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EXISTENCE AND PROPERTIES OF MULTIPLE POSITIVE SOLUTIONS FOR SEMI-LINEAR EQUATIONS WITH CRITICAL EXPONENTS

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1. Introduction and main results. In this paper we consider the following semi-linear elliptic problem

$$(1.1)_\mu \qquad -\Delta u + u = u^p + \mu f(u+\phi)$$
$$(1.2) \qquad u \in H^1(R^N), \quad u > 0 \quad \text{in } R^N$$

where $\mu \geq 0$ is a given constant, $p = (N+2)/(N-2)$ is the critical Sobolev's exponent. $\phi(x)$ is some given function in $L^1(R^N) \cap C^\alpha(R^N)$ and

$H_1)$ $\phi(x) \geq 0$, $\phi(x) \not\equiv 0$ in R^N, $|x|^{N-2}\phi(x)$ is bounded.

The hypotheses for f(t) are as follows:

$f_1)$ $f \in C^2(R^+)$, $f'(t) \geq 0$, $f''(t) \geq 0$ for all $t \geq 0$.

$f_2)$ There exists a $\delta > 0$ such that $tf'(t) \geq (1+\delta)f(t)$ for $t \geq 0$ if $N \geq 6$.

$f_3)$ $\lim_{t\to 0} f(t)/t = 0$, and $\lim_{t\to\infty} f(t)/t^q = 0$ for some $q \geq p$.

$f_4)$ $\lim_{t\to\infty} f(t)/t = +\infty$.

Critical semi-linear elliptic equations arise from widely diverse problems in differential geometry, quantum physics, astrophysics, and other scientific areas. Many researchers have studied the second order semi-linear elliptic boundary value problems involving critical exponents. Here we mention the articles written by Brezis and Nirenberg [**4**], Cerami, Fortunato and Struwe [**5**], Lions [**14**], Ambrosetti and Struwe [**2**]. In their papers, many interesting results about the existence and nonexistence have been obtained by using variational methods when nonlinear function is homogeneous. For the inhomogeneous case, Zhu

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and Zhou in their work [**20**] have obtained the existence of two positive solutions of the problem

$$-\Delta u + u = \lambda(g(u) + f(x)), \quad u \in H_0^1(\Omega)$$

by using variational and barrier methods when $\Omega = R^N \backslash \omega$, and $\omega \subset R^N$ is a bounded no-empty smooth domain. A similar result has also been obtained in [**17**] for problem

$$-\Delta u + u = \lambda f(u + \phi), \quad u \in H_0^1(\Omega).$$

They require, however, that f and g have lower growth than critical Sobolev's exponents.

Recently, we studied in [**12**] the existence and nonexistence of multiple positive solutions for problem

$$(*)_\mu \qquad \begin{cases} -\triangle u + u = f(x, u) + \mu h(x), & x \in R^N, \\ u \in H^1\left(R^N\right), \end{cases}$$

where $h \in H^{-1}\left(R^N\right)$, $N \geq 3$, $|f(x,u)| \leq C_1 u^p + C_2 u$ with $C_1 > 0$, $C_2 \in [0,1)$ being some constants and $1 < p < +\infty$. Under some assumptions on f and h, we proved that there exists a positive constant $\mu^* < +\infty$ such that problem $(*)_\mu$ has at least one positive solution u_μ if $\mu \in (0, \mu^*)$, there is no solution for $(*)_\mu$ if $\mu > \mu^*$, and u_μ is increasing with respect to $\mu \in (0, \mu^*)$. Furthermore, problem $(*)_\mu$ has at least two positive solutions for $\mu \in (0, \mu^*)$ if $p < (N+2)/(N-2)$ and a unique positive solution for $\mu = \mu^*$ if $p \leq (N+2)/(N-2)$. As you can find from above result that we still require nonlinear function $f(x,u)$ have lower growth than the critical exponents, $p < (N+2)/(N-2)$, when we try to find the second solution.

For the critical growth, for example $(1.1)_\mu$, (1.2), there are serious difficulties when trying to find solutions by using variational methods because the embedding $H^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ is not compact even if Ω is bounded. This double lack of compactness exhibits many interesting existence and non-existence scenarios. These kinds of phenomena for problem $(1.1)_\mu$ have been investigated by [**7–10**] when $f(u + \phi) = \phi$.

Deng and Li in their recent paper [**11**], studied the existence and nonexistence of multiple positive solutions for homogeneous problem

$$\begin{cases} \Delta u + K(x) u^p = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \quad u \in H^1_{\text{loc}}(\Omega) \cap C(\overline{\Omega}) \\ u\big|_{\partial\Omega} = 0 & u \to \mu > 0 \quad \text{as } |x| \to \infty \end{cases}$$

when $\Omega = \mathbf{R}^N \setminus \omega$ is an exterior domain in $\mathbf{R}^N$, $\omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary and $N > 2$. $\mu \geq 0$, $p > 1$ are some given constants which could be equal to the critical Sobolev's exponents. Some existence and nonexistence of multiple positive solutions have been discussed under different assumptions on K.

The main goal of this paper is to exhibit the existence results of $(1.1)_\mu$, (1.2) with a very general nonlinear term f which is nonhomogeneous. The results of this paper are stated in the following:

Theorem 1.1. *Suppose H_1), f_1) and f_3) hold. Then there exists a constant $\mu^* > 0$ such that*

(i) $(1.1)_\mu$, (1.2) *possesses a minimal solution u_μ for all $\mu \in (0, \mu^*)$, and u_μ is increasing with respect to μ if $\mu \in (0, \mu^*)$.*

(ii) $(1.1)_\mu$, (1.2) *possesses a unique solution for $\mu = \mu^*$ if $q = p$ in the assumption f_3).*

(iii) *There are no solutions of* $(1.1)_\mu$, (1.2) *for $\mu > \mu^*$. Furthermore*

$$\mu_1 \leq \mu^* < \mu_2 \tag{1.3}$$

where

(1.4)

$$\begin{cases} w_\varepsilon = (N(N-2)\varepsilon)^{(N-2)/4}\left(\frac{1}{\varepsilon+|x|^2}\right)^{(N-2)/2} \\ \varepsilon = N(N+2) \\ C = \left(\frac{3N+2}{N-2}\right)^{(N+2)/4}\frac{4}{N+2}\left(\frac{N-2}{N+2}\right)^{(N-2)/4} \\ \mu_1 = \min\left\{\frac{[N^2(N+2)(N-2)]^{(N-2)/4}}{\sup_{x\in R^N}\{(N(N+2)+|x|^2)^{(N-2)/2}f(w_\varepsilon+\phi)\}}, \frac{1}{\sup_{x\in R^N}\{f'(\phi)\}}\right\} \\ \mu_2 = \frac{CS^{N/2}}{\int_{R^N} f(\phi(x))w_\varepsilon^{p+1}\,dx} \end{cases}$$

and S is the Sobolev's constant for the embedding $H^1(R^N) \hookrightarrow L^{2^*}(R^N)$, $2^* = (2N)/(N-2)$.

It should be noted that we needn't the increase restriction on the nonlinear function f when we get the minimal solution of $(1.1)_\mu$, (1.2). That means the nonlinear function f may be supercritical.

Theorem 1.2. *Suppose H_1), f_1)–f_4) with $q = p$. Then there exists a constant μ^{**} with $\mu^* \geq \mu^{**} > 0$ such that $(1.1)_\mu$, (1.2) possesses at least two positive solutions for all $\mu \in (0,\ \mu^{**})$.*

Theorem 1.3. *Suppose H_1), f_1)–f_4) with $q = p$. Then there exists a constant μ_{**} with $\mu^* > \mu_{**} > 0$ such that $(1.1)_\mu$, (1.2) has at least two solutions if $\mu \in (\mu_{**}, \mu^*)$.*

Remark 1.1. We are not sure whether $\mu^{**} = \mu_{**}$ or not.

Theorem 1.4. *Suppose H_1), f_1)–f_4) with $q = p$. Define*

$$\Phi = \{(\mu, u) \in R^+ \times C^{2,\alpha}(R^N) \cap H^1(R^N) : \\ u \not\equiv 0,\ \text{and } (1.1)_\mu,\ (1.2) \text{ are satisfied}\}.$$

We have

(i) *For any $(\mu, u) \in \Phi$, both $u(x)$ and $|\nabla u(x)$ have uniform limits zero as $| x | \to \infty$;*

(ii) *u_μ is continuous with respect to μ;*

(iii) *u_μ is uniformly bounded in $L^\infty(R^N) \cap H^1(R^N)$ for all $\mu \in (0, \mu^*)$, U_μ is uniformly bounded in $H^1(R^N)$ for μ small enough and $u_\mu \to 0$ in $L^\infty(R^N) \cap H^1(R^N)$ as $\mu \to 0$ $\|U_\mu\|_{H^1(R^N)} \to S^{N/2}$ as $\mu \to 0$;*

(iv) *(μ^*, u_{μ^*}) is the bifurcation point for $(1.1)_\mu$, (1.2), where u_μ is the minimal solution of $(1.1)_\mu$, (1.2) and U_μ is the second solution of $(1.1)_\mu$, (1.2) constructed in Theorems 1.2 and 1.3.*

We shall organize this paper as follows. The minimal positive solution is obtained in Section 2 by means of the standard barrier method. The existence of the second positive solution is proved in Section 3 by the variational method and the concentration-compactness principle. Further analysis of the set of solutions are made in Section 4 according to the bifurcation theory.

2. The minimal positive solution. In this section we prove Theorem 1.1 by the standard barrier method. To this end, we need some lemmas.

Lemma 2.1. *Suppose* H_1), f_1) *and* f_3). *Then* $(1.1)_\mu$, (1.2) *possess a minimal solution for all* $\mu \in (0, \mu_1)$, *where* μ_1 *is given by* (1.4).

