ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 35, Number 6, 2005

THE HENSTOCK VARIATIONAL MEASURE, BAIRE FUNCTIONS AND A PROBLEM OF HENSTOCK

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1. Introduction. It is well known that the Lebesgue integral is a proper extension of the Riemann integral. Henstock [**9**] and Kurzweil [**11**] independently gave a slight, but ingenious, modification of the classical Riemann integral to obtain a Riemann-type definition of the Perron integral, which is an extension of the Lebesgue integral. This relatively new integral is now commonly known as the Henstock-Kurzweil integral [**1**–**3, 12, 15**], Kurzweil-Henstock integral [**10, 14**], the gauge integral [**18**] or the Henstock integral [**8, 13**]. In this paper, we shall use the term "Henstock-Kurzweil integral" for this integral.

The original definition of the Henstock-Kurzweil integral, see Definition 2.2, involves completely arbitrary positive gauge function δ. Bullen in [**4**] raised the question of determining how complicated δ need be. It turns out that a measurable positive gauge function can be selected for the one-dimensional Henstock-Kurzweil integral; see for example [**7, 8, 12, 13**]. For the importance of this result in topology, see [**12**]. Foran and Meinershagen went further to prove that if F is generalized absolutely continuous in the restricted sense on a compact interval $[a, b]$ in $\mathbf{R}$ with

$$f(x) = \begin{cases} F'(x) & \text{if } F'(x) \text{ exists,} \\ 0 & \text{otherwise,} \end{cases}$$

then the positive gauge function δ in the definition of the Henstock-Kurzweil integral of f can be chosen to be Baire 2 everywhere. See [**7**, Theorem 2] for more details. Since their method of proof is real-line dependent, Henstock in [**10**, pp. 53–54] asked whether an analogous result holds in higher dimensions. In this paper, we give an affirmative answer to the above problem of Henstock. Moreover, we deduce a full descriptive characterization of the Henstock-Kurzweil integral [**15**, Theorem 4.3]. It is worthwhile to note that, unlike the method

2000 AMS *Mathematics Subject Classification.* Primary 26B99, 26A39, 28A12.
Key words and phrases. Henstock variational measure, Henstock-Kurzweil integral, absolute continuity.
Received by the editors on January 28, 2003.

employed in [**15**], our current method does not seem to involve the measurability condition of f so that the Radon-Nikodym Theorem for the absolutely continuous Henstock variational measure [**15**, Theorem 4.2] can be avoided in our proofs.

2. Preliminaries. The set of all real numbers is denoted by $\mathbf{R}$, and the ambient space of this paper is $\mathbf{R}^m$, where m is a fixed positive integer. The norm in $\mathbf{R}^m$ is the maximum norm $|||\cdot|||$. For $x \in \mathbf{R}^m$ and $r > 0$, set $B(x, r) := \{y \in \mathbf{R}^m : |||y - x||| < r\}$. Let $E := \prod_{i=1}^{m}[a_i, b_i]$ be a fixed nondegenerate interval in $\mathbf{R}^m$. For a set $A \subset E$, we denote by $\chi_{_A}$, $\operatorname{diam}(A)$ and $\mu_m^*(A)$ the characteristic function, diameter and m-dimensional Lebesgue outer measure of A, respectively. If $Z \subseteq E$, we denote its interior, boundary and closure with respect to the subspace topology of E by $\operatorname{int}(Z)$, ∂Z and $\overline{Z}$, respectively. The distance between two nonempty subsets A, Z of E will be denoted by $\operatorname{dist}(A, Z)$. The expressions *measure, measurable, almost all* and *almost everywhere* refer to the m-dimensional Lebesgue measure μ_m. A set $Z \subset E$ is called *negligible* whenever $\mu_m(Z) = 0$. Given two subsets X, Y of E, the symmetric difference of X and Y is denoted by $X \Delta Y$. We say that X and Y are non-overlapping if their intersection is negligible. A function is always real-valued. When no confusion is possible, we do not distinguish between a function defined on a set Z and its restriction to a set $W \subset Z$. If Z is a measurable subset of E, $\mathcal{L}(Z)$ will denote the space of Lebesgue integrable functions on Z. If $f \in \mathcal{L}(Z)$, the Lebesgue integral of f over Z will be denoted by $(L)\int_Z f$.

