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# AN IMPLICIT FUNCTION THEOREM

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ABSTRACT. A nonsmooth variant of the implicit function theorem is proved.

Let $m, n \geq 1$ be integers. Denote by $M_n$ the linear space of the $n \times n$ matrices with real elements, by $I_n$ the unit matrix of $M_n$. Let $B^n(x, r)$ be the ball in $\mathbf{R}^n$ with center at the point $x$ and radius $r > 0$.

If $F(x, y)$ is any locally Lipschitz vector-function of variables $x \in \mathbf{R}^n$, $y \in \mathbf{R}^m$ and $(x, y)$ is a point of differentiability of $F$, then let $F'(x, y)$ be its Jacobi matrix, $F'_x(x, y)$ be the Jacobi matrix with respect to $x$ for any fixed $y$ and $F'_y(x, y)$ be the Jacobi matrix with respect to $y$ for any fixed $x$.

For an arbitrary matrix $C \in M_n$ we put

$$|C| = \max_{|h|=1} |Ch|.$$

If $C(x) : D \subset \mathbf{R}^m \to M_n$ is a matrix function, then set

$$\|C\|_D = \operatorname{ess\,sup}_{x \in D} |C(x)|.$$

For $P \subset \mathbf{R}^m$ let $K : P \subset \mathbf{R}^m \to M_n$ be an arbitrary matrix function. We set

$$\operatorname{osc}(K, P) = \operatorname{ess\,sup}_{x,y \in P} |K(x) - K(y)|.$$

We shall prove the following nonsmooth variant of the well-known implicit function theorem.

**Theorem.** *Let $x_0 \in \mathbf{R}^n$, $y_0 \in \mathbf{R}^m$. Let $D = B^n(x_0, r') \times B^m(y_0, r'')$ be a domain and $F : D \to \mathbf{R}^m$ be a locally Lipschitz mapping. Suppose that*

$$\mu \equiv \|F'_y - I_m\|_D + \operatorname{osc}(F'_x, D)\,(1 + \|F'\|_D) < 1. \tag{1}$$

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*Then there exist* $\rho = \rho(\mu, r', r'') > 0$ *and a (unique) Lipschitz mapping*

$$G(x) : B^n(x_0, \rho) \longrightarrow \mathbf{R}^m, \quad G(x_0) = y_0,$$

*such that*

$$F(x, G(x)) = F(x_0, y_0) \quad \textit{for all} \quad x \in B^n(x_0, \rho).$$

*Moreover, we can put*

$$\rho = \frac{r^*}{L}, \quad r^* = \min\{r', r''\}, \quad L = \left(1 + \|F'_x\|_D\right) / \left(1 - \mu\right),$$

*and* $G$ *satisfies the Lipschitz condition with a constant*

$$\text{Lip } (G, B^n(x_0, \rho)) \leq \sqrt{L^2 - 1}.$$

For other nonsmooth variants of the implicit function theorem (without bounds of $\rho$ and Lip $(G, B^n(x_0, \rho))$), see Pourciau [**3**], Warga [**4**], Cristea [**1**], Zhuravlev and Igumnov [**5**].

For the proof we need a simple condition for locally Lipschitz mappings to be one-to-one on convex regions.

**Lemma.** *Let* $D \subset \mathbf{R}^n$ *be a convex domain, and let* $f : D \to \mathbf{R}^n$ *be a locally Lipschitz mapping. If*

$$\|f' - I_n\|_D \equiv \Omega < 1, \tag{2}$$

*then* $f$ *is a homeomorphism in* $D$*. Moreover, for arbitrary points* $x', x'' \in D$ *we have*

$$(1 - \Omega)\,|x'' - x'| \leq |f(x'') - f(x')| \leq (1 + \Omega)\,|x'' - x'|. \tag{3}$$

*Proof*. Let $E$ be the set of the points $x \in D$ in which $f$ is differentiable. Since $f$ is locally Lipschitz then by the Stepanoff theorem we have $\mathcal{H}^n(D \setminus E) = 0$.

By (2) almost everywhere in $D$ the estimate

$$|f'(x) - I_n| \leq \Omega < 1 \tag{4}$$

holds.

