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SUBSPACES WITH NONINVERTIBLE ELEMENTS IN Re $C(X)$

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ABSTRACT. Let X be a compact Hausdorff space, and let M be a subspace of $\operatorname{Re} C(X)$ consisting only of noninvertible elements. We show that there exist closed sets $Y \subset X$ such that each element of M has a zero in Y and no closed subset of Y has this property; furthermore, such a Y is a singleton, or has no isolated points. If M has finite codimension n and Y is not a singleton, then Y is a union of at most n nontrivial connected components. We also show that positive functionals exist in $M^{\perp}$.

1. Introduction. Throughout this paper we assume that X is an arbitrary compact Hausdorff space. Denote by $C(X)$, respectively $\operatorname{Re} C(X)$, the space of all continuous complex, respectively real, functions on X.

In this section we discuss the motivation and a brief history of studying subspaces with noninvertible elements in $C(X)$ and $\operatorname{Re} C(X)$.

Plainly, every ideal of $C(X)$ or $\operatorname{Re} C(X)$ is a subspace consisting only of noninvertible elements. Let us call a subspace M of $C(X)$ or $\operatorname{Re} C(X)$ a $\mathcal{Z}$-subspace if M is consisting only of noninvertible elements. In other words, M is a $\mathcal{Z}$-subspace if for each $f \in M$ there exists $x \in X$ such that $f(x) = 0$.

So, every subspace of an ideal in $C(X)$ or $\operatorname{Re} C(X)$ is a $\mathcal{Z}$-subspace. It is easy to construct $\mathcal{Z}$-subspaces in $\operatorname{Re} C[0,1]$ which are not contained in maximal ideals. For example, let $M = \{f : f(0) + f(1) = 0\}$. Each $f \in M$ has a zero in $[0,1]$, by the intermediate value theorem, but clearly M is not contained in an ideal.

The situation for $C(X)$ is completely different. Studying $\mathcal{Z}$-subspaces begins with the following famous result due to Gleason [**2**] and Kahane and Zelazko [**5**]:

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Theorem 1.1. *A $\mathcal{Z}$-subspace of codimension* 1 *in a unital complex commutative Banach algebra is a maximal ideal.*

What about other codimensions, finite or infinite? Examples ([**3**, Section 2] and [**6**, Section 2]) show that in $C(X)$, arbitrary $\mathcal{Z}$-subspaces are not necessarily contained in maximal ideals. However, the following result of Jarosz [**3**] is interesting.

Theorem 1.2. *Every finite codimensional $\mathcal{Z}$-subspace of $C(X)$ is contained in a maximal ideal.*

Farnum and Whitley [**1**, Theorem 1], gave the following characterization of $\mathcal{Z}$-subspaces with codimension 1 in $\operatorname{Re} C(X)$. Recall that the dual space $\mathcal{M}(X)$ of $\operatorname{Re} C(X)$ is the space of all regular real Borel measures on X.

Theorem 1.3 [**1**]. *Let φ be a linear functional of norm* 1 *on* $\operatorname{Re} C(X)$ *such that $\varphi(f) \in \operatorname{Im}(f)$ for all $f \in \operatorname{Re} C(X)$ (this is equivalent to saying that $M = \ker \varphi$ is a $\mathcal{Z}$-subspace). Then φ is a positive measure supporting on a connected component of X.*

Corollary 1.4. *Let X be totally disconnected, for example, the Cantor set. Then a $\mathcal{Z}$-subspace of codimension* 1 *in* $\operatorname{Re} C(X)$ *is a maximal ideal.*

The author and Seddighi have shown the following combination of Theorem 1.2 and Corollary 1.4 ([**6**, Theorem 3.1]).

Theorem 1.5. *Let X be totally disconnected. Then each finite codimensional $\mathcal{Z}$-subspace of* $\operatorname{Re} C(X)$ *is contained in a maximal ideal.*

In the following two sections, we represent more general results for $\mathcal{Z}$-subspaces in $\operatorname{Re} C(X)$.