An *interval* in $\mathbf{R}^m$ is the cartesian product of m nondegenerate compact intervals in $\mathbf{R}$. $\mathcal{I}$ denotes the family of all subintervals of E. If $I \in \mathcal{I}$, we shall write $\mu_m(I)$ as $|I|$. For each $J \in \mathcal{I}$, the *regularity* of an m-dimensional interval $J \subseteq E$, denoted by $\operatorname{reg}(J)$, is the ratio of its shortest and longest sides. A function F defined on $\mathcal{I}$ is said to be additive if $F(I \cup J) = F(I) + F(J)$ for each non-overlapping interval $I, J \in \mathcal{I}$ with $I \cup J \in \mathcal{I}$. In particular, it is shown in [**14**, Corollary 6.2.4] that if F is an additive interval function on $\mathcal{I}$ with $J \in \mathcal{I}$, and $\{K_1, \dots, K_r\}$ is a collection of non-overlapping subintervals of J with $\cup_{i=1}^{r} K_i = J$, then

$$F(J) = \sum_{i=1}^{r} F(K_i).$$

A *partition* P is a finite collection $\{(I_i, \xi_i)\}_{i=1}^p$, where $I_1, I_2, \dots, I_p$ are non-overlapping intervals in $\mathcal{I}$, and $\xi_i \in I_i$ for $i = 1, 2, \dots, p$. Given $Z \subseteq E$, a positive function δ on Z is called a *gauge* on Z. We say that a partition $\{(I_i, \xi_i)\}_{i=1}^p$ is

(i) a partition *in* Z if $\cup_{i=1}^p I_i \subset Z$,

(ii) a partition *of* Z if $\cup_{i=1}^p I_i = Z$,

(iii) *anchored* in Z if $\{\xi_1, \xi_2, \dots, \xi_p\} \subset Z$,

(iv) δ*-fine* if $I_i \subset B(\xi_i, \delta(\xi_i))$ for each $i = 1, 2, \dots, p$,

(v) α*-regular* for some $\alpha \in (0, 1)$ if $\operatorname{reg}(I_i) \geq \alpha$ for each $i = 1, 2, \dots, p$.

The next lemma is important in this paper.

Lemma 2.1 [**14**, Lemma 6.2.6]. *Given a gauge δ on E, δ-fine partitions of E exist.*

Definition 2.2. A function $f : E \to \mathbf{R}$ is said to be *Henstock-Kurzweil integrable* on E if there exists $A \in \mathbf{R}$ with the following property: for each $\varepsilon > 0$ there exists a gauge δ on E such that

$$\left| \sum_{i=1}^{p} f(\xi_i)|I_i| - A \right| < \varepsilon \tag{1}$$

for each δ-fine partition $\{(I_i, \xi_i)\}_{i=1}^p$ of E. Here A is called the Henstock-Kurzweil integral of f over E, and we write A as $(HK)\int_E f$.

The linear space of Henstock-Kurzweil integrable functions on E is denoted by $\mathcal{HK}(E)$. If $f \in \mathcal{HK}(E)$, then $f \in \mathcal{HK}(J)$ for each subinterval J of E. The interval function $F : J \mapsto (HK)\int_J f$ is known as the *indefinite Henstock-Kurzweil integral*, or in short the indefinite $\mathcal{HK}$-integral, of f. Moreover F is an additive interval function on $\mathcal{I}$. For more properties of the Henstock-Kurzweil integral, see for example [**14**].

Let F be an additive interval function on $\mathcal{I}$ and X an arbitrary subset of E. If δ is a gauge on X, we set

$$V(F, X, \delta) := \sup_P \sum_{i=1}^{p} |F(I_i)|,$$

where the supremum is taken over all δ-fine partitions $P = \{(I_i, \xi_i)\}_{i=1}^{p}$ anchored in X.

We put

$$V_{\mathcal{HK}}F(X) := \inf_\delta V(F, X, \delta),$$

where the infimum is taken over all gauges δ on X. Then the extended real-valued set function $V_{\mathcal{HK}}F(\cdot)$ is a metric outer measure [**6**, Proposition 3.3], known as the *Henstock variational measure generated by* F. Moreover, we say that $V_{\mathcal{HK}}F$ is *absolutely continuous* if $V_{\mathcal{HK}}F(Z) = 0$ for each negligible set $Z \subset E$. The following theorems are now known for the Henstock variational measure.

Theorem 2.3 [**15**, Theorem 5.18]. *Let F be an additive interval function on $\mathcal{I}$. If $V_{\mathcal{HK}}F$ is absolutely continuous, then F is continuous on $\mathcal{I}$ in the following sense: for each $\varepsilon > 0$ there exists $\eta > 0$ such that*

$$|F(I) - F(J)| < \varepsilon$$

whenever $I, J \in \mathcal{I}$ with $\mu_m(I \Delta J) < \eta$.