Fix arbitrary points $x', x'' \in D$ and denote by $l(x', x'')$ the line segment joining $x'$ and $x''$. Since the region $D$ is convex then $l(x', x'')$ lies inside $D$. Let $l(\tilde{x}', \tilde{x}'')$ be a line segment with endpoints $\tilde{x}'$ and $\tilde{x}''$ formed by a parallel translation of $l(x', x'')$. For almost all such segments sufficiently close to $l(x', x'')$ we have

$$\mathcal{H}^1(l(\tilde{x}', \tilde{x}'') \setminus E) = 0. \tag{5}$$

Let $l(\tilde{x}'_k, \tilde{x}''_k)$ be a sequence of segments having (5) and such that

$$\tilde{x}'_k \to x', \quad \tilde{x}''_k \to x''.$$

Because $f$ is locally Lipschitz then $f$ is absolutely continuous on $l(\tilde{x}'_k, \tilde{x}''_k)$ and almost everywhere along $l(\tilde{x}'_k, \tilde{x}''_k)$ the derivative $f'$ exists. Integrating we find

$$\begin{aligned}
&|(f(x''_k) - x''_k) - (f(x'_k) - x'_k)| \\
&\qquad = \left| \int_0^1 (f'(x'_k + t(x''_k - x'_k)) - I)(x''_k - x'_k)\, dt \right| \\
&\qquad \leq \int_0^1 |f'(x'_k + t(x''_k - x'_k)) - I|\, |x''_k - x'_k|\, dt.
\end{aligned}$$

Then by (4) we obtain

$$|(f(x''_k) - x''_k) - (f(x'_k) - x'_k)| \leq \Omega\, |x''_k - x'_k|.$$

Letting $k \to \infty$ we arrive at the estimate

$$|(f(x'') - x'') - (f(x') - x')| \leq \Omega\, |x'' - x'|, \quad x', x'' \in D. \tag{6}$$

Let $\phi(x) = f(x) - x$. For arbitrary points $x', x'' \in D$ we have

$$f(x'') - f(x') = (\phi(x'') - \phi(x')) + (x'' - x').$$

Thus,
$$|f(x'') - f(x')| \le |\phi(x'') - \phi(x')| + |x'' - x'|.$$
Using (6) we can write
$$|f(x'') - f(x')| \le (1+\Omega)\,|x'' - x'|.$$
Analogously,
$$|f(x'') - f(x')| \ge |x'' - x'| - |\phi(x'') - \phi(x')|$$
and next,
$$|f(x'') - f(x')| \ge (1-\Omega)\,|x'' - x'|.$$
Thus (3) holds and the lemma is proved. □

*Proof of Theorem.* Consider the mapping $\Phi : D \to \mathbf{R}^n \times \mathbf{R}^m$ defined by
$$(x,y) \xrightarrow{\Phi} (X,Y) = (x_1, \dots, x_n, F_1(x,y), \dots, F_m(x,y)).$$
We need to prove that $\Phi(x,y)$ satisfies the assumptions of the lemma. The Jacobi matrix of $\Phi$ has the form
$$\Phi'(x,y) = \begin{pmatrix} I_n & Z^n_m \\ F'_x(x,y) & F'_y(x,y) \end{pmatrix},$$
where $Z^n_m$ is the zero $n \times m$ matrix.

Consider the $(n+m)\times(n+m)$ matrix
$$Q(x,y) = \begin{pmatrix} I_n & Z^n_m \\ -F'_x(x,y) & I_m \end{pmatrix}.$$
For almost every $(x_1,y_1)$, $(x_2,y_2) \in D$ we have
$$|Q(x_1,y_1) - Q(x_2,y_2)| \le |F'_x(x_1,y_1) - F'_x(x_2,y_2)| \le \operatorname{osc}(F'_x, D).$$
We observe now that
$$\begin{aligned} Q(x,y)\Phi'(x,y) - I_{m+n} &= \begin{pmatrix} I_n & Z^n_m \\ Z^m_n & F'_y(x,y) \end{pmatrix} - I_{n+m} \\ &= \begin{pmatrix} Z^n_n & Z^n_m \\ Z^m_n & F'_y(x,y) - I_m \end{pmatrix}. \end{aligned}$$

Hence,

$$\|Q\,\Phi' - I_{n+m}\|_D = \|F'_y - I_m\|_D. \tag{7}$$

For an arbitrary fixed point $(x^*, y^*) \in D$ we define the map

$$\Psi(x,y) = Q(x^*,y^*)\Phi(x,y) : D \to \mathbf{R}^n \times \mathbf{R}^m. \tag{8}$$

