to say about the billiard problem based on the planar curve, and Ahdout [**1**] has shown that the curve's curvature is a symplectic invariant.

It is probably a hopeless task to give global normal forms for pairs of submanifolds as this problem looks impossible even for one surface. Local normal forms are more approachable, see [**9**], and in fact something similar was done for the geometry of bihamiltonian structures [**4**]. These kinds of problems can be simplified by adding extra conditions, such as assuming that surfaces are of constant rank, see, for example, [**11**], or relaxing the requirements of the theorem, such as not requiring knowledge of the normal form near the intersection surface. This last approach was used by Gompf [**5**] and McDuff and Symington [**8**] in the development and application of symplectic surgery techniques requiring cutting and pasting along pairs of submanifolds.

3. Proof of main theorem. Our main theorem is the analog of the following result for submanifolds:

Theorem 5 [**3**]. *Let there be given a smooth submanifold $N \subset V$ and two germs of coincident symplectic structures. If these symplectic forms can be deformed into one another inside the class of symplectic structures coincident with them, then the germs are symplectomorphic.*

The proof of this and the main theorem uses the relative Poincaré lemma, which is stated below for the reader's convenience as it does not seem to be well known [**7**].

Theorem 6 (Relative Poincaré lemma). *Let (F, π, N) be a vector bundle. We identify its base N with the closed submanifold $i(N)$ of F by means of the zero-section $i : N \to F$. For every real t, let λ_t denote the map from F into itself which is multiplication by t on the fibers. Let O be an open subset of F such that $\lambda_t(O)$ is contained in O for every element $t \in [0, 1]$. Let τ be a differential k-form on O. We assume that τ is closed and that the form induced on N by τ vanishes identically,*

$$d\tau = 0, \qquad i^\star \tau = 0.$$

Then there exists a differential $(k-1)$-form σ which vanishes at points belonging to N and such that

$$d\sigma = \tau.$$

If, in addition, τ vanishes at points belonging to N, we may choose σ so that the first-order partial derivatives of its components with respect to the local coordinates, in any chart, vanish identically on N.

Note. In this paper, i will denote inclusion and ι will denote contraction.

The form σ is defined as $\sigma = -H\tau$ where $H\tau$ is the homotopy operator $\int_0^1 \xi_t^\star(\iota(\eta_t)\tau)\, dt$ and where the isotopy ξ_t and time-dependent vector field η_t are defined as follows: $\xi(t,x) = \xi_t(x) = \lambda_{1-t}x$ and ξ_t is the reduced flow of the time-dependent (and vertical) vector field η_t (wherever all this makes sense).

It is an open problem as to whether or not the relative Poincaré lemma can be extended from submanifolds to varieties. The proof of our main theorem can be viewed as a proof of one version of this lemma for 2-forms on a pair of transversal submanifolds.

Proof of Theorem 5. The proof of this theorem introduces notation and illustrates *Moser's homotopy method* and so will form the base of the argument proving the main theorem. The notation as well as the proof is taken from Vaisman [**11**, Lemma 3.4.13].

By assumption there exists a smooth one-parameter family of symplectic forms ω_t, $0 \leq t \leq 1$, connecting ω_0 to ω_1 which induce constant forms on the subtangent bundle of N, i.e., $i^\star\omega_t = i^\star\omega_0$ where $i : N \hookrightarrow V$ is the inclusion map. Since the family of 2-forms

$$\tau_t = \frac{d}{dt}(\omega_t - \omega_0) = \frac{d}{dt}\omega_t$$

is closed and induces the zero form on the tangent bundle of TN, the relative Poincaré lemma (F can be any normal bundle of N in V; we will identify normal bundles with an image in V) guarantees the existence of a smooth family of 1-forms σ_t defined near N such that

$$\begin{aligned} \tau_t &= d\sigma_t \\ \sigma_t &= 0 \quad \text{on} \quad T_N V. \end{aligned}$$