Theorem 2.4 [**15**, Theorem 4.1]. *Let F be an additive interval function on $\mathcal{I}$. If $V_{\mathcal{HK}}F$ is absolutely continuous, then there exists an increasing sequence $\{X_n\}$ of closed sets such that $E = \cup_{n=1}^{\infty} X_n$ for which $V_{\mathcal{HK}}F(X_n) < \infty$ for $n = 1, 2, \dots$. In particular, $V_{\mathcal{HK}}F$ is σ-finite.*

Following [**17**], we say that an additive interval function F on $\mathcal{I}$ is derivable in the ordinary sense, or simply derivable, at $x \in E$ to $F'(x) \in \mathbf{R}$ if for each $\varepsilon > 0$ and $0 < \alpha \leq 1$, there exists $\eta = \eta(\varepsilon, \alpha, x) > 0$ such that

$$\left| F'(x) - \frac{F(I)}{|I|} \right| < \varepsilon$$

whenever $x \in I \subset B(x, \eta)$ with $I \in \mathcal{I}$ and $\operatorname{reg}(I) \geq \alpha$.

Theorem 2.5 [**15**, Theorem 4.2]. *Let F be an additive interval function on $\mathcal{I}$. If $V_{\mathcal{HK}}F$ is absolutely continuous, then F is derivable almost everywhere on E with*

$$V_{\mathcal{HK}}F(Y) = (L)\int_Y |F'|$$

for each measurable subset Y of E, even if one of the sides is equal to ∞.

Theorem 2.6 [**15**, Theorem 4.3]. *Let F be an additive interval function on $\mathcal{I}$. Then the following conditions are equivalent*:

(i) *F is the indefinite $\mathcal{HK}$-integral of some function on E*;

(ii) *the variational measure $V_{\mathcal{HK}}F$ is absolutely continuous.*

Theorem 2.7 [**15**, Theorem 4.7]. *Let F be an additive interval function on $\mathcal{I}$. Then the following conditions are equivalent*:

(i) *F is the indefinite $\mathcal{L}$-integral of some function on E*;

(ii) *the variational measure $V_{\mathcal{HK}}F$ is absolutely continuous and finite.*

3. Some estimates involving the Henstock variational measure. In this section we shall obtain some crucial estimates for Henstock variational measures $V_{\mathcal{HK}}F$. See Theorems 3.5, 3.6 and 3.11 for more details. To begin with, we have the following lemma, whose proof is left to the reader.

Lemma 3.1 *Let $\{X_n\}$ be an increasing sequence of nonempty closed subsets of E. If, for each positive integer k, there exists an upper semicontinuous gauge Δ_k on X_k such that $\Delta_k(x) \leq 1/k$ for each $x \in X_k$, then the non-negative function Γ defined on E by*

$$\Gamma(x) = \begin{cases} \Delta_1(x) & \text{if } x \in X_1, \\ \min\{\Delta_k(x), \operatorname{dist}(\{x\}, X_{k-1})\} & \text{if } x \in X_k \backslash X_{k-1} \\ & \text{for some integer } k \geq 2, \\ 0 & \text{otherwise,} \end{cases}$$

is upper semicontinuous on E.

The next two definitions are given in [**15**, Section 5].

Definition 3.2. Let $m = 1$ with $[u, v] \subseteq [a, b] \subset \mathbf{R}$. An interval-point pair $([u, v], x)$ is said to be 0-special if one of the following conditions holds:

(i) $[u, v] \subseteq [a, b]$ with $x \in (u, v)$,

(ii) $[u, v] \subseteq [a, b]$ with $u = x = a$,

(iii) $[u, v] \subseteq [a, b]$ with $v = x = b$.

Definition 3.3. An interval-point pair $(\prod_{k=1}^{m}[u_k, v_k], (x_1, x_2, \ldots, x_m))$ is said to be **0**-special if $([u_i, v_i], x_i)$ is of 0-special for each $i = 1, 2, \ldots, m$.

Following [**15**], given that F is an interval function on $\mathcal{I}$, $X \subseteq E$ and δ is a gauge on X, we set

$$V_0(F, X, \delta) := \sup_{Q} \sum_{i=1}^{q} |F(I_i)|,$$

where the supremum is taken over all δ-fine **0**-special partitions $Q = \{(I_i, \xi_i)\}_{i=1}^{q}$ anchored in X. The next lemma is a special case of [**15**, Theorem 5.11].

Lemma 3.4. *Let F be a continuous additive interval function on $\mathcal{I}$. If δ is a gauge on some nonempty set $X \subseteq E$, then*

$$V(F, X, \delta) \leq 3^m V_0(F, X, \delta).$$

We shall now give the next theorem, which is crucial in this paper.