Proof. For any $\varepsilon > 0$, let

$$w_\varepsilon = (N(N-2)\varepsilon)^{(N-2)/4}\left(\frac{1}{\varepsilon + |x|^2}\right)^{(N-2)/2}. \tag{2.1}$$

It is well known that $w_\varepsilon(x)$ satisfies the following problem

$$\begin{cases} \Delta w_\varepsilon = w_\varepsilon^p & \text{in } R^N, \\ w_\varepsilon(x) \to 0 & \text{as } |x| \to \infty \end{cases} \tag{2.2}$$

and

$$|\nabla w_\varepsilon|_2^2 = |w_\varepsilon|_{p+1}^{p+1} = S^{N/2}, \tag{2.3}$$

where S is the best Sobolev's constant. Setting $\bar{u} = w_\varepsilon$, we have

$$\begin{aligned} &-\Delta\bar{u} + \bar{u} - \bar{u}^p - \mu f(\bar{u} + \phi) \\ &\quad = w_\varepsilon - \mu f(w_\varepsilon + \phi) \\ &\quad = (\varepsilon + |x|^2)^{-(N-2)/2} \\ &\qquad \times [(N(N-2)\varepsilon)^{(N-2)/4} - \mu f(w_\varepsilon + \phi)(\varepsilon + |x|^2)^{(N-2)/2}]. \end{aligned} \tag{2.4}$$

From f_3) and (H_1), we deduce that

$$\begin{aligned} f(w_\varepsilon + \phi) &\le C[(w_\varepsilon + \phi) + (w_\varepsilon + \phi)^p] \\ &\le C[w_\varepsilon + \phi + 2^p w_\varepsilon^p + 2^p \phi^p] \\ &\le C[w_\varepsilon + \phi + w_\varepsilon^p + \phi^p] \end{aligned}$$

and $|x|^{N-2}\phi$, $|x|^{N-2}\phi^p$, $|x|^{N-2}w_\varepsilon^p$ are all bounded. Thus $(\varepsilon + |x|^2)^{(N-2)/2} f(w_\varepsilon(x) + \phi(x))$ is bounded. By H_1) and f_3), we also can conclude that $f'(\phi)$ is bounded. Set

$$M = \sup_{x \in R^N}\{(\varepsilon + |x|^2)^{(N-2)/2} f(w_\varepsilon(x) + \phi(x))\}, \tag{2.5}$$

$$\mu_1 = \min\left\{\frac{[(N+2)N^2(N-2)]^{(N-2)/4}}{M}, \quad \frac{1}{\sup_{x\in R^N}\{f'(\phi)\}}\right\}. \tag{2.6}$$

Then for any $\mu \in (0, \mu_1]$, $-\Delta \bar{u} + \bar{u} - \bar{u}^p - \mu f(\bar{u} + \phi) \geq 0$ which shows that $\bar{u} = w_\varepsilon$ is a supersolution of $(1.1)_\mu$ if $\mu \in (0, \mu_1]$. On the other hand, it is easy to verify that $\underline{u} = 0$ is a subsolution for all $\mu > 0$ and $\underline{u} < \bar{u}$. By the standard barrier method [**1**], there exists a solution u_μ of $(1.1)_\mu$ such that $0 \leq u_\mu \leq \bar{u}$. Since 0 is not a solution of $(1.1)_\mu$ and $f(u + \phi) \geq 0$, the maximum principle implies that $0 < u_\mu \leq \bar{u}$. Again, using a result of Amann [**1**], we can choose a minimal solution u_μ in the order interval $[0, \bar{u}]$. Furthermore, u_μ can be obtained by an iteration scheme with initial value $u(0) = \underline{u} = 0$. The same argument in [**10**], we can show that u_μ is minimal among all solutions of $(1.1)_\mu$ and

$$\begin{aligned}
\int_{R^N} (|\nabla u_\mu|^2 + u_\mu^2)\, dx &= \int_{R^N} u_\mu^{p+1} + \mu f(u_\mu + \phi) u_\mu \, dx \\
&\leq \int_{R^N} u_\mu^{p+1}\, dx + \varepsilon \int_{R^N} u_\mu^2 \, dx + \varepsilon \int_{R^N} u_\mu \phi \, dx \\
&\quad + C_\varepsilon \int_{R^N} \phi^p u_\mu \, dx + C_\varepsilon \int_{R^N} u_\mu^{p+1}\, dx \\
&\leq C \int_{R^N} w_\varepsilon^{p+1}\, dx + \varepsilon \int_{R^N} u_\mu^2 \, dx \\
&\quad + C \max_{x \in R^N} w_\varepsilon(x) \int_{R^N} (\phi + \phi^p)\, dx,
\end{aligned}$$

i.e.,

$$(1 - \varepsilon) \int_{R^N} (|\Delta u_\mu|^2 + u_\mu^2)\, dx \leq C S^{N/2} + C \max_{x \in R^N} w_\varepsilon(x) \int_{R^N} (\phi + \phi^p)\, dx.$$

Choosing ε small enough, we may deduce $\|u_\mu\|_{H^1(R^N)} < \infty$. Thus $u_\mu \in H^1(R^N)$.

Remark 2.1. From the proof of Lemma 2.1, we can see that

$$1 - p\, u_\mu^{p-1} - \mu f'(u_\mu + \phi) \geq 0, \quad \text{for} \quad \mu \in (0, \mu_1].$$

In fact, we can choose ε large enough, $\mu > 0$ small enough such that

$$1 - p\, u_\mu^{p-1} - \mu f'(u_\mu + \phi) \geq 1 - p\, w_\varepsilon^{p-1} - \mu f'(w_\varepsilon + \phi) \geq 0.$$

Lemma 2.2. *Suppose* H_1), f_1) *and* f_3). *Then problem* $(1.1)_\mu$, (1.2) *has no solution if* $\mu > \mu_2$, *where* μ_2 *is given by* (1.4).

Proof. Let u be a positive solution of $(1.1)_\mu$, (1.2). Then

$$-\Delta u w_\varepsilon^{p+1} + w_\varepsilon^{p+1} u = u^p w_\varepsilon^{p+1} + \mu f(u+\phi) w_\varepsilon^{p+1}. \tag{2.7}$$

Since $p > 1$, we may conclude that, for any $M > 0$, there exists a constant $C > 0$ such that

$$u^p \geq Mu - C, \quad u > 0. \tag{2.8}$$

It follows from (2.7), (2.8) that

$$\begin{aligned} -\int_{R^N} \Delta u w_\varepsilon^{p+1}\,dx &+ \int_{R^N} u w_\varepsilon^{p+1}\,dx \\ &\geq \int_{R^N} (Mu - c) w_\varepsilon^{p+1} + \mu f(u+\phi) w_\varepsilon^{p+1}\,dx. \end{aligned} \tag{2.9}$$

From [**7**], we have

$$\int_{R^N} \Delta u w_\varepsilon^{p+1}\,dx = \int_{R^N} u \Delta w_\varepsilon^{p+1}\,dx \tag{2.10}$$

which gives us

$$\begin{aligned} \mu \int_{R^N} f(u+\phi) w_\varepsilon^{p+1}\,dx \leq &- \int_{R^N} u \Delta w_\varepsilon^{p+1}\,dx - M \int_{R^N} w_\varepsilon^{p+1} u\,dx \\ &+ C \int_{R^N} w_\varepsilon^{p+1}\,dx + \int_{R^N} w_\varepsilon^{p+1} u\,dx. \end{aligned} \tag{2.11}$$

We may choose $\varepsilon = N(N-2)$. From [**7**], we get

$$\Delta(w_\varepsilon^{p+1}) \leq 2p\, w_\varepsilon^{p+1}$$

since $p > 1, w_\varepsilon \leq 1$. So (2.11) becomes

$$\mu \int_{R^N} f(u+\phi) w_\varepsilon^{p+1}\,dx \leq C \int_{R^N} w_\varepsilon^{p+1}\,dx + (2p+1-M) \int_{R^N} w_\varepsilon^{p+1} u\,dx.$$

If we choose $M = 2p+1$, then by (2.3), we have

$$\mu \leq \frac{C \int_{R^N} w_\varepsilon^{p+1}\,dx}{\int_{R^N} f(u+\phi) w_\varepsilon^{p+1}\,dx} \leq \frac{C S^{N/2}}{\int_{R^N} f(\phi) w_\varepsilon^{p+1}\,dx} \equiv \mu_2.$$

By the same argument in [**7**], we can get the expression of C which is

$$C = \sup_{u \geq 0} h(u) = \left(\frac{3N+2}{N-2}\right)^{(N+2)/4} \frac{4}{N+2} \left(\frac{N-2}{N+2}\right)^{(N-2)/4},$$

where $h(u) = (2p+1)u - u^p$.

Proof of Theorem 1.1. From Lemma 2.2, we set

$$\mu^* = \sup\{\mu > 0 \mid (1.1)_\mu, (1.2) \text{ possesses at least one solution}\}. \tag{2.12}$$

By Lemma 2.1 and Lemma 2.2, we have

$$0 < \mu_1 \leq \mu^* < \mu_2 < +\infty.$$

For any $\mu \in (0, \mu^*)$, by the definition of μ^* we can find a $\bar{\mu} \in (\mu, \mu^*)$ such that $(1.1)_{\bar{\mu}}$, (1.2) have a minimal positive solution $u_{\bar{\mu}}$ and

$$-\Delta u_{\bar{\mu}} + u_{\bar{\mu}} - u_{\bar{\mu}}^p - \mu f(u_{\bar{\mu}} + \phi) = \bar{\mu} f(u_{\bar{\mu}} + \phi) - \mu f(u_{\bar{\mu}} + \phi) \geq 0.$$

Thus $\bar{u} = u_{\bar{\mu}}$ is a supersolution of $(1.1)_\mu$, (1.2). From $f(u+\phi) \geq 0$, we can deduce that $u \equiv 0$ is a subsolution of $(1.1)_\mu$, (1.2) for all $\mu > 0$. By the standard Barrier method there exists a solution u_μ of $(1.1)_\mu$, (1.2) such that $0 \leq u_\mu \leq u_{\bar{\mu}}$. Since 0 is not a solution of $(1.1)_\mu$, $\bar{\mu} > \mu$ and $f(u+\phi) \geq 0$, the maximum principle implies that $0 < u_\mu < u_{\bar{u}}$. Again by the results of Amann [**1**], we can obtain a minimal solution u_μ by the iteration scheme with initial value $u(0) = 0$. Thus u_μ is a minimal solution of $(1.1)_\mu$, (1.2) and

$$\begin{aligned}\int_{R^N} |\nabla u_\mu|^2 + u_\mu^2 \, dx &= \int_{R^N} u_\mu^{p+1} + \mu f(u_\mu + \phi) \, dx \\ &\leq \int_{R^N} u_{\bar{\mu}}^{p+1} + \bar{\mu} f(u_{\bar{\mu}} + \phi) \, dx \\ &= \int_{R^N} |\nabla u_{\bar{\mu}}|^2 + u_{\bar{\mu}}^2 \, dx < +\infty.\end{aligned}$$

Thus $u_\mu \in H^1(R^N)$. By the definition of μ^*, we can conclude that $(1.1)_\mu$ and (1.2) have no solution for $\mu > \mu^*$. Therefore the proofs of Theorem 1.1 (i) and (iii) are completed.

In order to show the existence and uniqueness of $(1.1)_{\mu^*}$ and (1.2), we shall use the following Lemmas 2.3–2.5. The proofs of the lemmas are similar to those in [**11**]. For simplicity, we will omit the detailed proofs here.