2. $\mathcal{Z}$-subspaces and $\mathcal{Z}$-supports. Let M be a $\mathcal{Z}$-subspace of $\operatorname{Re} C(X)$. We call a closed set $Y \subseteq X$ a $\mathcal{Z}$-support for M if every element of M has a zero in Y. In particular, X is a $\mathcal{Z}$-support for M. A $\mathcal{Z}$-support for M is called a minimal $\mathcal{Z}$-support, if Y is a $\mathcal{Z}$-support and no proper closed subset of Y is a $\mathcal{Z}$-support for M. In other words, Y is a minimal $\mathcal{Z}$-support for M if and only if for each proper closed set $F \subset Y$, there exists $f \in M$ such that $f \neq 0$ everywhere on F. One can easily verify that the uniform closure of a $\mathcal{Z}$-subspace is a $\mathcal{Z}$-subspace, and both have the same $\mathcal{Z}$-supports.

Theorem 2.1. *Let M be a $\mathcal{Z}$-subspace of* $\operatorname{Re} C(X)$. *Then each $\mathcal{Z}$-support for M contains a minimal $\mathcal{Z}$-support. In particular, since X is a $\mathcal{Z}$-support, minimal $\mathcal{Z}$-supports exist for M.*

Proof. Suppose Y is a $\mathcal{Z}$-support for M. Consider the class

$$\mathcal{T} = \{S \subseteq Y : \ S \ \text{ is a } \mathcal{Z}\text{-support for } \ M\}.$$

Since $Y \in \mathcal{T}$, $\mathcal{T} \neq \varnothing$. Let $\{S_\alpha\}$ be a chain in $\mathcal{T}$. Compactness of Y implies that $\cap S_\alpha \in \mathcal{T}$. It follows that minimal elements, in the sense of inclusion, exist in $\mathcal{T}$. They are minimal $\mathcal{Z}$-supports for M. □

Lemma 2.2. *Let $Y = Y_1 \cup Y_2$ be a $\mathcal{Z}$-support for a $\mathcal{Z}$-subspace M in* $\operatorname{Re} C(X)$, *where Y_1 and Y_2 are disjoint closed nonvoid sets. Furthermore assume that there exists $f \in M$ such that f is constantly zero on Y_1 and $f \neq 0$ everywhere on Y_2. Then Y_1 is a $\mathcal{Z}$-support for M. In particular Y cannot be a minimal $\mathcal{Z}$-support for M.*

Proof. Let $g \in M$. We have to show that g has a zero in Y_1. Since $|f| > 0$ on Y_2, and Y_2 is compact, we can choose a real scalar β so large that $|\beta f + g| > 0$ on Y_2. But $\beta f + g \in M$ and hence $(\beta f + g)(x) = 0$, for some $x \in Y$. Clearly, $x \notin Y_2$, so that $x \in Y_1$. This gives $g(x) = 0$; the desired result. □

Theorem 2.3. *A minimal $\mathcal{Z}$-support for a $\mathcal{Z}$-subspace in* $\operatorname{Re} C(X)$ *is either a singleton, or has no isolated points.*

Proof. Let $Y = \{a\} \cup S$ be a $\mathcal{Z}$-support for a $\mathcal{Z}$-subspace M, where S is closed and $a \notin S$. If there exists $f \in M$ such that $|f| > 0$ on S, then $f(a) = 0$. Therefore by Lemma 2.2, we conclude that a is a common zero for all elements of M. Otherwise, each element of M has a zero in S. So, either $\{a\}$ or S is a $\mathcal{Z}$-support for M. Therefore, a minimal $\mathcal{Z}$-support for M, if not a singleton, cannot have any isolated points. □

Theorem 2.4. *Let M be a $\mathcal{Z}$-subspace of $\mathrm{Re}\, C(X)$ with finite codimension n and Y a minimal $\mathcal{Z}$-support for M. Then one and only one of the following statements holds*:

1. *Y is a singleton*;
2. *Y is a union of at most n nontrivial connected components.*

Proof. In view of Theorem 2.3, it suffices to prove that Y has not more than n connected components. On the contrary, suppose $Y = C_1 \cup \cdots \cup C_{n+1}$, where C_i's are disjoint closed nonvoid sets. For $1 \leq i \leq n+1$, the characteristic function $\mathcal{X}_i$ of C_i is continuous on Y; let f_i be a continuous extension of $\mathcal{X}_i$ to whole X. Since M is of codimension n, there exist scalars $c_1, \ldots, c_{n+1}$, not all zero, such that $f = c_1 f_1 + \cdots + c_{n+1} f_{n+1} \in M$. Clearly, $f = c_i$ identically on C_i. Now we have the decomposition $Y = Y_1 \cup Y_2$, where $Y_1 = \cup_{c_j=0} C_j$ and $Y_2 = \cup_{c_j \neq 0} C_j$. The set Y_1 is closed and not empty, since f vanishes on Y. Also Y_2 is closed and not empty, since some scalars $c_i \neq 0$. So f is constantly zero on Y_1 and $|f| > 0$ on Y_2. Lemma 2.2 implies that Y_1 is a $\mathcal{Z}$-support for M. This contradicts the minimality of Y. □

We now get Theorem 1.5 as a simple corollary:

Corollary 2.5. *Let X be totally disconnected. Then each finite codimensional $\mathcal{Z}$-subspace of $\mathrm{Re}\, C(X)$ is contained in a maximal ideal.*

Proof. Suppose M is a finite codimensional $\mathcal{Z}$-subspace in $\mathrm{Re}\, C(X)$. Let Y be a minimal $\mathcal{Z}$-support for M. Since it is not possible to have any nontrivial component for Y, necessarily it is a singleton. That is, Y is contained in a maximal ideal. □

The next theorem shows that the number n cannot be reduced in case 2 of Theorem 2.4 above. More precisely, if X has at least n nontrivial connected components, then there exist a $\mathcal{Z}$-subspace M with codimension n such that each $\mathcal{Z}$-support for M has at least n components. It follows that each minimal $\mathcal{Z}$-support for such an M has exactly n components.

Theorem 2.6. *Let $X = X_1 \cup \cdots \cup X_n$, where X_i's, $1 \leq i \leq n$, are nontrivial disjoint connected components in X. Then there exists a $\mathcal{Z}$-subspace M of codimension n in $\operatorname{Re} C(X)$ such that each minimal $\mathcal{Z}$-support for M intersects each X_i, for $1 \leq i \leq n$.*

Proof. If $n = 1$, there is nothing to prove; so assume $n > 1$. Since each X_i is nontrivial, we can choose two distinct points x_i and y_i in X_i, $1 \leq i \leq n$. Define the subspace M of $\operatorname{Re} C(X)$ by

$$M = \{f : \ f(x_i) + f(y_{i+1}) = 0, \text{ if } i \neq n \text{ and } f(x_n) + (-1)^{n+1} f(y_1) = 0\}.$$

We claim that M has the desired properties. Plainly M is of codimension n. Now we show that M is a $\mathcal{Z}$-subspace. To this end, let $f \in M$ and suppose, to get a contradiction, that f has no zero in X. So, by replacing f with $-f$, if necessary, we can assume that f is strictly positive on X_1, since X_1 is connected; specially $f(x_1) > 0$. We have $f(x_1) + f(y_2) = 0$. This gives $f(y_2) < 0$. It follows that f is always negative on X_2, by connectedness of X_2. Continuing in this way, and using the equalities $f(x_i) + f(y_{i+1}) = 0$, for $1 \leq i \leq n-1$, we conclude that f is strictly positive on X_i, if i is odd, and strictly negative on X_i, if i is even.