Theorem 3.5. *Let F be a continuous additive interval function on $\mathcal{I}$. If there exists a gauge δ_0 on some nonempty set $X \subseteq E$ such that $V(F, X, \delta_0)$ is finite, and the gauge δ is defined on $\cup_{k=1}^{\infty}\overline{Y_k}$, where $Y_k := \{x \in X : \delta_0(x) \geq 1/k\}$ for each positive integer k, by*

$$\delta(\xi) = \begin{cases} 1 & \text{if } \xi \in \overline{Y_1}, \\ \min\{1/(k+1), \operatorname{dist}(\{\xi\}, \overline{Y_k})\} & \textit{if } \xi \in \overline{Y_{k+1}} \setminus \overline{Y_k} \\ & \textit{for some integer } k \geq 2, \end{cases}$$

then

$$V\left(F, \bigcup_{k=1}^{\infty} \overline{Y_k}, \delta\right) \leq 3^m V_0(F, X, \delta_0).$$

Proof. An application of Lemma 3.4 gives

$$V\left(F, \bigcup_{k=1}^{\infty} \overline{Y_k}, \delta\right) \leq 3^m V_0(F, \bigcup_{k=1}^{\infty} \overline{Y_k}, \delta).$$

Since

$$V_0\left(F, \bigcup_{k=1}^{\infty} \overline{Y_k}, \delta\right) \leq V_0\left(F, \bigcup_{k=1}^{\infty} Y_k, \delta\right) \leq V_0(F, X, \delta_0),$$

the theorem follows. □

Let F be an additive interval function on $\mathcal{I}$. For each $I \in \mathcal{I}$, we write

$$\omega(F; I) := \sup\{|F(J)| : J \subseteq I \text{ with } J \in \mathcal{I}\}.$$

The next theorem sharpens [**15**, Lemma 6.2].

Theorem 3.6. *Let F be an additive interval function on $\mathcal{I}$ such that $V_{\mathcal{HK}}F$ is absolutely continuous. If X is a nonempty closed subset of E such that $V_{\mathcal{HK}}F(X)$ is finite, then for each $\varepsilon > 0$ there exists an upper semicontinuous gauge δ on X and $\eta > 0$ such that*

$$\sum_{i=1}^{p} \omega(F; I_i) < \varepsilon$$

for each δ-fine partition $\{(I_i, \xi_i)\}_{i=1}^{p}$ anchored in X with $\sum_{i=1}^{p} |I_i| < \eta$.

Proof. For each $\varepsilon > 0$ there exists a gauge δ_0 on X and a constant $\eta > 0$ corresponding to $\varepsilon/9^m$ in [**15**, Lemma 6.2]. By using the compactness of X and Lemma 2.1, it is easy to check that $V(F, X, \delta_0)$ is finite. In view of Theorem 2.3, we may let δ be given as in Theorem 3.5.

The compactness of X and Lemma 3.1 imply that δ is an upper semicontinuous gauge on X. By using [**15**, Lemma 5.5], followed by a modification of the proof of [**15**, Theorem 5.9], [**15**, Lemma 5.13] and [**15**, Remark 5.14], we obtain the desired result. The proof is complete. $\square$

If a continuous additive interval function F on $\mathcal{I}$ is almost everywhere derivable in the ordinary sense, then we have the following crucial lemma for this paper.

Lemma 3.7. *Let F be a continuous additive interval function on $\mathcal{I}$. If there exists a negligible set $Z \subset E$ such that F is derivable at each $x \in E\backslash Z$, then for each $\varepsilon_0 > 0$ there exists an increasing sequence $\{X_n\}$ of nonempty closed subsets of E whose union contains $E\backslash Z$. Moreover, the following conditions hold for each positive integer n:*

(i) *the inequality $|F'(x)|I| - F(I)| < \varepsilon_0|I|$ holds whenever $I \in \mathcal{I}$ and $\{(I,x)\}$ is a $1/n$-fine, $1/2$-regular partition anchored in $\{x\} \subset X_n\backslash Z$;*

(ii) *$|F'(x)| \leq (n+2)\varepsilon_0$ for all $x \in X_n\backslash Z$.*

Proof. Since F is derivable at each $x \in E\backslash Z$, for each $\varepsilon_0 > 0$ there exists a gauge δ_0 on $E\backslash Z$ such that the inequality

$$|F'(x)|I| - F(I)| < \frac{\varepsilon_0}{4}\,|I| \tag{2}$$

holds for each δ_0-fine, $1/2$-regular partition $\{(I,x) : I \in \mathcal{I}\}$ anchored in $\{x\} \subset E\backslash Z$. For each positive integer n, we define

$$Y_n = \left\{x \in E\backslash Z : |F'(x)| < n\varepsilon_0 \text{ and } \delta_0(x) > \frac{1}{n}\right\},$$

and X_n denotes the closure of Y_n. Then it is clear that first part of the theorem holds.