Lemma 2.3. *Suppose H_1), f_1) and f_2) with $q = p$. Let u_μ be the minimal positive solution given by Theorem* 1.1 (i). *The corresponding eigenvalue problem*

$$(2.13_\mu) \qquad \begin{cases} -\Delta\varphi + \varphi = \lambda\,[p\,u_\mu^{p-1} + \mu f'(u_\mu + \phi)]\varphi \\ \varphi \in H^1(R^N) \end{cases}$$

has the first eigenvalue $\lambda_1 > 1$ and the corresponding eigenfunction $\phi_1 > 0$ in R^N.

Lemma 2.4. *Suppose H_1), f_1) and f_3) with $q = p$. Let u_μ be the minimal solution of $(1.1)_\mu$, (1.2) for $\mu \in [0, \mu^*)$. Then for any $g(x) \in H^{-1}(R^N)$, problem*

$$(2.14) \qquad -\Delta W + W = [p\,u_\mu^{p-1} + \mu f'(u_\mu + \phi)]\,W + g(x)$$

has a solution.

Lemma 2.5. *Suppose H_1), f_1) and f_3) with $q = p$. If u_{μ^*} is a solution of $(1.1)_{\mu^*}$, (1.2), then problem $(2.13)_{\mu^*}$ has its first eigenvalue $\lambda_1(\mu^*) = 1$. Moreover, the solution u_{μ^*} is unique.*

Now we are going to prove (ii) of Theorem 1.1. From Lemma 2.3 we have

$$\int_{R^N} (|\nabla u_\mu|^2 + u_\mu^2)\,dx > \int_{R^N} p\,u_\mu^{p+1}\,dx + \mu \int_{R^N} f'(u_\mu + \phi)u_\mu\,dx$$

and also we have

$$\int_{R^N} (|\nabla u_\mu|^2 + u_\mu^2)\,dx = \int_{R^N} u_\mu^{p+1}\,dx + \mu \int_{R^N} f(u_\mu + \phi)u_\mu\,dx.$$

Thus

$$\begin{aligned}\int_{R^N} (|\nabla u_\mu|^2 + u_\mu^2)\, dx \\ &< \frac{1}{p}\int_{R^N} (|\nabla u_\mu|^2 + u_\mu^2)\, dx + \mu^* \int_{R^N} f(w_\varepsilon + \phi) u_\mu \, dx \\ &\le \left(\frac{1}{p} + \frac{\delta}{2}\,\mu^*\right)\|u_\mu\|^2 + \frac{\mu^*}{2\delta}\int_{R^N} (w_\varepsilon^2 + \phi^2 + \phi^{2p} + w_\varepsilon^{2p})\, dx\end{aligned}$$

for any $\delta > 0$, since $p > 1$, we can obtain that

$$\|u_\mu\|_{H^1(R^N)} \le C < +\infty$$

for all $\mu \in (0, \mu^*)$ by taking δ small enough. Since u_μ is monotone increasing with respect to μ, we may suppose that $u_\mu \to u_{\mu^*}$ weakly in $H^1(R^N)$ as $\mu \to \mu^*$ and hence u_{μ^*} is a solution of $(1.1)_\mu$. The uniqueness of u_{μ^*} is obtained by Lemma 2.5.

Remark 2.2. Set $\mu^{**} = \sup\{\mu \in (0, \mu^*) : 1 - p\, u_\mu^{p-1} - \mu f'(u_\mu + \phi) \ge 0\}$. From Remark 2.1, we have

$$0 < \mu^{**} \le \mu^*$$

and

$$1 - p\, u_\mu^{p-1} - \mu f'(u_\mu + \phi) \ge 0$$

for all $\mu \in (0, \mu^{**})$.

3. Existence of the second positive solution. Let u_μ be the minimal positive solution of $(1.1)_\mu$, (1.2) for $\mu \in (0, \mu^*)$. In order to find a second solution of $(1.1)_\mu$ (1.2), we introduce the following problem:

$$(3.1)_\mu \qquad \begin{cases} -\Delta v + v = (u_\mu + v)^p - u_\mu^p + \mu[f(\phi_\mu + v) - f(\phi_\mu)], \\ v \in H^1(R^N), v > 0 \quad \text{in } R^N, \end{cases}$$

where $\phi_\mu = u_\mu + \phi$. Clearly we can get another solution $U_\mu = u_\mu + v_\mu$ of $(1.1)_\mu$, (1.2) if $(3.1)_\mu$ possesses a solution v_μ. To this end we set

$$\bar{f}(v) = \begin{cases} f(v) & \text{for } v \ge 0, \\ -f(-v) & \text{for } v < 0, \end{cases}$$

$$a(x) = 1 - [p\,u_\mu^{p-1} + \mu f'(u_\mu + \phi)],$$

$$h(x,v) = (v+u_\mu)^p - u_\mu^p - v^p + \mu[f(v+\phi_\mu - f(\phi_\mu)] - [p\,u_\mu^{p-1} + \mu f'(\phi_\mu)]v$$

and

$$\bar{h}(x,v) = \begin{cases} h(x,v) & \text{for } v \geq 0, \\ -h(x,-v) & \text{for } v < 0. \end{cases} \tag{3.2}$$

Then $(3.1)_\mu$ is equivalent to

$$\begin{cases} -\Delta v + a(x)v = |v|^{p-1}\,v + \bar{h}(x,v), \\ v \in H^1(R^N), \quad v > 0 \quad \text{in } R^N. \end{cases} \tag{3.3}$$

The corresponding variational function is

$$I(v) = \frac{1}{2}\int_{R^N} |\nabla v|^2 + a(x)v^2\,dx - \frac{1}{p+1}\int_{R^N} |v|^{p+1}\,dx - \int_{R^N} \overline{H}(x,v)\,dx,$$

where $\overline{H}(x,v) = \int_0^v \overline{h}(x,s)\,ds$. Set

$$I^\infty(v) = \frac{1}{2}\int_{R^N} |\nabla v|^2 + v^2\,dx - \frac{1}{p+1}\int_{R^N} |v|^{p+1}\,dx - \mu\int_{R^N} \overline{F}(v)\,dx$$

$$M^\infty = \left\{ v \in H^1(R^N) \,\middle|\, \int_R^N |\nabla v|^2 + v^2\,dx = \int_{R^N} |v|^{p+1}\,dx + \mu \int_{R^N} \bar{f}(v)v\,dx, \right\}$$

and

$$J^\infty = \inf\{I^\infty(v) \mid v \in M^\infty\},$$

where $\overline{F}(v) = \int_0^v \bar{f}(t)\,dt$. From [**8**], we can get the following theorem and corollary.

Theorem A. *Suppose $a(x), h(x,v)$ is a continuous function satisfying the following conditions*

(a) $a(x) \geq 0$ *for any* $x \in R^N$, $a(x) \to \bar{a} > 0$ *as* $|x| \to \infty$,

(b) $\lim_{t\to 0} \bar{h}(x,t)/t = 0$, $\lim_{t\to\infty}(\bar{h}(x,t))/(|t|^{p-1}t) = 0$.

(c) $(1/2)t\bar{h}(x,t) \geq \overline{H}(x,t)$ *for any* $x \in R^N$, $t \in R^1$.

(d) $\bar{h}(x,t) \to \bar{g}(t)$ *as* $|x| \to \infty$ *for* t *bounded uniformly. Then* $I(v)$ *satisfies the* $(p.s)_c$ *condition for any* $c \in (0, \min\{J^\infty, (1/N)S^{N/2}\})$.

Corollary A. *Suppose* $H_1)$, $f_1)$–$f_4)$ *with* $q = p$. *Then* J^∞ *can be attained by a function* $w \in H^1(R^N)$ *if* $J^\infty < (1/N)S^{N/2}$.

Remark 3.1. By Remark 2.2 and $f_1)$, $f_2)$, it is easy to verify that $\bar{h}(x,v)$ defined by (3.2) satisfies conditions (b) and (d) of Theorem A. Since $u_\mu \to 0$, $\phi(x) \to 0$ as $|x| \to +\infty$, $a(x) \to 1 > 0$ as $|x| \to \infty$, $a(x)$ satisfies condition (a) of Theorem A.

The following lemmas can be found in [**8**].

Lemma 3.1. *Suppose* $f_1)$ *holds. Then*

$$\frac{s}{2}\,(f(s+\phi_\mu) - f(\phi_\mu)) \geq \int_0^s (f(t+\phi_\mu) - f(\phi_\mu))\,dt \quad \text{for all } s \geq 0.$$

Lemma 3.2. *Let* $3 \leq N \leq 6$. *Then*

$$\frac{1}{2}\,s((s+u_\mu)^p - u_\mu^p - s^p) \geq \int_0^s ((t+u_\mu)^p - u_\mu^p - t^p)\,dt \quad \text{for } s \geq 0.$$

Lemma 3.3. *Suppose* $f_1)$–$f_4)$ *with* $q = p$. *Then* $I(v)$ *satisfies* $(p.s)_c$ *conditions for all* $c \in (0, \min\{J^\infty, (1/N)S^{N/2}\})$ *if* $3 \leq N \leq 6$ *and* $\mu \in (0, \mu^{**})$.

Lemma 3.4.

$$\frac{1}{2}\,s((s+u_\mu)^p - u_\mu^p) \geq \int_0^s ((t+u_\mu)^p - u_\mu^p)\,dt \quad \text{for all } s \geq 0.$$

Lemma 3.5. *Let* $3 \leq N \leq 5$ *and* $\mu \in (0, \mu^{**})$. *Then*

$$(v+u_\mu)^p - v^p - u_\mu^p \geq p\,u_\mu v^{p-1} \quad \text{for all } v \geq 0, \quad x \in R^N.$$

Remark 3.2. By f_2), we can conclude that there exists a $\theta \in (0, 1/2)$ such that $\theta t f(t) \geq F(t)$, for $t \geq 0$ if $N \geq 6$, where $F(t) = \int_0^t f(s)\, ds$.