Using (7) we find

$$\begin{aligned} ||\Psi' - I_{n+m}||_D &= \|Q(x^*,y^*)\Phi' - I_{n+m}\|_D \\ &= \|Q\,\Phi' - I_{n+m} + (Q(x^*,y^*) - Q)\,\Phi'\|_D \\ &\le \|Q\,\Phi' - I_{n+m}\|_D + \|Q(x^*,y^*) - Q\|_D\|\Phi'\|_D \\ &\le \|F'_y - I_m\|_D + \operatorname{osc}(F'_x, D)\,\|\Phi'\|_D. \end{aligned}$$

Taking into consideration that

$$\begin{aligned} ||\Phi'||_D &= \operatorname{esssup}_{(x,y)\in D}\left\{\max_{|h|=1}|\Phi'(x,y)\cdot h|\right\} \\ &\le \operatorname{esssup}_{(x,y)\in D}\left\{\max_{|h|=1}\big(|h| + |(F'_x(x,y) + F'_y(x,y))\cdot h|\big)\right\} \\ &\le 1 + \|F'\|_D, \end{aligned}$$

we obtain

$$\|\Psi' - I_{n+m}\|_D \le \|F'_y - I_m\|_D + \operatorname{osc}(F'_x, D)\,(1 + \|F'\|_D).$$

Thus by (1) for the fixed point $(x^*, y^*)$ we have

$$||\Psi' - I_{m+n}||_D \le \mu < 1. \tag{9}$$

The domain $D = B^n(x_0, r') \times B^m(y_0, r'')$ is convex and we may use the lemma. By the inequality (9) we conclude that the map

$$\Psi(x,y) = Q(x^*,y^*)\Phi(x,y)$$

is a homeomorphism. By Horn, Johnson [**2**, Corollary 5.6.16] from (9) it follows also that the matrix $\Psi'(x,y)$ is nonsingular. Therefore from this relation it follows that matrices $\Phi'(x,y)$ and $Q^* \equiv Q(x^*,y^*)$ are nonsingular. Thus the map $\Phi = Q^{*-1}\Psi : D \to \mathbf{R}^{n+m}$ is also a homeomorphism.

For the evaluation of $\rho = \rho(\mu, r', r'')$ we shall need some special information about $\Psi$. Using (9) and (3) we find

$$(1-\mu)\,|(x-x_0, y-y_0)| \le |\Psi(x,y) - \Psi(x_0,y_0)| \\ \le (1+\mu)\,|(x-x_0,y-y_0)|.$$

Then by $\Psi = Q^*\Phi$ we may write

$$\frac{1-\mu}{|Q^*|}\,|(x-x_0,y-y_0)| \le |\Phi(x,y)-\Phi(x_0,y_0)| \\ \le (1+\mu)\,|Q^{*-1}|\,|(x-x_0,y-y_0)|. \tag{10}$$

However,

$$\begin{pmatrix} I_n & Z^n_m \\ -F'_x(x^*_0,y^*_0) & I_m \end{pmatrix} = \begin{pmatrix} I_n & Z^n_m \\ Z^m_n & I_m \end{pmatrix} + \begin{pmatrix} Z^n_n & Z^n_m \\ -F'_x(x^*_0,y^*_0) & Z^m_m \end{pmatrix}$$

and hence $|Q^*| \le 1 + \|F'_x\|_D$. Let $a = (x_0,y_0)$. Since the ball $B^{n+m}(a,r^*)$ is contained in $D$ from (10) we find

$$B' \equiv B^{n+m}\left(\Phi(a), r^*(1-\mu)/|Q^*|\right) \subset \Phi(B^{n+m}(a,r^*)). \tag{11}$$

Further,

$$B'' \equiv B^{n+m}(\Phi(a), r^*(1-\mu)/(1+\|F'_x\|_D)) \subset B' \subset \Phi\,(D)\,. \tag{12}$$