We shall next prove that assertion (i) holds. In view of the definition of Y_n, the continuity of $F(\cdot)$ and $\mu_m(\cdot)$, it suffices to prove that

$$|F'(x)|I| - F(I)| < \frac{3\varepsilon_0}{4}\,|I| \tag{3}$$

for each $1/n$-fine, $1/2$-regular partition $\{(I, x)\}$ anchored in $\{x\} \subset (X_n \cap \operatorname{int}(I)) \backslash (Y_n \cup Z)$. In view of (2) and our choice of X_n, we may clearly assume that $\delta_0(x) \leq 1/n$.

For such a choice of x, we may choose an interval $J_x \subset \operatorname{int}(I)$ so that $x \in \operatorname{int}(J_x)$ with $J_x \subset B(x, \delta_0(x))$ and $\operatorname{reg}(J_x) \geq 1/2$. Since Y_n is dense in X_n, we may fix $y \in J_x \cap Y_n$ so that $\max\{\delta_0(x), \operatorname{diam}(J_x)\} \leq 1/n < \delta_0(y)$. Then it is clear that $\{x, y\} \subset J_x \subset B(x, \delta_0(x)) \cap B(y, \delta_0(y)) \cap E$. Consequently, it follows from (2) that

$$|F'(x) - F'(y)| \leq \left|F'(x) - \frac{F(J_x)}{|J_x|}\right| + \left|F'(y) - \frac{F(J_x)}{|J_x|}\right| < \frac{\varepsilon_0}{2}$$

and hence (3) holds. Assertion (ii) is now obvious. The proof is complete. $\square$

The next lemma, which is considerably simpler than the lemmas given in [**15**, Lemmas 6.5–6.7], will help us to decompose any interval $I \in \mathcal{I}$ into a finite union of non-overlapping $1/2$-regular intervals so that Lemma 3.7 can be applied in the proof of Theorem 3.11.

Lemma 3.8. *If $I \in \mathcal{I}$, then it can be written as a finite union of non-overlapping $1/2$-regular intervals.*

Proof. Let $I := \prod_{i=1}^{m} [u_i, v_i]$ and $t(I) := \min\{v_i - u_i : i = 1, 2, \ldots, m\}$. For each $i = 1, 2, \ldots, m$, there exists a finite collection $\mathcal{D}_i$ of one-dimensional non-overlapping intervals whose union is $[u_i, v_i]$. Moreover, we may assume that each element of $\mathcal{D}_i$ has length at least $t(I)$ but less than $2t(I)$. It is now clear that the following collection

$$\left\{ \prod_{i=1}^{m} J_i : J_i \in \mathcal{D}_i \quad \text{for each} \quad i = 1, 2, \ldots, m \right\}$$

has the required properties. The proof is complete. $\square$

Using the proof of Lemma 3.8, it is now easy to simplify some of the notation given in [**15**, Notation 6.4].

Fix $I := \prod_{i=1}^{m} [u_i, v_i] \in \mathcal{I}$ and let $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_m$ be given as in the proof of Lemma 3.8. For each positive integer k and $T \subseteq [u_k, v_k]$, we set

$$\Phi_{I,k}(T) := \prod_{i=1}^{m} X_i,$$
$$\text{where} \quad X_i = \begin{cases} T & \text{if } i = k, \\ [u_i, v_i] & \text{if } i \in \{1, 2, \dots, m\} \backslash \{k\}. \end{cases}$$

Let $X \subseteq E$ be nonempty and closed. We define

$$\mathcal{A}_{I,k}(X) := \{\Phi_{I,k}(T) : T \in \mathcal{D}_k \text{ and } X \cap \operatorname{int}(\Phi_{I,k}(T)) \neq \varnothing\}.$$

Let $\mathcal{C}_{I,k}(X)$ denote the set of all connected components of the figure, i.e. finite union of non-overlapping intervals from $\mathcal{I}$,

$$\overline{I \backslash \bigcup \{\Phi_{I,k}(T) : \Phi_{I,k}(T) \in \mathcal{A}_{I,k}(X)\}}$$

of the form $\cup\{\Phi_{I,k}(J) : J \in \mathcal{D}_k\}$. The next two useful lemmas, which are slight modifications of [**15**, Lemma 6.8] and [**15**, Lemma 6.9], respectively, will also be crucial in the proof of our main result.