Lemma 3.6. *Suppose f_1)–f_4) with $q = p$. Then $I(v)$ satisfies the $(p.s)_c$ condition for any $c \in (0, \min\{J^\infty,\ (1/N)S^{N/2}\})$ if $\mu \in (0, \mu^{**})$.*

Proof. From Lemma 3.3 we are only required to prove this lemma for $N \geq 7$. Let $c \in (0, \min\{J^\infty, (1/N)S^{N/2}\})$ and $\{v_n\} \subset H^1(R^N)$ be a $(p.s)_c$ sequence of I. That means $I(v_n) \to c$ as $n \to \infty$, $I'(v_n) \to 0$ in $H^{-1}(R^N)$ as $n \to \infty$. Thus

$$
\begin{aligned}
&\int_{R^N} |\nabla v_n|^2 + v_n^2\, dx \\
&- \int_{R^N} [(|v_n| + u_\mu)^p - u_\mu^p + \mu f(|v_n| + \phi_\mu) - \mu f(\phi_\mu)]|v_n|\, dx \\
&\qquad\qquad = \langle \zeta_n, v_n \rangle,
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
&\frac{1}{2}\int_{R^N} |\nabla v_n|^2 + v_n^2\, dx \\
&- \int_{R^N}\int_0^{|v_n|} [(s + u_\mu)^p - u_\mu^p + \mu f(s + \phi_\mu) - \mu f(\phi_\mu)]\, ds\, dx \\
&\qquad\qquad = c + 0(1),
\end{aligned}
\tag{3.5}
$$

where $\zeta_n = I'(v_n)$. By Remark 3.2, we have

$$
\begin{aligned}
0(1) + c \geq\ & \frac{1}{2}\int_{R^N} |\nabla v_n|^2 + v_n^2\, dx \\
& - \theta^* \int_{R^N} (|v_n| + u_\mu)[(|v_n| + u_\mu)^p + \mu f(|v_n| + \phi_\mu)]\, dx \\
& - \theta^* \int_{R^N} \mu\phi f(|v_n| + \phi_\mu)\, dx + \int_{R^N} u_\mu^p |v_n| + \mu f(\phi_\mu)|v_n|\, dx,
\end{aligned}
$$

where $\theta^* = \max\{1/(p+1), \theta\}$. Taking $\bar{\theta} \in (\theta^*, 1/2)$ from (3.4), we have

$$\begin{aligned}
0(1) + c \geq\ & \bar{\theta} < \zeta_n, v_n > + \left(\frac{1}{2} - \bar{\theta}\right) \int_{R^N} |\nabla v_n|^2 + v_n^2\, dx \\
& + (\bar{\theta} - \theta^*) \int_{R^N} |v_n|(|v_n| + u_\mu)^p\, dx \\
& + (\bar{\theta} - \theta^*) \int_{R^N} |v_n|[\mu f(|v_n| + \phi_\mu)]\, dx \\
& - \theta^* \int_{R^N} u_\mu[(|v_n| + u_\mu)^p] + \mu(u_\mu + \phi) f(|v_n| + \phi_\mu)\, dx \\
& + (1 - \bar{\theta}) \int_{R^N} u_\mu^p |v_n| + \mu f(\phi + u_\mu)|v_n|\, dx \\
\geq\ & \bar{\theta}\langle \zeta_n, v_n\rangle + \left(\frac{1}{2} - \bar{\theta}\right) \int_{R^N} |\nabla v_n|^2 + v_n^2\, dx + (\bar{\theta} - \theta^*) \\
& \times \int_{R^N} \left[|v_n| - \frac{\theta^*}{(\bar{\theta} - \theta^*)} \phi_\mu\right] [(|v_n| + u_\mu)^p + \mu f(|v_n| + \phi_\mu)]\, dx.
\end{aligned}$$

Setting $\tau = \theta^*/(\bar{\theta} - \theta^*)$, we have

$$\begin{aligned}
0(1) + c \geq\ & \bar{\theta}\langle \zeta_n, v_n\rangle + \left(\frac{1}{2} - \bar{\theta}\right) \int_{R^N} |\nabla v_n|^2 + v_n^2\, dx \\
& + (\bar{\theta} - \theta^*) \int_{\{|v_n| \leq \tau(u_\mu + \phi)\}} [|v_n| - \tau(u_\mu + \phi)] \\
& \qquad \times [(|v_n| + u_\mu)^p + \mu f(|v_n| + \phi + u_\mu)]\, dx \\
\geq\ & \bar{\theta}\langle \zeta_n, v_n\rangle + \left(\frac{1}{2} - \bar{\theta}\right) \int_{R^N} |\nabla v_n|^2 + v_n^2\, dx \\
& - \theta^* \int_{\{|v_n| \leq \tau(u_\mu + \phi)\}} (u_\mu + \phi)[(|v_n| + u_\mu)^p + \mu f(|v_n| + u_\mu + \phi)]\, dx \\
\geq\ & \bar{\theta}\langle \zeta_n, v_n\rangle + \left(\frac{1}{2} - \bar{\theta}\right) \int_{R^N} |\nabla v_n|^2 + v_n^2\, dx \\
& - \theta^* \int_{R^N} (u_\mu + \phi)[\tau(u_\mu + \phi) + u_\mu]^p + \mu f[(\tau + 1)(u_\mu + \phi)]\, dx.
\end{aligned}$$

Thus

$$\left(\frac{1}{2} - \bar{\theta}\right) \int_{R^N} |\nabla v_n|^2 + v_n^2\, dx \leq c + 0(1) + \bar{\theta}\langle \zeta_n, v_n\rangle + \theta^* c.$$

By using Young's Inequality we can deduce

$$\|v_n\|_{H^1(R^N)} \leq C < \infty$$

By taking a subsequence we suppose that

$$\begin{aligned} v_n &\longrightarrow v \quad \text{weakly in } H^1(R^N) \\ v_n &\longrightarrow v \quad \text{a.e. in } R^N. \end{aligned}$$

Letting $\rho_n = |\nabla v_n|^2 + a(x)v_n^2$, we may assume that

$$\int_{R^N} \rho_n(x)\,dx \longrightarrow l \geq 0 \quad \text{as } n \to \infty.$$

First, we show $l > 0$. In fact, if $l = 0$, by Remark 3.1 and Sobolev embedding, we have $\int_{R^N} |v_n|^{p+1}\,dx \to 0$, $\int_{R^N} \overline{H}(x, |v_n|)\,dx \to 0$ as $n \to \infty$. Then by (3.5), we get $c = 0$, a contradiction.

Applying the concentration-compactness Lemma due to Lions [**14**], there exists a subsequence (still denoted by ρ_n) satisfying one of the following three possibilities: (i) compactness, (ii) vanishing, (iii) dichotomy.

We should rule out (ii) and (iii) that couldn't occur by contradiction. If (ii) (vanishing) occurs, i.e., for all $R < +\infty$, $\lim_{n\to\infty} \sup_{y\in R^N} \times \int_{y+B_R} (|\nabla v_n|^2 + a(x)|v_n|^2)\,dx = 0$. By a lemma due to Lions [**14**], we have

$$v_n \longrightarrow 0 \quad \text{in } L^q(R^N), \qquad 2 < q < \frac{2N}{N-2}. \tag{3.6}$$

By Remark 3.1, for all $\varepsilon > 0$, there exists a C_ε such that for fixed $q \in (2, 2^*)$,

$$\left| \int_{R^N} |v_n| h(x, |v_n|)\,dx \right| \leq \varepsilon \int_{R^N} (|v_n|^2 + |v_n|^{2*})\,dx + C_\varepsilon \int_{R^N} |v_n|^q\,dx.$$

Since $\{v_n\}$ is bounded in $H^1(R^N)$ and (3.6) holds, we have

$$\int_{R^N} |v_n| h(x, |v_n|)\,dx \longrightarrow 0 \quad \text{as } n \to \infty.$$

Similarly $\int_{R^N} \overline{H}(x, v_n)\, dx \to 0$ as $n \to \infty$. By (3.4) and (3.5)

$$l = \lim_{n\to\infty} \|v_n\|^2 = \lim_{n\to\infty} \int_{R^N} |v_n|^{2^*}\, dx$$

and

$$\frac{1}{2}\, l - \frac{1}{2^*}\, l = c, \quad \text{i.e.} \quad \frac{1}{N}\, l = c.$$

Hence

$$S\|v_n\|^2_{L^{2^*}(R^N)} \le \|\nabla v_n\|^2_{L^2(R^N)} \le \|v_n\|^2_{H^1(R^N)}.$$

Thus $Sl^{2/2^*} \le l$, i.e., $l^{N/2} \ge S$. Therefore, $c \ge (1/N)S^{N/2}$, a contradiction.

If (iii) occurs, we denote Q_n to be the concentration function of ρ_n

$$Q_n(t) = \sup_{y\in R^N} \int_{y+B_t} \rho_n(x)\, dx, \quad t \ge 0.$$

Without loss of generality, we may assume

$$\lim_{n\to\infty} Q_n(t) = Q(t) \quad \text{for any} \quad t \ge 0$$

and

$$\lim_{t\to\infty} Q(t) = \alpha \in (0, l) \quad \text{(because dichotomy occurs).}$$

By Lemma 3.4, using the same argument in [**8**], we can rule out (iii).

Thus, only case (i) occurs, i.e., there exists a sequence $\{y_n\} \in R^N$ such that for any $\varepsilon > 0$, there exists an $R < +\infty$ such that

$$\int_{\{x-y_n\ge R\}} (|\nabla v_n|^2 + a(x)v_n^2)\, dx \le \varepsilon. \tag{3.7}$$

Same as in ruling out dichotomy, we may prove that $\{y_n\}$ is bounded. Choosing R large enough such that $\{y_n\} \subset B_R$, applying Sobolev's compact embedding on a bounded domain and (3.7), we may deduce that there exists a subsequence of $\{v_n\}$ such that

$$\begin{aligned} v_n &\longrightarrow v \quad \text{strongly in } L^q(R^N) \text{ for } 2 \le q < 2N/(N-2)\\ v_n &\longrightarrow v \quad \text{a.e. on } R^N\\ v_n\bar{h}(x,v_n) &\longrightarrow v\bar{h}(x,v) \quad \text{a.e. in } R^N\\ \overline{H}(x,v_n) &\longrightarrow \overline{H}(x,v) \quad \text{a.e. in } R^N\\ a(x)v_n &\longrightarrow a(x)v \quad \text{a.e. in } R^N\\ |v_n|^{p-1}v_n &\longrightarrow |v|^{p-1}v \quad \text{weakly in } (L^{P+1}(R^N))^*. \end{aligned}$$

From $I'(v_n) \to 0$ in $H^{-1}(R^N)$ we can conclude that

$$-\Delta v + a(x)v = v^p + h(x,v) \quad \text{in } H^{-1}(R^N). \tag{3.8}$$

Next, we shall show that $v_n \to v$ strongly in $H^{-1}(R^N)$. Indeed, by Strauss's lemma [**16**], for any $R < +\infty$,

$$\int_{B_R} |\bar{h}(x,v_n)v_n - \bar{h}(x,v)v|\,dx \longrightarrow 0 \quad \text{as } n \to +\infty.$$

By tightness and continuity of integral, for any $\varepsilon > 0$ we may choose R sufficiently large such that

$$\int_{\{R^N - B_R\}} |v_n\bar{h}(x,v_n)|\,dx < \varepsilon, \qquad \int_{\{R^N - B_R\}} |\,v\bar{h}(x,v)|\,dx < \varepsilon.$$