By (3) the mapping $\Psi^{-1}$ satisfies the Lipschitz condition in $\Psi(D)$ with a constant

$$\mathrm{Lip}\,\left(\Psi^{-1},\Psi(D)\right) \le \frac{1}{1-\mu}.$$

The mapping $\Phi(x,y)$ had been defined such that its inverse map has the form

$$x = X, \quad y = \Theta(X,Y). \tag{13}$$

Moreover, $\Phi^{-1} = \Psi^{-1}Q^*$ where $Q^*$ is the invertible linear transformation and $\Psi^{-1}$ is the Lipschitz map. Hence, $\Phi^{-1}$ satisfies the Lipschitz condition in $\Phi(D)$ with a constant

$$\text{Lip}\left(\Phi^{-1}, \Phi(D)\right) \leq |Q^*| \,\text{Lip}\left(\Psi^{-1}, \Psi(D)\right) \leq L. \tag{14}$$

Next we observe that

$$(X, Y) = \Phi(\Phi^{-1}(X, Y)) = (X, F(X, \Theta(X, Y))).$$

From this relation it follows that

$$F(X, \Theta(X, Y)) = Y. \tag{15}$$

From (11) it follows that $B'$ lies in $\Phi(D)$. The intersection $\Pi$ of the ball $B'$ and the plane $Y_1 = F_1(x_0, y_0), \dots, Y_m = F_m(x_0, y_0)$ is a connected set with the codimension $m$ containing the point $(X_0, Y_0) = (x_0, F(a))$. Denote by $j$ the orthogonal projection from $\mathbf{R}^n \times \mathbf{R}^m$ onto $\mathbf{R}^n$. For any set $A \subset \mathbf{R}^n \times \mathbf{R}^m$ we have

$$j(A) = \cup_{y \in \mathbf{R}^m} \{x \in \mathbf{R}^n : (x, y) \in A\}.$$

By the definition of $\Phi$ we may write

$$\begin{aligned} j(\Phi(A')) &= \Phi(j(A')) \qquad \forall\ A' \subset D, \\ j(\Phi^{-1}(A'')) &= \Phi^{-1}(j(A'')) \qquad \forall\ A' \subset \Phi(D). \end{aligned} \tag{16}$$

The equation of the connected piece of the surface $\Phi^{-1}(\Pi)$ containing $a = (x_0, y_0)$ can be rewritten in the nonparametric form. Namely, let

$$(X, Y) = (x, \Theta(x, Y_0)), \quad x \in \Phi^{-1}(j(B')).$$

We put $G(x) = \Theta(x, Y_0)$.

By (15) we now find

$$F(x, G(x)) = Y_0 = F(x_0, y_0),$$

where

$$G(x_0) = \Theta(x_0, Y_0) = \Theta(X_0, Y_0) = y_0.$$

Uniqueness of the map $G$ follows from the bijectivity of $\Phi(x, y)$. In fact, if $(x, y_1), (x, y_2) \in D$ and $F(x, y_1) = F(x, y_2)$ then $\Phi(x, y_1) = \Phi(x, y_2)$. Thus $y_1 = y_2$.

The relation (12) states that the ball $B''$ is contained in $B'$ and guarantees, together with (16), that the ball $B^n(x_0, \rho)$ lies in $\Phi^{-1}(j(B'))$. This implies the necessary bound for $\rho$.

Now we estimate the Lipschitz constant of the function $\Theta$. For arbitrary

$$(X', Y'), (X'', Y'') \in B^{n+m}(a, \rho)$$

from (14) it follows

$$\left|\Phi^{-1}(X', Y') - \Phi^{-1}(X'', Y'')\right| \le L \,|(X'' - X', Y'' - Y')|\,.$$

By the relations (13), which describe $\Phi^{-1}$, we may rewrite this inequality in the following form

$$|(X'' - X', \Theta(X'', Y'') - \Theta(X', Y'))| \le L\,|(X'' - X', Y'' - Y')|\,.$$

Further,

$$|X'' - X'|^2 + |\Theta(X'', Y'') - \Theta(X', Y')|^2 \le L^2\,|X'' - X'|^2 + L^2\,|Y'' - Y'|^2\,,$$

and

$$|\Theta(X'', Y'') - \Theta(X', Y')|^2 \le \left(L^2 - 1\right) \cdot |X'' - X'|^2 + L^2\,|Y'' - Y'|^2\,.$$

Using the definition of $G$, we may choose $Y'' = Y' = Y_0$ and put $X = x$. Then we obtain

$$|G(x'') - G(x')|^2 \le \left(L^2 - 1\right)|x'' - x'|^2\,.$$

That is,

$$\mathrm{Lip}\,(G, B^n(x_0, \rho)) \le \sqrt{L^2 - 1}.$$

The theorem is proved completely. □

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