Lemma 3.9. *Let F be an additive interval function on $\mathcal{I}$, and let X be a given nonempty closed subset of E. If $I \in \mathcal{T}_k(I)$ and $\mathcal{C}_{I,k}(X) \neq \varnothing$ for some $k \in \{1, 2, \dots, m\}$, then*

$$F(I) = \sum_{J \in \mathcal{A}_{I,k}(X)} F(J) + \sum_{J \in \mathcal{C}_{I,k}(X)} F(J).$$

Let $\mathcal{B}_{I,k}(X)$ be the collection of all intervals $\Phi_{I,k}([c, d])$ such that $[c, d] \subseteq [u_k, v_k]$ with $d - c > 0$, $X \cap \Phi_{I,k}([c, d]) \neq \varnothing$ and $X \cap \operatorname{int}(\Phi_{I,k}([c, d])) = \varnothing$. Then we have

Lemma 3.10. *Let F be an additive interval function on $\mathcal{I}$, and let X be a given nonempty closed subset of E. If $I \in \mathcal{T}_k(I)$ and $J \in \mathcal{C}_{I,k}(X)$ for some $k \in \{1, 2, \dots, m\}$, then there exists $L \in \mathcal{B}_I(X)$ such that $J \subseteq L \subseteq I$ and*

$$|F(J)| \leq \omega(F; L).$$

In view of Theorems 2.4 and 2.7, the function f appearing in Theorem 3.11 need not be Lebesgue integrable on E.

Theorem 3.11. *Let F be an additive interval function on $\mathcal{I}$ such that $V_{\mathcal{HK}}F$ is absolutely continuous. If the following conditions are satisfied*:

(i) $f : E \to \mathbf{R}$ *is a function such that $f = F'$ almost everywhere on* E;

(ii) f *is bounded on a nonempty closed set $X \subseteq E$ and $V_{\mathcal{HK}}F(X) < \infty$,*

then for each $\varepsilon > 0$ there exists an upper semicontinuous gauge δ on X such that

$$\sum_{i=1}^{p} |f(\xi_i)|I_i| - F(I_i)| < \varepsilon$$

for each δ-fine partition $\{(I_i, \xi_i)\}_{i=1}^{p}$ anchored in X.

Proof. If $\mu_m(X) = 0$, then the theorem follows from the boundedness of f on X and Theorem 3.6. Thus we may assume that $\mu_m(X) > 0$. $\square$

For each $\varepsilon > 0$ choose an upper semicontinuous gauge Δ_1 on X and $0 < \eta_0 < \mu_m(X)$ corresponding to $\varepsilon_0 := \varepsilon/(2m + 5)$ in Theorem 3.6.

By Theorem 2.5, there exists a negligible G_δ-set $Z \subset E$ such that $F'(x) = f(x)$ for each $x \in E \backslash Z$. By Lemma 3.7, there exists an increasing sequence $\{X_n\}$ of nonempty closed subsets of E whose union contains $E \backslash Z$. For a fixed positive integer n, we have

$$|f(\xi)|I| - F(I)| < \frac{\varepsilon_0 |I|}{|E|} \tag{4}$$

whenever $\{(I, \xi)\}$ is $1/n$-fine, $1/2$-regular partition anchored in $\{\xi\} \subset X_n \backslash Z$. Hence

$$|f(x) - f(y)| < \frac{2\varepsilon_0}{|E|} \tag{5}$$

whenever $x \in X_n \backslash Z$ and $y \in B(x, 1/n) \cap (X_n \backslash Z)$.

Since Z is a G_δ-set and X is closed, there exists an increasing sequence $\{Z_n\}$ of nonempty closed sets such that $X\backslash Z = \cup_{k=1}^{\infty} Z_k$. We set

$$K_n := X_n \cap \left\{ x \in X : \Delta_1(x) \geq \frac{1}{n} \right\} \cap Z_n.$$

It follows from our choices of $\{X_n\}$, $\{Z_n\}$ and the upper semicontinuity of the gauge Δ_1 on X that $\{K_n\}$ is an increasing sequence of closed subsets of E. Since it is clear that $\mu_m(X)$ is the limit of the increasing sequence $\{\mu_m(K_n)\}$, we may choose an open set $G \supset X$ and a positive integer N such that $\mu_m(G\backslash K_N) < \kappa_0 := \min\{\eta_0, \varepsilon_0/(\|f\chi_X\|_\infty + 1)\}$. Our choice of η_0 implies that $K_N \subset X\backslash Z$ is a nonempty set of positive measure. Define a gauge δ on X as follows:

$$\delta(\xi) = \begin{cases} 1/N & \text{if } \xi \in K_N, \\ \min\{\Delta_1(\xi), \operatorname{dist}(\{\xi\}, K_N), \operatorname{dist}(X, E\backslash G)\} & \text{if } \xi \in X\backslash K_N. \end{cases}$$