Then we deduce

$$\int_{R^N} v_n\bar{h}(x,v_n)\,dx \longrightarrow \int_{R^N} v\bar{h}(x,v)\,dx. \tag{3.9}$$

Similarly,

$$\begin{aligned} \int_{R^N} a(x)v_n^2\,dx &\longrightarrow \int_{R^N} a(x)v^2\,dx,\\ \int_{R^N} \overline{H}(x,v_n)\,dx &\longrightarrow \int_{R^N} \overline{H}(x,v)\,dx. \end{aligned} \tag{3.10}$$

Since $\{v_n\}$ is a $(p.s)_c$ sequence

(3.11)
$$\frac{1}{2}\int_{R^N} |\nabla v_n|^2 + a(x)v_n^2\,dx - \frac{1}{p+1}\int_{R^N} |v_n|^{p+1}\,dx - \int_{R^N} \overline{H}(x,v_n)\,dx = c + o(1).$$

(3.12)
$$\int_{R^N} |\nabla v_n|^2 + a(x)v_n^2\,dx - \int_{R^N} |v_n|^{p+1}\,dx - \int_{R^N} \bar{h}(x,v_n)v_n\,dx = o(1).$$

Denote $W_n = v_n - v$. By (3.9), (3.10) and using the Brezis-Lieb lemma, as $n \to \infty$, equation (3.11) becomes

$$I(v) + \frac{1}{2}\int_{R^N} |\nabla W_n|^2 - \frac{1}{p+1}\int_{R^N} |W_n|^{p+1}\,dx = c + o(1). \tag{3.13}$$

Similarly, (3.12) becomes

$$\int_{R^N} |\nabla v|^2 dx + \int_{R^N} |\nabla W_n|^2\,dx - \int_{R^N} |v|^{p+1}\,dx - \int_{R^N} |W_n|^{p+1}\,dx + \int_{R^N} a(x)v^2\,dx - \int_{R^N} vh(x,v)\,dx = o(1).$$

By (3.8) we have

$$\int_{R^N} |\nabla W_n|^2\,dx = \int_{R^N} |W_n|^{p+1}\,dx + o(1). \tag{3.14}$$

Thus

$$I(v) + \frac{1}{N}\int_{R^N} |W_n|^{p+1}\,dx = c + o(1). \tag{3.15}$$

Because $\{W_n\}$ is bounded, we may suppose $\int_{R^N} |\nabla W_n|^2\,dx \to \bar{l} \geq 0$. By Sobolev's inequality,

$$\int_{R^N} |\nabla W_n|^2\,dx \geq S\left(\int_{R^N} |W_n|^{p+1}\right)^{2/(p+1)}.$$

Let $n \to +\infty$ and, noting (3.14), we have $\bar{l} \geq S(\bar{l})^{2/(p+1)}$, i.e., $\bar{l} \geq S^{N/2}$ or $\bar{l} = 0$. If $\bar{l} \geq S^{N/2}$, from (3.15) we have $c = I(v) + (1/N)\bar{l}$. On the other hand, by Lemma 3.1,

$$\begin{aligned} I(v) = \frac{1}{2}\bigg(&\int_{R^N} |\nabla v|^2 + a(x)v^2 \, dx - \int_{R^N} |v|^{p+1} \, dx - \int_{R^N} \bar{h}(x,|v|)|v| \, dx \bigg) \\ &+ \frac{1}{2}\int_{R^N} |v|^{p+1} + h(x,|v|)|v| \, dx \\ &- \frac{1}{p+1}\int_{R^N} |v|^{p+1} \, dx - \int_{R^N} H(x,|v|) \, dx \\ = &\int_{R^N} \frac{1}{2}[(|v| + u_\mu)^p - u_\mu^p]|v| - \int_0^{|v|} [(s+u_\mu)^p - u_\mu^p] \, ds \, dx \\ &+ \int_{R^N} \frac{\mu}{2}|v|[f(|v| + \phi + u_\mu) - f(\phi + u_\mu)] \\ &- \mu \int_0^{|v|} [f(s+\phi+u_\mu) - f(\phi+u_\mu)] \, ds \, dx + \frac{1}{N}\int_{R^N} |v|^{p+1} \, dx \\ \geq &\frac{1}{N}\int_{R^N} |v|^{p+1} \, dx \geq 0. \end{aligned}$$

So we deduce

$$c \;=\; I(v) + \frac{1}{N}\,\bar{l} \;\geq\; \frac{1}{N}\,\bar{l} \;\geq\; \frac{1}{N}\,S^{N/2},$$

a contradiction with $c \in (0, \min\{J^\infty, (1/N)S^{N/2}\})$. Hence $\bar{l} = 0$, and $v_n \to v$ strongly in $H^1(R^N)$ as $n \to \infty$.

In the following we are going to verify the existence of the nontrivial positive solution of (3.3) by the Mountain Pass lemma. We check the conditions of the Mountain Pass lemma by the following obvious lemmas.

Lemma 3.7. *Suppose H_1), f_1)–f_4) with $q = p$. Then there exist $\alpha > 0$, $\rho > 0$ such that*

$$I(v)|_{\partial B_\rho} \geq \alpha > 0,$$

where $B_\rho = \{u \in H^1(R^N) \mid \|u\| \leq \rho\}$.

Lemma 3.8. *Suppose H_1), f_1)–f_4) with $q = p$. For any $v \in H^1(R^N)$, $v \not\equiv 0$, there exists an $R_0 > 0$ such that $I(Rv) \le 0$ for all $R \ge R_0$.*

Now we give the existence result.

Theorem 3.1. *Suppose H_1), f_1)–f_4) with $q = p$. If there exists a $v_0 \in H^1(R^N)$, $v_0 \not\equiv 0$, such that*

$$\sup_{t\ge 0} I(tv_0) < \min\Big\{J^\infty,\ \frac{1}{N}\,S^{N/2}\Big\}, \tag{3.16}$$

then $(3.1)_\mu$ possesses at least one solution for $\mu \in (0,\ \mu^{**})$.

Proof. By Lemma 3.7 and Lemma 3.8, there exists an $R_1 > 0$ such that $e = R_1 v_0 \notin B_\rho$ and $I(e) \le 0$. Define

$$c = \inf_{\Gamma\in\aleph}\sup_{v\in\Gamma} I(v), \tag{3.17}$$

where $\aleph$ denotes the class of continuous paths joining 0 to e in $H^1(R^N)$. Clearly

$$0 < \alpha \le c = \inf_{\Gamma\in\aleph}\sup_{v\in\Gamma} I(v) \le \sup_{t\ge 0} I(tv_0) < \min\Big\{J^\infty, \frac{1}{N}\,S^{N/2}\Big\}.$$

By Lemma 3.6, $I(v)$ satisfies the $(p.s)_c$ condition. So c can be achieved by some function $v \in H^1(R^N)$ via the Mountain Pass lemma. Because $\bar{h}(x,v)$ is an odd function with respect to v, we have $|v|$ which also achieves c. So we may suppose $v \ge 0$. By applying the strong maximum principle we have $v > 0$ in R^N. Thus v is a positive solution of $(3.1)_\mu$.

In the following, we shall verify that condition (3.16) holds naturally. To this end we set

$$\psi_\varepsilon(x) = \phi(x) w_\varepsilon(x),$$

where $\phi(x) \in C_0^\infty(R^N)$ is a cutoff function and w_ε is as in (1.4). For $R > 0$, let $\phi(x) \equiv 1$ if $|x| < R$; $\phi(x) \equiv 0$ if $|x| \ge 2R$. From [**7**] we have the following estimates

$$|\nabla\psi_\varepsilon|^2 = S^{N/2} + o(\varepsilon^{(N-2)/2}). \tag{3.18}$$

$$|\psi_\varepsilon|_{2^*}^{2^*} = S^{N/2} + o(\varepsilon^{N^2/(2N-2)}). \tag{3.19}$$

$$|\psi_\varepsilon|^2 = \begin{cases} k_1\varepsilon + o(\varepsilon^{(N-2)/2}) & N \geq 5, \\ k_1\varepsilon|\ln\varepsilon| + o(\varepsilon^{(N-2)/2}) & N = 4, \\ o(\varepsilon^1/2) & N = 3, \end{cases} \tag{3.20}$$

where S is the best Sobolev's constant and k_1 is a positive constant independent of ε.

Lemma 3.9. *Suppose* f_1)*. Then*

$$\mu[f(s+\phi+u_\mu) - f(\phi+u_\mu)] \geq \mu f(s) \geq 0 \quad \textit{for all } \mu \geq 0, \quad s \geq 0, \tag{3.21}$$

where u_μ *is the minimal solution of* $(1.1)_\mu$, (1.2) *given by Theorem* 1.1.

Proof. Set $g(s) = \mu[f(s+\phi+u_\mu) - f(\phi+u_\mu] - \mu f(s)$. Then $g''(s) = \mu[f'(s+\phi+u_\mu) - f'(s)]$. Since $f''(s) > 0$, $g''(s) > 0$. Because $g(0) = 0$ and $g'(0) = 0$, we have $g(s) \geq 0$ for $s \geq 0$. This gives (3.21).