Now the equality $f(x_n) + (-1)^{n+1} f(y_1) = 0$ implies that $f(y_1) < 0$. But we had $f > 0$ on X_1. This contradiction shows that M is a $\mathcal{Z}$-subspace.

Next we show that every $\mathcal{Z}$-support for M contains x_1. Suppose Y is a $\mathcal{Z}$-support for M. If $x_1 \notin Y$, the Uryson lemma provides a continuous function g on X_1 such that $g(x_1) = 1$ and $g = -1$ constantly on $(Y \cap X_1) \cup \{y_1\}$. Extend g so that it is equal to $(-1)^i$ on each X_i, $2 \leq i \leq n$. Evidently, $g \in M$, but g has no zero on Y. This contradiction shows that Y must contain x_1. Similarly Y contains all other x_i's (and y_i's, of course), for $1 \leq i \leq n$. Therefore Y intersects all X_i. $\square$

Example 2.7. Let $X = [0,1] \cup [2,3]$, and

$$M = \{f \in \operatorname{Re} C(X) : \; f(0) + f(2) = 0 \text{ and } f(1) = f(3)\}.$$

As in the proof of the above theorem, we see that M is a $\mathcal{Z}$-subspace with codimension 2 and if Y is a $\mathcal{Z}$-support for M, then $0, 1, 2, 3 \in Y$. It is now easy to see that the only $\mathcal{Z}$-support for M is X, which would be of course minimal.

Example 2.8. Let $X = I^2$, the closed unit square, and

$$M = \{f \in \operatorname{Re} C(X) : \; f(0,0) + f(1,1) = 0\}.$$

Clearly M is a $\mathcal{Z}$-subspace. The graph of the functions $y = x^m$, $x \in [0,1]$, for positive integers m are all different minimal $\mathcal{Z}$-supports for M. In fact, minimal $\mathcal{Z}$-supports for M are minimal connected subsets of X containing $(0,0)$ and $(1,1)$, and they are the graphs of continuous 1-1 curves inside I^2 which connect $(0,0)$ to $(1,1)$.

We can define a maximal $\mathcal{Z}$-subspace to be a $\mathcal{Z}$-subspace M such that no subspace containing M is a $\mathcal{Z}$-subspace. In this sense, the subspace constructed in Theorem 2.6 is a maximal $\mathcal{Z}$-subspace (Examples 2.7 and 2.8 are special cases). The reason for maximality of M is the following. If M' is a $\mathcal{Z}$-subspace properly containing M, then M' has codimension $< n$. Let Y be a minimal $\mathcal{Z}$-support for M'. Then Y has at most $n-1$ connected components. But Y is also a $\mathcal{Z}$-support for M, and this contradicts the fact that every $\mathcal{Z}$-support for M has at least n components.

It is easy to see that every $\mathcal{Z}$-subspace is contained in a, not necessarily unique, maximal $\mathcal{Z}$-subspace.

3. $\mathcal{Z}$-subspaces and positive functionals. In this section we investigate the relationship between the $\mathcal{Z}$-subspaces and positive functionals (measures).

Note that a regular Borel positive measure μ of norm 1 on X is supported on $Y \subseteq X$ if and only if $\mu \in \overline{\mathrm{co}}\widetilde{Y} \subseteq \mathcal{M}(X)$. Here $\widetilde{Y}$ denotes the set of Dirac measures $\tilde{y}$ supported on points $y \in Y$, and $\overline{\mathrm{co}}\widetilde{Y}$ is the weak* closure of the convex hull of $\widetilde{Y} \subset \mathcal{M}(X)$. The Banach-Alaoglu theorem implies that $\overline{\mathrm{co}}\widetilde{Y}$ is weak* compact. For a subspace$M$ of $\mathrm{Re}\, C(X)$, denote

$$M^{\perp} = \Big\{\mu \in \mathcal{M}(X) : \int f\, d\mu = 0, \text{ for all } f \in M\Big\}.$$