We will define a family of diffeomorphisms ϕ_t satisfying $\phi_t^\star \omega_t = \omega_0$ by representing them as the time-dependent flow of a smooth family of vector fields X_t defined near N, with ϕ_1 as our sought after symplectomorphism. Define ϕ_t near N by the equation (we will define X_t momentarily)

$$\frac{d}{dt}\,\phi_t = X_t \circ \phi_t, \quad \phi_0^\star = id\,.$$

Differentiating $\phi_t^\star \omega_t$ with respect to time yields (see, for example, [**11**, pp. 90–91] for an explanation of the notation: there is no missing $\phi_t^\star$)

$$\begin{aligned} 0 &= \frac{d}{dt}\,\phi_t^\star\,\omega_t = \phi_t^\star \frac{d}{dt}\,\omega_t + \iota_{Xt} d\,\omega_t + d(\iota_{Xt}\,\omega_t) \\ &= \phi_t^\star d\sigma_t + 0 + d(\iota_{Xt}\,\omega_t) \end{aligned}$$

(here $\iota_{Xt}\,\omega_t$ denotes the contraction of ω_t by plugging X_t into the first slot of ω_t) and so we only require that $d(\iota_{Xt}\,\omega_t) = -\phi_t^\star d\sigma_t$. Now define X_t by the equation

$$\iota_{Xt}\,\omega_t = -\sigma_t.$$

The non-degeneracy of ω_t guarantees that X_t is uniquely defined. Since X_t vanishes along N the maps ϕ_t restrict to the identity map there, and so ϕ_1 is our sought after symplectomorphism. □

Proof of main theorem. By assumption there exists a smooth one-parameter family of coincident symplectic forms ω_t, $0 \leq t \leq 1$, connecting ω_0 to ω_1. If we simply ignore the submanifold M for the moment and apply the above construction to N, then we will obtain a family of diffeomorphisms ϕ_t on a neighborhood of N which induce the identity map on N and which also satisfy $\phi_t^\star \omega_t = \omega_0$ on and near $T_N V$. As M is almost certainly moved off of itself by these transformations we now need to alter ϕ_t by judiciously composing this family of maps with another.

The relative Poincaré lemma guarantees that not only will σ_t vanish along N, but that its 1-jet vanishes along $M \cap N$ as well (since $\tau_t = 0$ there). Lemma 1 below shows that $\phi_t^\star$ is the identity transformation along $M \cap N$. By Lemma 2 we may compose ϕ_t with a family of maps ψ_t, defined on a neighborhood of N, which bend $\phi_t(M)$ back to M so that $\psi_t \circ \phi_t$ induces the identity map on M, fixes N, and whose Jacobian maps are the identity along N. We extend the family $\psi_t \circ \phi_t$

of diffeomorphisms near N to a neighborhood of $M \cup N$ by the isotopy extension theorem [**6**]:

Theorem 7. *Let $O \subset V$ be an open set and $N \subset O$ a compact set. Let $F : O \times I \to V$ be an isotopy of O such that $\hat{F}(O \times I) \subset V \times I$ is open, where we define the track of the isotopy F to be the embedding $\hat{F} : O \times I \to V \times I$, $(x,t) \mapsto (F(x,t),t)$. Then there is a diffeotopy of V having compact support, which agrees with F on a neighborhood of $N \times I$.*

The construction of this diffeotopy, as given in [**6**], will fix all points of M. We will then have a family of diffeomorphisms η_t on a neighborhood of M and N which induce the identity map on both M and N and so that $\eta_t^\star \omega_t = \omega_0$ on $T_N V$.

So we now have two germs of coincident symplectic structures which have the stronger property that they are isotopic to each other via coincident forms which actually agree with each other on $T_N V$. We again apply Moser's homotopy method, not to N this time, but to M. We will then obtain a family of symplectomorphisms on a tubular neighborhood of M that fixes N (by definition of the homotopy operator H and by choosing the vector bundle F so that M will be contained in the fibers of F), a family which leaves the forms invariant on TN. These forms can then be trivially extended so that they are defined on TN, and so that we have a family of symplectomorphic forms which are identical in a neighborhood of M and which induce the same form on all of TN. We do one final application of Moser's homotopy method to N again: The forms will be unchanged near M and so we will be done, having constructed an isotopy of ω_0 to ω_1 through coincident forms, an isotopy which leaves M and N fixed. □

4. Lemmas 1 and 2.

Lemma 1. *$\phi_t^\star(x)$ is the identity transformation for $x \in M \cap N$ and for each $t \in [0,1]$.*