Lemma 3.10. *Assume* H_1 *and* f_1)–f_4) *with* $q = p$*. Then there exists a constant* $t_\varepsilon > 0$ *such that*

$$\sup_{t\geq 0} I(t\psi_\varepsilon) = I(t_\varepsilon\psi_\varepsilon) \tag{3.22}$$

$$I(t_\varepsilon\psi_\varepsilon) \leq \frac{1}{N} S^{N/2} - \int_{R^N} G(x, t_\varepsilon\psi_\varepsilon)\,dx + \begin{cases} o(\varepsilon) & N \geq 5 \\ o(\varepsilon|\ln\varepsilon|) & N = 4 \\ o(\varepsilon^{1/2}) & N = 3 \end{cases} \tag{3.23}$$

where

$$G(x, t_\varepsilon\psi_\varepsilon) = \int_0^{t_\varepsilon\psi_\varepsilon} [(s+u_\mu)^p - u_\mu^p - s^p + \mu f(s+\phi+u_\mu) - \mu f(\phi+u_\mu)]\,ds.$$

Proof. By Lemma 3.8, we can easily show that there exists $t_\varepsilon > 0$ such that $\sup_{t\geq 0} I(t\psi_\varepsilon) = I(t_\varepsilon\psi_\varepsilon)$. We claim that there exist some constants $c_1,\ c_2$ such that

$$0 < c_1 \leq t_\varepsilon \leq c_2 < \infty \quad \text{for } \varepsilon \text{ small enough.} \tag{3.24}$$

In fact, since $I(t_\varepsilon\psi_\varepsilon) = sup_{t\geq 0} I(t\psi_\varepsilon)$, it follows that

$$I'(t_\varepsilon\psi_\varepsilon) = t_\varepsilon \int_{R^N} |\nabla\psi_\varepsilon|^2 + \psi_\varepsilon^2\, dx - \int_{R^N} [(t_\varepsilon\psi_\varepsilon + u_\mu)^p\psi_\varepsilon - u_\mu^p\psi_\varepsilon]\, dx \\ - \mu\int_{R^N} [f(t_\varepsilon\psi_\varepsilon + \phi_\mu) - f(\phi_\mu)]\psi_\varepsilon\, dx = 0.$$

Thus

$$(3.25)\quad \frac{|\nabla\psi_\varepsilon|^2 + |\psi_\varepsilon|_2^2}{|\psi_\varepsilon|_{p+1}^{p+1}} - t_\varepsilon^{p-1} \\ = \frac{\int_{R^N}\{(t_\varepsilon\psi_\varepsilon + u_\mu)^p - u_\mu^p - (t_\varepsilon\psi_\varepsilon)^p + \mu[f(t_\varepsilon\psi_\varepsilon + \phi_\mu) - f(\phi_\mu)]\}\psi_\varepsilon\, dx}{|\psi_\varepsilon|_{p+1}^{p+1} t_\varepsilon} \\ \geq 0.$$

By (3.18)–(3.20),

$$(3.26)\qquad t_\varepsilon^{p-1} \leq \frac{|\nabla\psi_\varepsilon|^2 + |\psi_\varepsilon|_2^2}{|\psi_\varepsilon|_{p+1}^{p+1}} \leq C_2^{p-1} < \infty \quad \text{for } \varepsilon \text{ small enough.}$$

On the other hand, using

$$\lim_{u\to\infty} \frac{(u+u_\mu)^p - u^p - u_\mu^p + \mu[f(u+\phi_\mu) - f(\phi_\mu)]}{u^p} = 0$$

and (3.18)–(3.20), we see that for any $\delta > 0$ there exists a constant $C_\delta > 0$ such that

$$[|\psi_\varepsilon|_{p+1}^{p+1}]^{-1} \\ \cdot \int_{R^N} \frac{\{(t_\varepsilon\psi_\varepsilon + u_\mu)^p - u_\mu^p - (t_\varepsilon\psi_\varepsilon)^p + \mu[f(t_\varepsilon\psi_\varepsilon + \phi_\mu) - f(\phi_\mu)]\}\psi_\varepsilon}{t_\varepsilon}\, dx \\ \leq [|\psi_\varepsilon|_{p+1}^{p+1}]^{-1} \int_{R^N} \frac{\delta t_\varepsilon^p \psi_\varepsilon^{p+1} + t_\varepsilon C_\delta \psi_\varepsilon^2}{t_\varepsilon}\, dx \\ = \delta t_\varepsilon^{p-1} + o(\varepsilon^{1/2}).$$

Again by (3.18)–(3.20) and (3.25)

$$1 - t_\varepsilon^{p-1} - \delta t_\varepsilon^{p-1} + o(\varepsilon^{1/2}) \leq 0.$$

Choosing δ, ε small enough we can find a constant $c_1 > 0$ such that $t_\varepsilon \geq c_1$. Therefore we obtain (3.24). Thus

$$\begin{aligned} I(t_\varepsilon \psi_\varepsilon) &= \frac{1}{2}\, t_\varepsilon^2 \int_{R^N} |\nabla \psi_\varepsilon|^2 + \psi_\varepsilon^2 \, dx - \frac{1}{p+1} \int_{R^N} t_\varepsilon^{p+1} \psi_\varepsilon^{p+1} \, dx \\ &\quad - \int_{R^N} G(x, t_\varepsilon \psi_\varepsilon) \, dx \\ &\leq \left(\frac{1}{2} - \frac{1}{p+1} \right) S^{N/2} + \frac{1}{2}\, c_2^2 \int_{R^N} \psi_\varepsilon^2 \, dx \\ &\quad - \int_{R^N} G(x, t_\varepsilon \psi_\varepsilon) \, dx + 0(\varepsilon^{(N-2)/2}) \\ &\leq \frac{1}{N}\, S^{N/2} - \int_{R^N} G(x, t_\varepsilon \psi_\varepsilon) \, dx + \begin{cases} o(\varepsilon) & N \geq 5, \\ o(\varepsilon |\ln \varepsilon|) & N = 4, \\ o(\varepsilon^{1/2}) & N = 3. \end{cases} \end{aligned}$$

Lemma 3.11. *Suppose H_1) and f_1)–f_4) with $q = p$. Then there exists a function $v_0 \in H_0^1(R^N)$, $v_0 \not\equiv 0$, such that* (3.16) *holds.*

Proof. If $N \geq 6$, by f_2) we have

$$\left(\frac{\mu f(t)}{t^{1+\delta}} \right)' = \frac{\mu[t f'(t) - (1+\delta) f(t)]}{t^{2+\delta}} > 0$$

for all $t > 0$, $\mu > 0$. Thus there exists a constant $C > 0$ such that

$$\mu F(t) \geq C\, t^{2+\delta} \quad \text{if } t > 1. \tag{3.27}$$

From (3.24) and Lemma 3.9

$$\begin{aligned} &\lim_{\varepsilon \to o^+} \varepsilon^{-1} \int_{R^N} G(x, t_\varepsilon \psi_\varepsilon) \, dx \\ &\qquad \geq \lim_{\varepsilon \to 0^+} \varepsilon^{-1} \int_{B_R} \mu F(t_\varepsilon \psi_\varepsilon) \, dx \\ &\qquad \geq \lim_{\varepsilon \to 0^+} \varepsilon^{(N-2)/2} w_N \int_0^{R\varepsilon^{-1/2}} \mu F \left[\frac{c_1 \varepsilon^{(2-N)/4}}{(1+s^2)^{(N-2)/2}} \right] s^{N-1} \, ds, \end{aligned}$$

where w_N denotes the area of unit sphere and $s = r\varepsilon^{-1/2}$, $r = |x|$. We can choose ε small enough such that

$$R\varepsilon^{-1/2} \geq (\varepsilon^{-1/2}c_1^{2/(N-2)} - 1)^{1/2}.$$

By (3.27)

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \varepsilon^{-1} \int_{R^N} G(x, t_\varepsilon\psi_\varepsilon)\, dx \\
&\geq \lim_{\varepsilon\to 0^+} \varepsilon^{(N-2)/2} w_N \int_0^{(\varepsilon^{-1/2}c_1^{2/(N-2)}-1)^{1/2}} \mu F\bigg(\frac{c_1\varepsilon^{(2-N)/4}}{(1+s^2)^{(N-2)/2}}\bigg) s^{N-1}\, ds \\
&\geq \lim_{\varepsilon\to 0^+} \varepsilon^{(N-2)/2} w_N \int_0^{(\varepsilon^{-1/2}c_1^{2/(N-2)}-1)^{1/2}} C\bigg(\frac{c_1\varepsilon^{(2-N)/4}}{(1+s^2)^{(N-2)/2}}\bigg)^{2+\delta} s^{N-1}\, ds \\
&\geq \lim_{\varepsilon\to 0^+} C\varepsilon^{[(2-N)\delta]/4} \int_{\{|y|\leq(\varepsilon^{-1/2}c_1^{2/(N-2)}-1)^{1/2}\}} \bigg(\frac{1}{1+|y|^2}\bigg)^{[(N-2)(2+\delta)]/2} dy \\
&= +\infty.
\end{aligned}$$

Thus from (3.23) we have

$$I(t_\varepsilon\psi_\varepsilon) < \frac{1}{N} S^{N/2} \quad \text{if } \varepsilon \text{ is small enough}, \quad N \geq 6 \tag{3.28}$$

if $3 \leq N \leq 5$. Applying (3.24), Lemma 3.5 and Lemma 3.9, we have

$$\begin{aligned}
\int_{R^N} G(x, t_\varepsilon\psi_\varepsilon) &\geq \int_{R^N} u_\mu(x)(t_\varepsilon\psi_\varepsilon)^p dx \\
&\geq \min_{x\in B_R} u_\mu(x) c_1 \int_{B_R} \psi_\varepsilon^p\, dx \\
&= c\int_{B_R} \psi_\varepsilon^p\, dx.
\end{aligned} \tag{3.29}$$

For $N = 3$ we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \varepsilon^{-1/2} \int_{R^N} G(x, t_\varepsilon\psi_\varepsilon)\, dx \\
&\quad\geq \lim_{\varepsilon\to 0^+} c\varepsilon^{-1/2} \int_{B_R} \psi_\varepsilon^p\, dx \\
&\quad= \lim_{\varepsilon\to 0^+} c\varepsilon^{-1/2} \int_{B_R} \bigg[\frac{\varepsilon^{1/4}}{(\varepsilon+|x|^2)^{1/2}}\bigg]^5 dx \\
&\quad= \lim_{\varepsilon\to 0^+} c\varepsilon^{-1/4} \int_{\{|y|\leq R\varepsilon^{-1/2}\}} \bigg(\frac{1}{1+|y|^2}\bigg)^{5/2} dy = +\infty.
\end{aligned} \tag{3.30}$$

Similarly, we can prove

$$\lim_{\varepsilon\to 0^+}(\varepsilon|\ln\varepsilon|)^{-1}\int_{R^N} G(x,t_\varepsilon\psi_\varepsilon)\,dx=+\infty \quad \text{if } N=4, \tag{3.31}$$

$$\lim_{\varepsilon\to 0^+}\varepsilon^{-1}\int_{R^N} G(x,t_\varepsilon\psi_\varepsilon)\,dx=+\infty \quad \text{if } N=5. \tag{3.32}$$

By (3.30)–(3.32) and (3.23), we have

$$I(t_\varepsilon\psi_\varepsilon)<\frac{1}{N}S^{N/2} \quad \text{if } \varepsilon \text{ small enough and } 3\le N\le 5. \tag{3.33}$$

Case (i). If $(1/N)S^{N/2}\le J^\infty$, we can take $v_0=\psi_\varepsilon$, ε small enough by (3.22), (3.28) and (3.33) we have

$$\sup_{t\ge 0} I(tv_0)=\sup_{t\ge 0} I(t\psi_\varepsilon)=I(t_\varepsilon\psi_\varepsilon)<\frac{1}{N}S^{N/2}$$

for small ε, which gives (3.16).