Theorem 3.1. *If M is a $\mathcal{Z}$-subspace of $\mathrm{Re}\, C(X)$ and $Y \subseteq X$ is a $\mathcal{Z}$-support for M, then $M^{\perp}$ contains positive measures supported on Y.*

Proof. We have to prove that $M^{\perp} \cap \overline{\mathrm{co}}\widetilde{Y} \neq \varnothing$. Let $M^{\perp} \cap \overline{\mathrm{co}}\widetilde{Y} = \varnothing$. Since $M^{\perp}$ is weak* closed and $\overline{\mathrm{co}}\widetilde{Y}$ is weak* compact in $\mathcal{M}(X)$, there exists $f \in \mathrm{Re}\, C(X)$ and $a \in \mathbf{R}$ such that

$$\int f d\mu < a < \int f\, ds,$$

for all $\mu \in M^{\perp}$ and all $s \in \overline{\mathrm{co}}\widetilde{Y}$. The left side of the above equality is identically zero for all μ in $M^{\perp}$, because $M^{\perp}$ is a subspace. This shows that f is an element of $\overline{M}$, the uniform closure of M. So there exists $s_0 \in Y$ such that $f(s_0) = \int f\, d\widetilde{s_0} = 0$. This is impossible in the above inequality, since then the right side would also be zero for $s = \widetilde{s_0}$. This contradiction shows that $M^{\perp} \cap \overline{\mathrm{co}}\widetilde{Y} \neq \varnothing$. □

From Theorems 2.4 and 3.1 we establish Theorem 1.3 of Farnum and Whitley.

Corollary 3.2. *If φ is a linear functional on $\mathrm{Re}\, C(X)$ such that $\varphi(f) \in \mathrm{Im}\,(f)$ for all $f \in \mathrm{Re}\, C(X)$, then φ is positive of norm one and is supported on a connected component of X.*

Proof. That φ is positive with norm one is obvious. If Y is a minimal $\mathcal{Z}$-support for $M = \ker \varphi$, then Y is either a singleton or has only one connected component, by Theorem 2.4, and there exists a positive functional in $M^\perp$ of norm one, supported on Y. This linear functional is necessarily φ, since $M^\perp$ has dimension 1. $\square$

Note that the converse of the above corollary is also true, i.e., a positive functional of norm one supported on a connected component of X has the property $\varphi(f) \in \mathrm{Im}\,(f)$ for all $f \in \mathrm{Re}\, C(X)$ (in other words its kernel is a $\mathcal{Z}$-subspace). This fact is an easy consequence of the intermediate value theorem.

Corollary 3.3. *Let X be connected. A subspace M of* $\mathrm{Re}\, C(X)$ *is a $\mathcal{Z}$-subspace if and only if there exists a positive functional in $M^\perp$.*

Remarks 3.4. 1. All the results mentioned above for $\mathrm{Re}\, C(X)$ can be slightly modified so that be true for unital real Banach algebras via the Gelfand transformation.

2. Theorems 2.1, 2.3, 2.4 and 3.1 do hold for $C(X)$. Theorem 1.2 of Jarosz states that case 2 cannot happen in Theorem 2.4, for a finite codimensional $\mathcal{Z}$-subspace of $C(X)$. However, Theorems 2.1, 2.3 and 3.1 are worth mentioning for $C(X)$.

3. Finite codimensional $\mathcal{Z}$-subspaces in complex Banach algebras are studied by many authors ([**4, 6**]), and it is not known if every finite codimensional $\mathcal{Z}$-subspace of a complex unital Banach algebra is contained in a maximal ideal, [**4**, Problem 3].

4. If X is connected, Corollary 3.3 implies that every maximal $\mathcal{Z}$-subspace in $\mathrm{Re}\, C(X)$ is of codimension 1. The following conjecture seems to be true:

If X has $n < \infty$ connected components, then every maximal $\mathcal{Z}$-subspace in $\mathrm{Re}\, C(X)$ is of codimension $\leq n$.

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