It is not difficult to check that δ is upper semicontinuous on X. The rest of the proof is similar but simpler than that of [**15**, Theorem 4.2]. For completeness sake, we give the details. Given any δ-fine partition $P = \{(I_i, \xi_i)\}_{i=1}^p$ anchored in X, we want to show that

$$\sum_{i=1}^{p} |f(\xi_i)|I_i| - F(I_i)| < \varepsilon. \tag{6}$$

Put $Q = \{(I, \xi) \in P : \xi \in K_N\}$. Then it follows from Lemmas 3.9 and 3.10, our choice of G and η_0 that

$$\begin{aligned} \sum_{i=1}^{p} |f(\xi_i)|I_i| - F(I_i)| &\leq \sum_{(I,\xi)\in Q} |f(\xi)|I| - F(I)| \\ &\quad + \sum_{(I,\xi)\in P\backslash Q} |f(\xi)|I| - F(I)| \\ &< \sum_{(I,\xi)\in Q} |f(\xi)|I| - F(I)| \\ &\quad + \|f\chi_{_X}\|_\infty \mu_m(G\backslash K_N) + \sum_{(I,\xi)\in P\backslash Q} |F(I)| \end{aligned} \tag{7}$$

$$
\begin{aligned}
&< \sum_{(I,\xi)\in Q} |f(\xi)|I| - F(I)| + \|f\chi_X\|_\infty \kappa_0 + \varepsilon_0 \\
&\le \sum_{(I,\xi)\in Q} |f(\xi)|I| - F(I)| + 2\varepsilon_0.
\end{aligned}
$$

If Q is empty, then the theorem is proved. Thus we may assume that $Q \neq \varnothing$. For each $(I,\xi) \in Q$, we use induction to construct two sequences $\{P_k[I]\}_{k=0}^m$, $\{Q_k[I]\}_{k=0}^m$ as follows:

$$P_0[I] := \{I\}; \quad Q_0[I] := \varnothing,$$

and for $k = 1, 2, \ldots, m$, put

$$
\begin{aligned}
P_k[I] &:= \{J \in \mathcal{A}_{S,k}(K_N) : S \in P_{k-1}[I]\}; \\
Q_k[I] &:= \{J \in \mathcal{C}_{S,k}(K_N) : \ S \in P_{k-1}[I]\}.
\end{aligned}
$$

Without loss of generality, we may assume that $P_k[I]$ is nonempty for all $k = 1, 2, \ldots, m$. By mimicking the proof of (7), we have by Lemmas 3.9 and 3.10, our choice of G and η_0,

$$
\begin{aligned}
&\sum_{(I,\xi)\in Q} \sum_{J\in P_k[I]} |f(\xi)|J| - F(J)| \\
&\qquad < \sum_{(I,\xi)\in Q} \sum_{J\in P_{k+1}[I]} |f(\xi)|J| - F(J)| + 2\varepsilon_0
\end{aligned}
\tag{8}
$$

for each $k = 0, 1, 2, \ldots, m-1$. Summing both sides of (8) over all the possible values of k gives

$$
\begin{aligned}
&\sum_{(I,\xi)\in Q} \sum_{J\in P_0[I]} |f(\xi)|J| - F(J)| \\
&\qquad < \sum_{(I,\xi)\in Q} \sum_{J\in P_m[I]} |f(\xi)|J| - F(J)| + 2m\varepsilon_0.
\end{aligned}
\tag{9}
$$

Next we observe that, if $J \in P_m[I]$ for some $(I,\xi) \in Q$, then it follows from Lemma 3.8 that $\operatorname{reg}(J) \geq 1/2$. For each $J \in P_m[I]$, the definition of $P_m[I]$ allows us to fix $x_J \in K_N \cap \operatorname{int}(J)$. Consequently, it follows from (4) and (5) that

$$\begin{aligned}
&\sum_{(I,\xi)\in Q} \sum_{J\in P_m[I]} |f(\xi)|J| - F(J)| \\
&\quad \le \sum_{(I,\xi)\in Q} \sum_{J\in P_m[I]} |(f(\xi) - f(x_J))||J| \\
&\qquad + \sum_{(I,\xi)\in Q} \sum_{J\in P_m[I]} |f(x_J)|J| - F(J)| \\
&\quad < 3\varepsilon_0.
\end{aligned} \tag{10}$$

Since $P_0[I] := \{I\}$, we see that (6) follows from (7), (9), (10) and our choice of $\varepsilon_0 := \varepsilon/(2m+5)$. The proof is complete. $\square$

4. Henstock's problem. The proof of [**7**, Theorem 2] is real-line dependent. In view of [**1**, Theorem 1], we can reformulate the above result of Foran and Meinershagen in terms of the Henstock variational measure so that it is real-line independent. As a result, we can state and prove a multidimensional version of [**7**, Theorem 2], answering a problem of Henstock [**10**, pp. 53–54].

Theorem 4.1. *Let F be an additive interval function on $\mathcal{I}$ such that $V_{\mathcal{HK}}F$ is absolutely continuous. If the function $f : E \to \mathbf{R}$ is given by*

$$f(x) = \begin{cases} F'(x) & \text{if } F'(x) \text{ exists,} \\ 0 & \text{otherwise,} \end{cases}$$

then for each $\varepsilon > 0$ there exists a Baire 2 gauge function δ on E such that

$$\left| \sum_{i=1}^{p} f(\xi_i)|I_i| - F(E) \right| < \varepsilon$$

for each δ-fine partition $\{(I_i, \xi_i)\}_{i=1}^{p}$ of E.