Case (ii). If $(1/N)S^{N/2}>J^\infty$, by Corollary A there exists a $w\in H^1(R^N)$, $w\not\equiv 0$ such $J^\infty=I^\infty(w)$ and

$$\int_{R^N}(|\nabla w|^2+w^2)\,dx=\int_{R^N}(w^{p+1}+\mu f(w)w)\,dx. \tag{3.34}$$

Because $\bar{f}(t)$ is an odd function we may suppose $w\ge 0$. By f_1) we deduce $(f(t)/t)'\ge 0$ for $t\ge 0$. Setting $q(t)=I^\infty(tw)$, by (3.34) we can easily deduce that $t=1$ is the unique critical point of $q(t)$ in $(0,\infty)$ and $q''(1)\le 0$. Because $\lim_{t\to 0}q(t)=0$ and $\lim_{t\to\infty}q(t)=-\infty$, we have $I^\infty(w)=\sup_{t\ge 0}I^\infty(tw)$. From Lemma 3.8 there exists $t_0>0$ such that

$$I(t_0w)=\sup_{t\ge 0} I(tw).$$

Thus by Lemma 3.9

$$\sup_{t\ge 0}\ I(tw)=I(t_0w)<I^\infty(t_0w)\le I^\infty(w)=J^\infty.$$

Taking $v_0 = w$, we obtain (3.16).

Proof of Theorem 1.2. From Theorem 1.1, $(1.1)_\mu$ and (1.2) possesses a minimal positive solution u_μ if $\mu \in (0, \mu^*)$. By Theorem 3.1 and Lemma 3.11 we have that $(3.1)_\mu$ possesses a solution v_μ if $\mu \in (0, \mu^{**})$. Setting $U_\mu = u_\mu + v_\mu$, we can easily verify that U_μ is the second solution of $(1.1)_\mu$, (1.2) for $\mu \in (0, \mu^{**})$.

4. Properties and bifurcation of solutions. In this section, we give some further properties and bifurcation of solutions for problem $(1.1)_\mu$ and (1.2). In particular, we will prove Theorem 1.3.

Proposition 4.1. *Suppose H_1), f_1)–f_4) with $q = p$. Let u be a weak solution of $(1.1)_\mu$, (1.2). Then $u \in L^q_{\text{loc}}(R^N)$ for all $q \in (1, \infty)$ and $u(x)$, $|\nabla u(x)|$ have uniform limits zero as $|x| \to \infty$.*

Proof. Let $\varphi(x) \equiv 0$; if $|x - x_0| < 2$, $\varphi(x) \equiv 0$ if $|x - x_0| \geq 3$ is a smooth cutoff function with $|\nabla \varphi| \leq 1$. For $s \geq 0$, $l > 0$, testing $(1.1)_\mu$ with $\varphi^2 u \min\{u^{2s}, L^2\}$, we obtain

$$\begin{aligned}\int_{R^N} \nabla u \nabla (\varphi^2 u \min\{u^{2s}, L^2\})\, dx + \int_{R^N} \varphi^2 u^2 \min\{u^{2s}, L^2\}\, dx \\ = \int_{R^N} \varphi^2 u^{p+1} \min\{u^{2s}, L^2\} + \mu f(u + \phi) \varphi^2 u \min\{u^{2s}, L^2\}\, dx.\end{aligned}$$

Suppose $u \in L^{2s+2}_{\text{loc}}(R^N)$. Then we may conclude by applying the Holder's and Sobolev's inequalities that, for $k > 1$,

$$\begin{aligned}& f(u+\phi) \\ & \leq C[(u+\phi) + (u+\phi)^p] \\ & \leq C[u + \phi + u^p + \phi^p] \int_{R^N} f(u+\phi)\varphi^2 u \min\{u^{2s}, L^2\}\, dx \\ & \leq C\bigg[\int_{R^N} u^{p-1} u^2 \varphi^2 \min\{u^{2s}, L^2\}\, dx + \int_{|x-x_0|<3} u^2 \min\{u^{2s}, L^2\}\, dx \\ & \qquad + \int_{R^N} \phi \varphi^2 u \min\{u^{2s}, L^2\}\, dx + \int_{R^N} \varphi^2 \phi^p u \min\{u^{2s}, L^2\}\, dx\bigg].\end{aligned}$$

and

$$\begin{aligned}
&\int_{R^N} |\nabla(\varphi u \min\{u^s, L\})|^2\,dx + \int_{R^N} (\varphi u \min\{u^s, L\})^2\,dx \\
&\leq C\bigg[\int_{R^N} u^{P-1}\varphi^2 u^2 \min\{u^{2s}, L^2\}\,dx + \int_{\{|x-x_0|<3\}} u^2 \min\{u^{2s}, L^2\}\,dx \\
&\qquad\qquad + \int_{R^N} \mu f(u+\phi)\varphi^2 u \min\{u^{2s}, L^2\}\,dx\bigg] \\
&\leq C\bigg[\int_{R^N} u^{p-1}\varphi^2 u^2 \min\{u^{2s}, L^2\}\,dx + \int_{\{|x-x_0|<3\}} u^2 \min\{u^{2s}, L^2\}\,dx \\
&\qquad + \int_{R^N} \phi\varphi^2 \min\{u^{2s}, L^2\}\,dx + \int_{R^N} \phi^p \varphi^2 u \min\{u^{2s}, L^2\}\,dx\bigg] \\
&\leq C\bigg[k\int_{\{|x-x_0|<3\}} u^{2s+2}\,dx + \|\phi\|^{p(2s+2)}_{L^\infty(B_3(x_0))} + \|\phi\|^{2s+2}_{L^\infty(B_3(x_0))} \\
&\qquad\qquad + \int_{\{x\in R^N, u^{p-1}\geq k\}} u^{p-1}\varphi^2 u^2 \min\{u^{2s}, L^2\}\,dx\bigg] \\
&\leq C\bigg\{k\int_{|x-x_0|<3} u^{2s+2}\,dx + \|\phi\|^{p(2s+2)}_{L^\infty(B_3(x_0))} + \|\phi\|^{2s+2}_{L^\infty(B_3(x_0))} \\
&\qquad + \bigg[\int_{\{|x-x_0|<3, u^{p-1}>k\}} (u^{p-1})^{N/2}\,dx\bigg]^{2/N} \\
&\qquad \times \bigg[\int_{R^N} (\varphi u \min\{u^s, L\})^{2N/(N-2)}\,dx\bigg]^{(N-2)/N}\bigg\} \\
&\leq C\bigg[k\int_{|x-x_0|<3} u^{2s+2}\,dx + \|\phi\|^{p(2s+2)}_{L^\infty(B_3(x_0))} + \|\phi\|^{2s+2}_{L^\infty(B_3(x_0))} \\
&\qquad\qquad + \varepsilon(k)\int_{R^N} |\nabla(\varphi u \min\{u^s, L\})|^2\,dx\bigg],
\end{aligned}$$

where

$$\varepsilon(k) = \sup_{x_0}\bigg[\int_{\{|x-x_0|<3, u^{p-1}\geq k\}} (u^{p-1})^{N/2}\,dx\bigg]^{2/N}.$$

Because $u \in H^1(R^N)$, we deduce that $\varepsilon(k) \to 0$ as $k \to \infty$. We may now conclude that

$$\int_{\{|x-x_0|<2,u^s<L\}} |\nabla(u^{s+1})|^2 + (u^{s+1})^2\,dx \le Ck\int_{\{|x-x_0|<3\}} u^{2s+2}\,dx + C\big(\|\phi\|^{2s+2}_{L^\infty(B_3(X_0))} + \|\phi\|^{p(2s+2)}_{l^\infty(B_3(x_0))}\big)$$

remains uniformly bounded in L. Hence we may let $L\to\infty$ to derive that

$$u^{s+1}\in H^1(\{|x-x_0|<2\}) \longrightarrow L^{2^*}(\{|x-x_0|<2\})$$

with

$$\int_{\{|x-x_0|<2\}} u^{(2s+2)N/(N-2)}\,dx \le Ck\int_{\{|x-x_0|<3\}} u^{2s+2}\,dx + C\left(\|\phi\|^{2s+2}_{L^\infty(B_3(x_0))} + \|\phi\|^{p(2s+2)}_{L^\infty(B_3x_0))}\right).$$

Let $q = [(2s+2)N]/(N-2)$. Hence $u\in L^q(R^N)$ for $q>0$ large. Obviously u satisfies the linear problem

$$-\Delta u + u = F(x) = u^p + \mu f(u+\phi),\quad x\in R^N,\quad u\in H^1(R^N).$$

Choosing $q>\max\{N/2, 2N/(N-2)\}$, by Holder's inequality in $B_2(x)$ we get

$$\|u\|_{L^2(B_2(x))} \le C\|u\|_{L^q(B_2(x))}.$$

By f_3), we have

$$\|F\|_{L^{q/p}(B_2(x))} \le C\big(\|u\|_{L^q(B_2(x))} + \|\phi\|_{L^q(B_2(x))}\big).$$

It's deduced by the elliptic regular theory that $u\in L^{2,\alpha}(R^N)$. By [**13**, Theorem 8.24], we have

$$\|u\|_{C^\alpha(B_1(x))} \le C\big(\|u\|_{L^q(B_2(x))} + \|\phi\|_{L^q(B_2(x))}\big) \tag{4.1}$$

then $u(x)\to 0$ as $\|x\|\to\infty$ since $u\in L^q(R^N)$. By [**13**, Theorem 8.32]

$$\|u\|_{C^{1,\alpha}(B_1(x))} \le C\big(\|u\|_{C^\alpha(B_2(x))} + \|\phi\|_{L^\infty(B_2(x))}\big) \tag{4.2}$$

(4.1), (4.2) give $\|\nabla u(x)\|\to 0$ as $\|x\|\to\infty$.