Proof. We will show that $D\phi_t(Y) = Y$ for all $Y \in T_x V$ and for all $0 \le t \le 1$. Extend Y to a smooth vector field defined near $x \in M \cap N$, and call it Y also. If $D\phi_t(Y)$ is constant along $M \cap N$ (so that its time derivative is 0), then $D\phi_t(Y) = D\phi_0(Y) = Y$ for all t, and our lemma will be proved. Now since Y is defined independently of time, we can write

$$\frac{d}{dt}\left[D\phi_t(Y)\right]_{t=t_0} = D\phi_t(t_0)\left(\frac{dY}{dt}|_{t=t_0} + [X_{t_0}, Y]\right) = D\phi_t(t_0)\left([X_{t_0}, Y]\right)$$

where X_t is the vector field constructed above. It remains to show that $[X_{t_0}, Y]$ vanishes along $M \cap N$, but this is a consequence of the fact that the 1-jet of σ_t vanishes there and that X_t is defined by $\omega_t(X_t, \star) = \sigma_t$. $\square$

Lemma 2. *There exists a family of diffeomorphisms ψ_t defined on a neighborhood of N so that the maps $\psi_t \circ \phi_t$ induce the identity map on N and M (for points of M near N), $(\psi_t \circ \phi_t)^\star \omega_t = \omega_0$ on N, and $i_M^\star((\psi_t \circ \phi_t)^\star \omega_t) = i_M^\star(\omega_0)$ where $i_M : M \to V$ is the inclusion map.*

Proof. Our argument will closely follow the proof of the isotopy extension theorem that is given in [**6**] and which we slightly modify for our purposes here.

Let V have dimension v, M dimension m and N dimension n. Since M and N are transversal, the dimension of $M \cap N$ is $m + n - v$. Let $E_N \to M \cap N$ be any normal sub-bundle of TN over $M \cap N$ (via an appropriate choice of metric) and then let $E \to M$ be any smooth extension of E_N to a normal bundle over M. So the dimension of any fiber of E is $v - m$. A similar definition defines the bundles F_M and F whose fibers have dimension $v-n$. We will freely identify neighborhoods of the base spaces of these bundles with their embeddings into V.

Our choice of a smooth metric on E allows us to define an ε neighborhood $D_\varepsilon(p)$ of each base point $p \in M$ in the fibers of E for p near $M \cap N$. Furthermore, we can assume that $D_\varepsilon(p)$ intersects each submanifold $\phi_t(M)$ in precisely one point. We define an open neighborhood $O = \bigcup D_\varepsilon(p)$ and let $U \subset O$ be a neighborhood so that ${\phi_t}^{-1}(U) \subset O$ for each $t \in [0,1]$ and so that U contains $M \cap N$.

Define the diffeotopy, that is, ambient isotopy, $\Phi^{-1} : U \times I \to V$ where $\Phi^{-1}(x,t) = \phi_t^{-1}(x)$. Then the tangent vectors to the curves $\hat{\Phi}^{-1} : x \times I \to V \times I$ $(x \in U)$ define a vector field X on $\hat{\Phi}^{-1}(U \times I)$

of the form $X(y,t) = (H(y,t),1)$. Here $H : \hat{\Phi}^{-1}(U \times I) \to TV$ with $H(y,t) \in T_yV$.

By means of a partition of unity argument we can construct a time-dependent vector field $G : V \times I \to TV$ which agrees with H on a neighborhood of $(M \cap N) \times I$. Since U is a compact neighborhood we can make G have compact support. Furthermore G may be constructed so that $G : N \times I \to T_NV$ is the zero-section: This is possible since $H(y,t)$ is the zero vector of T_yV for $y \in N$ and we can use the fiber structure of the normal bundle F to extend H. Finally, our extension G may be chosen so that the Jacobian transformations which are induced on T_NV are the identity transformations (by Lemma 1 this property is true for H). In this way we have defined a diffeotopy $\Psi(y,t) = \psi_t(y)$ on a neighborhood O of N (shrink O if necessary) such that $\psi_t \circ \phi_t$ fixes M and whose Jacobian maps on T_NV are the identity transformations. □

Acknowledgments. I would like to thank Serge Tabachnikov for his helpful suggestions on the contents of this paper.

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