Proof. An application of Theorem 2.5 shows that F is derivable at almost all $x \in E$, so we may choose $\{X_n\}$ to be a sequence of closed sets corresponding to $\varepsilon_0 = 1$ in Lemma 3.7. For each positive integer

k, f is bounded on X_k. An application of Theorem 3.11 shows that for each $\varepsilon > 0$ there exists an upper semicontinuous gauge δ_k on X_k so that

$$\sum_{i=1}^{q} |f(x_i)|J_i| - F(J_i)| < \frac{\varepsilon}{2^{k+1}} \tag{11}$$

for each δ_k-fine partition $\{(J_i, x_i)\}_{i=1}^{q}$ anchored in X_k. Moreover, we may assume that $\delta_k(x) \leq 1/k$ for each $x \in X_k$.

From our construction of X_k, it is clear that the set $Z_0 := E \backslash \cup_{k=1}^{\infty} X_k$ is negligible. As $V_{\mathcal{HK}}F$ is absolutely continuous, there exists a gauge Δ on Z_0 such that

$$V(F, Z_0, \Delta) < \frac{\varepsilon}{4(3^m)}. \tag{12}$$

For each positive number k, set

$$U_k = \Big\{ x \in Z_0 : \Delta(x) \geq \frac{1}{k} \Big\}.$$

Define a gauge δ_0 on $\cup_{k=1}^{\infty} \overline{U_k}$ by

$$\delta_0(x) = \begin{cases} 1 & \text{if } x \in \overline{U_1}, \\ \min\{1/(k+1), \operatorname{dist}(\{\xi\}, \overline{U_k})\} & \text{if } \xi \in \overline{U_{k+1}} \backslash \overline{U_k} \\ & \text{for some integer } k \geq 2 \end{cases}$$

so that the inequality

$$V\left(F, \bigcup_{k=1}^{\infty} \overline{U_k}, \delta_0\right) < \frac{\varepsilon}{4} \tag{13}$$

follows from (12) and Theorem 3.5. Now, we define a gauge δ on E as follows:

$$\delta(\xi) = \begin{cases} \delta_1(\xi) & \text{if } \xi \in X_1, \\ \min\{\delta_k(\xi), \operatorname{dist}(\{\xi\}, X_{k-1})\} & \text{if } \xi \in X_k \backslash X_{k-1} \\ & \text{for some integer } k \geq 2, \\ \delta_0(\xi) & \text{if } \xi \in Z_0. \end{cases}$$

It follows from (11) and (13) that, given any δ-fine partition $\{(I_i, \xi_i)\}_{i=1}^p$ of E, we have

$$\left|\sum_{i=1}^p f(\xi_i)|I_i| - F(E)\right| < \sum_{k=1}^\infty \frac{\varepsilon}{2^{k+1}} + \frac{\varepsilon}{4} < \varepsilon.$$

It remains to verify that δ is a Baire 2 function. In view of [**16**, Theorem 4.5.4], it suffices to prove that for each $a > 0$,

$$S_1(a) := \{x \in E : \delta(x) > a\}, \quad S_2(a) := \{x \in E : \delta(x) < a\}$$

are $G_{\delta\sigma}$-sets.

Recall that, for each positive integer k, we have δ_k is upper semicontinuous on E with $\delta_k(x) \leq 1/k$ for each $x \in X_k$. Thus there exist positive integers p, q such that

$$S_1(a) = X \cup (Y \cap Z_0)$$

for some F_σ subsets X, Y of $X_p, \overline{U_q}$, respectively, showing that $S_1(a)$ is a $G_{\delta\sigma}$-set. Similarly, one can also check that $S_2(a)$ is a $G_{\delta\sigma}$-set. The proof is complete. $\square$

The next theorem, which is a mild generalization of [**15**, Theorem 4.3], see Theorem 2.6, follows from [**5**, Proposition 2] and Theorem 4.1. Moreover, we also deduce a result of Buczolich [**2**] that the gauge function in the definition of the multidimensional Henstock-Kurzweil integral can be chosen to be nearly upper semicontinuous on E, that is, upper semicontinuous when restricted to some suitable subset whose complement has measure zero.

Theorem 4.2. *Let F be an additive interval function on $\mathcal{I}$. Then $V_{\mathcal{HK}}F$ is absolutely continuous if and only if F is the indefinite $\mathcal{HK}$-integral of some function f on E. Moreover, the gauge function in Definition* 2.2 *can be chosen to be nearly upper semicontinuous for each $\varepsilon > 0$.*

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