Proposition 4.2. *Suppose H_1), f_1)–f_4) with $q = p$. Let u_μ be the minimal solution of* $(1.1)_\mu$, (1.2). *Then u_μ is uniformly bounded in $L^\infty(R^N) \cap H^1(R^N)$, for all $\mu \in [0, \mu^*]$ and $u_\mu \to 0$ in $L^\infty(R^N) \cap H^1(R^N)$ as $\mu \to 0$.*

Proof. From Proposition 4.1 we can deduce that $\|u^*_\mu\|_\infty \le C$. Then by Theorem 1.1 (i) (ii), we can deduce that $\|u_\mu\|_\infty \le \|u_{\mu^*}\|_\infty \le C$. From Lemma 2.3

$$\int_{R^N} (\|\nabla u_\mu\|^2 + u_\mu^2)\,dx \ge \lambda_1 \left[\int_{R^N} p\, u_\mu^{p+1}\,dx + \int_{R^N} \mu f'(u_\mu + \phi) u_\mu^2\,dx \right]$$

and

$$\int_{R^N} (\|\nabla u_\mu\|^2 + u_\mu^2)\,dx = \int_{R^N} u_\mu^{p+1}\,dx + \mu \int_{R^N} f(u_\mu + \phi) u_\mu\,dx.$$

Thus

$$\begin{aligned} &\int_{R^N} (\|\nabla u_\mu\|^2 + u_\mu^2)\,dx \\ &\qquad \le \frac{1}{p\lambda_1} \int_{R^N} (\|\nabla u_\mu^2 + u_\mu^2)\,dx + \mu \int_{R^N} f(u_\mu + \phi) u_\mu\,dx \\ &\qquad \le \frac{1}{p\lambda_1} \|u_\mu\|^2 + C\mu \int_{R^N} (w_\varepsilon + w_\varepsilon^p + \phi + \phi^p) u_\mu\,dx. \end{aligned}$$

By the Holder and Young inequalities we deduce

$$\begin{aligned} &\left(1 - \frac{1}{\lambda_1 p} - \frac{\delta}{2}\right) \|u_\mu\|^2 \\ &\quad \le \frac{\mu}{2\delta} (\|w_\varepsilon\|_{H^{-1}(R^N)} + \|w_\varepsilon^p\|_{H^{-1}(R^N)} + \|\phi\|_{H^{-1}(R^N)} + \|\phi^p\|_{H^{-1}(R^N)}) \end{aligned}$$

for all $\delta > 0$. Taking δ small enough so that

$$\left(1 - \frac{1}{\lambda_1 p} - \frac{\delta}{2}\right) > 0, \tag{4.3}$$

we hence have $\|u_\mu\|^2 \le C\mu$. From Theorem 1.1, we have

$$\left(\int_{R^N} u_\mu^q\,dx\right)^{2/q} \le \left(w_\varepsilon^{q-2^*}(0) \int_{R^N} u_\mu^{2^*}\,dx\right)^{2/q} \le C\|u_\mu\|^2 \le C\mu \tag{4.4}$$

for any $q \in (2^*, \infty)$ if $\mu \in (0, \mu_1)$. By (4.3), (4.4), we can deduce our proposition.

Proposition 4.3. *Let U_μ be the second solution of $(1.1)_\mu$, (1.2) constructed in Section 3. Then U_μ is uniformly bounded for μ small enough and*

$$\lim_{\mu\to 0} \|U_\mu\|_{H^1(R^N)} = S^{N/2}.$$

Proof. Define

$$I_1(u) = \frac{1}{2}\int_{R^N} (|\nabla u|^2 + u^2)\,dx - \frac{1}{p+1}\int_{R^N} u^{p+1}\,dx - \mu \int_{R^N}\int_0^u f(t+\phi)\,dt\,dx.$$

By Lemma 3.7, Lemma 3.11, we can find a positive constant α independent of $\mu \in (0, \mu_1)$ such that

$$0 < \alpha < I_1(U_\mu) - I_1(u_\mu) < \frac{1}{N} S^{N/2}. \tag{4.5}$$

From $I_1(U_\mu) = c + I_1(u_\mu)$ and $I_1'(U_\mu) = 0$, we deduce

$$\int_{R^N} |\nabla U_\mu|^2 + U_\mu^2\,dx = \int_{R^N} U_\mu^{p+1}\,dx + \mu \int_{R^N} f(U_\mu + \phi)U_\mu\,dx$$

$$I_1(U_\mu) = \frac{1}{2}\int_{R^N} |\nabla U_\mu|^2 + U_\mu^2\,dx - \frac{1}{p+1}\int_{R^N} U_\mu^{p+1}\,dx - \mu\int_{R^N}\int_o^{U_\mu} f(t+\phi)\,dt\,dx.$$

By f_3) we have

$$\int_0^{U_\mu} f(t+\phi)\,dt \le C' \int_0^{U_\mu} (t + t^p + \phi + \phi^p)\,dt \le C'\Big[\frac{1}{2} U_\mu^2 + \frac{1}{p+1} U_\mu^{p+1} + (\phi + \phi^p)U_\mu\Big],$$

$$I_1(U_\mu) \geq \frac{1}{2}(1-\mu C')\|U_\mu^2\| - \frac{1+\mu C'}{p+1}\,|U_\mu|_{p+1}^{p+1} - \mu C' \int_{R^N} (\phi+\phi^p) U_\mu \, dx.$$

$$\left[\frac{1}{2}(1-\mu C') - \frac{1+\mu C'}{p+1}\right]\|U_\mu\|^2 \leq C + I_1(u_\mu) + \mu C' \int_{R^N} (\phi+\phi^p) U_\mu \, dx.$$

By Holder's and Young's inequalities, we deduce

$$\left[\frac{1}{2}(1-\mu C') - \frac{1+\mu C'}{p+1} - \frac{\delta}{2}\right]\|U_\mu\|^2 \leq \frac{1}{2/\delta}\,\mu^2 (C')^2 \|\phi+\phi^p\|_{H^{-1}(R^N)}^2 + C + I_1(u_\mu).$$

Choose $\mu < [(1/2)p - (1/2)]/[(1/2)pC' + (3/2)]$, δ small enough so that

$$\left[\frac{1}{2}(1-\mu C') - \frac{1+\mu C'}{p+1} - \delta/2\right] > 0.$$

We have $\|U_\mu\|^2 \leq C$ for μ small enough. From (4.5), we can conclude that

$$\begin{aligned}\alpha + &I_1(u_\mu)\\ &\leq \frac{1}{N}\,\|U_\mu\|^2 - \mu\left[\int_{R^N}\int_0^{U_\mu} f(t+\phi)\,dt\,dx - \frac{1}{p+1}\int_{R^N} f(U_\mu+\phi)U_\mu\,dx\right]\\ &\leq I_1(u_\mu) + \frac{1}{N}\,S^{N/2}.\end{aligned}$$

Because U_μ is uniformly bounded in $H^1(R^N)$ for μ small enough, we have

$$\lim_{\mu\to 0}\mu\left[\int_{R^N}\int_0^{U_\mu} f(t+\phi)\,dt\,dx - \frac{1}{p+1}\int_{R^N} f(U_\mu+\phi)U_\mu\,dx\right] = 0.$$

Thus, as $\mu \to 0$,

$$0 < \alpha + o(1) \leq \frac{1}{N}\,\|U_\mu\|^2 \leq \frac{1}{N}\,S^{N/2} + o(1). \tag{4.6}$$

On the other hand, by Sobolev's inequality we have

$$S\|U_\mu\|_{P+1}^{P+1} \leq \|U_\mu\|^2 = \|U_\mu\|_{p+1}^{p+1} + O(1).$$

Thus,

$$\|U_\mu\|_{p+1}^{p+1} \geq S^{N/2} + O(1). \tag{4.7}$$

From (4.6), (4.7) we have

$$\lim_{\mu\to 0} \|U_\mu\|^2_{H^1(R^N)} = S^{N/2}.$$

Proposition 4.4. *(μ^*, u_{μ^*}) is the $H^1(R^N)$-bifurcation point for $(1.1)_\mu$ and (1.2).*

Proof. Define

$$F : R^1 \times H^1(R^N) \longrightarrow H^{-1}(R^N)$$

by

$$F(\mu, u) = \Delta u - u + u^p + \mu f(u + \phi)$$

From Lemma 2.5, we deduce that $F_u(\mu^*, u_{\mu^*})\phi = 0$ has a solution $\phi_1 > 0$. This implies that $N(F_u(\mu^*, u_{\mu^*})) = \text{span}\,\{\phi_1\} = 1$ is one-dimensional and $\text{codim}\, R(F_u(\mu^*, u_{\mu^*})) = 1$. In the following we shall check that $F_\mu(\mu^*, u_{\mu^*}) \notin R(F_u(\mu^*, u_{\mu^*}))$. Assuming the contrary would imply existence of $v(x) \not\equiv 0$ such that

$$\Delta v - v + p\, u_{\mu^*}^{p-1} v + \mu f'(u_{\mu^*} + \phi)v = f(u_{\mu^*} + \phi).$$

From $F_u(\mu^*, u_{\mu^*})\phi_1 = 0$, we conclude that $\int_{R^N} f(u+\phi)\phi_1\, dx = 0$. This is impossible because $f(u+\phi) \geq 0$, $f(u+\phi) \not\equiv 0$ and $\phi_1(x) > 0$ in R^N.

Applying the bifurcation Theorem in [**6**], we conclude that (μ^*, u_{μ^*}) is the bifurcation point near which the solution of $(1.1)_\mu$, (1.2) form a curve $(\mu^* + \tau(s),\ u_{\mu^*} + s\phi_1 + z(s))$ with s near $s = 0$ and $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$. We claim that $\tau''(0) < 0$, which implies that the bifurcation curve turns strictly to the left in the (μ, u) plane. Substitute $\mu = \mu^* + \tau(s)$, $u = u_{\mu^*} + s\phi_1 + z(s)$ to

$$-\Delta u + u - u^p - \mu f(u + \phi) = 0, \quad u > 0. \tag{4.8}$$

Differentiating (4.8) in s twice we have

$$-\Delta u_{ss}+u_{ss}-p(p-1)u^{p-2}u_s^2-p\,u^{p-1}u_{ss}-\mu_{ss}f(u+\phi)-2\mu_s f'(u+\phi)u_s \\ -\mu f''(u+\phi)u_s^2-\mu f'(u+\phi)u_{ss}=0.$$

Setting here $s=0$ and using that $\tau'(0)=0$, $u_s=\phi_1(x)$ and $u=u_{\mu^*}$ as $s=0$, we obtain

$$(4.9)\quad -\Delta u_{ss}+u_{ss}-p(p-1)u_{\mu^*}^{p-1}\phi_1^2-p\,u_{\mu_*}^{p-1}u_{ss}-\tau''(0)f(u_{\mu^*}+\phi) \\ -\mu^* f''(u_{\mu^*}+\phi)\phi_1^2-\mu^* f'(u_{\mu^*}+\phi)u_{ss}=0.$$

Multiplying

$$F_u(\mu^*,u_{\mu^*})\phi_1=0$$

by u_{ss}, and (4.9) by ϕ_1, integrating and subtracting the result we obtain

$$p(p-1)\int_{R^N}u_{\mu^*}^{p-2}\phi_1^2\,dx+\tau''(0)\int_{R^N}f(u_{\mu^*}+\phi)\,dx+\mu\int_{R^N}f''(u_{\mu^*}+\phi)\phi_1^2\,dx=0$$

which immediately gives $\tau''(0)<0$.

Proof of Theorem 1.3. From Proposition 4.4 and its proof we can immediately get the result of Theorem 1.3.

Proof of Theorem 1.4. The conclusions (i), (iii) (iv), (v) come immediately from Propositions 4.1, 4.2, 4.3, 4.4. As for (ii) we can verify it by applying the implicit function theorem.

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