ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 36, Number 1, 2006

DIFFERENTIAL INEQUALITIES AND CRITERIA FOR STARLIKE AND UNIVALENT FUNCTIONS

M. OBRADOVIĆ, S. PONNUSAMY, V. SINGH AND P. VASUNDHRA

ABSTRACT. The main aim of this paper is to use the method of differential subordination to obtain a number of sufficient conditions for a normalized analytic function to be univalent or starlike in the unit disc. In particular, we find a condition on β so that each normalized analytic function f satisfying the condition

$$\left|1+\frac{zf''(z)}{2f'(z)}-\frac{zf'(z)}{f(z)}\right|<\beta, \quad z\in\Delta$$

implies that f is univalent or starlike in the unit disc.

1. Introduction. Throughout the text, $\Delta=\{z:|z|<1\}$ denotes the unit disc and $\mathcal{H}$ denotes the class of all analytic functions in Δ. A function $f\in\mathcal{H}$ is said to be convex if $f(\Delta)$ is a convex domain. It is well known that f is convex if and only if $f'(0)\neq 0$ and $\operatorname{Re}(zf''(z)/f'(z)+1)>0$ for $z\in\Delta$. A function $f\in\mathcal{H}$ is said to be starlike if f is univalent and $f(\Delta)$ is a starlike domain, with respect to $z=0$. It is well known that f is starlike if and only if $f(0)=0$, $f'(0)\neq 0$ and $\operatorname{Re}(zf'(z)/f(z))>0$ for $z\in\Delta$. Let $\mathcal{A}$ be the class of all functions $f\in\mathcal{H}$ such that $f(0)=f'(0)-1=0$. The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. In the following, we denote by $\mathcal{K}$ and $\mathcal{S}^*$ the normalized subclasses of functions in $\mathcal{S}$ for which $f(\Delta)$ is convex and starlike, respectively. We denote by $\mathcal{S}^*(\beta)$, the class of all starlike functions f of order β, $\beta<1$, if and only if $f\in\mathcal{A}$ and $\operatorname{Re}(zf'(z)/f(z))>\beta$ for $z\in\Delta$. Similarly, f is said to belong to $\mathcal{K}(\beta)$, the class of all convex functions of order β, if and only if $zf'(z)\in\mathcal{S}^*(\beta)$. Note that $\mathcal{S}^*(0)=\mathcal{S}^*$, and $\mathcal{K}(0)=\mathcal{K}$. Define

$$\mathcal{R}=\{f\in\mathcal{A}:\operatorname{Re} f'(z)>0,\ z\in\Delta\}.$$

2000 AMS *Mathematics Subject Classification.* Primary 30C45,30C55.
Key words and phrases. Univalent, starlike and convex functions, Hadamard product and subordination.
The work of the second and fourth authors was supported by NBHM grant (Ref. No. 48/1/98-R & D-II) and was initiated during the third author's visit to the Department of Math., Indian Institute of Technology, IIT-Madras, Chennai.
Received by the editors on February 28, 2002, and in revised form on January 28, 2004.

Finally, let us recall an important class that was studied recently in [**5**]:

$$\mathcal{U}(\lambda) = \left\{ f \in \mathcal{A} : \left| f'(z) \left(\frac{z}{f(z)} \right)^2 - 1 \right| < \lambda, \ z \in \Delta \right\}, \tag{1.1}$$

where $f(z) \neq 0$ for $z \in \Delta \setminus \{0\}$. According to a result due to Ozaki and Nunokawa [**7**], we have the inclusion $\mathcal{U}(\lambda) \subset \mathcal{S}$ for $0 < \lambda \leq 1$. We see that the Koebe function $z/(1-z)^2$ belongs to $\mathcal{U}(1)$ but does not belong to the class of starlike functions of order α, $\alpha > 0$. Similarly, $z + z^2/2 \in \mathcal{U}(1)$ but not in $\mathcal{S}^*(\alpha)$ with $\alpha > 0$. In [**5**] Ponnusamy and Obradović obtained a condition on λ so that $\mathcal{U}(\lambda)$ is in $\mathcal{S}^*$ or $\mathcal{R}$ (in fact in smaller subclasses) respectively. We recall [**6**]

Lemma 1.1. *Let $f \in \mathcal{U}(\lambda)$. Then we have*

(i) $f \in \mathcal{S}^*$ *for*

$$0 < \lambda \leq \frac{-|f''(0)| + \sqrt{8 - |f''(0)|^2}}{4}$$

(ii) $f \in \mathcal{R}$ *for*

$$0 < \lambda \leq \frac{\sqrt{4|f''(0)| + 9} - (2|f''(0)| + 1)}{4}.$$

This lemma was proposed as conjectures by Obradović and Ponnusamy [**5**] and has been proved in [**6**] in a more general form. In particular, if $f \in \mathcal{A}$ with $f''(0) = 0$, then from Lemma 1.1 one has the following implications:

$$\mathcal{U}(\lambda) \subset \mathcal{S}^* \quad \text{for } 0 < \lambda \leq 1/\sqrt{2}$$

(see also [**9**]) and

$$\mathcal{U}(\lambda) \subset \mathcal{R} \quad \text{for } 0 < \lambda \leq 1/2.$$

Let $\psi : \mathbf{C}^2 \to \mathbf{C}$ and let h be univalent in Δ. If $p \in \mathcal{H}$ and satisfies the first order differential subordination

$$\psi(p(z), z\,p'(z)) \prec h(z), \tag{1.2}$$

then p is called a solution of the differential subordination. A univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1.2). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.2) is said to be the best dominant of (1.2). Note that the best dominant is unique up to rotation. For a detailed collection of works on differential subordination, we refer to the recent monograph due to Miller and Mocanu in [**2**]. For the proof of our results, we also need the following lemmas on differential subordination.

Lemma 1.2 [**2**, Theorem 3.4h, p. 132]. *Let q be univalent in Δ, θ and ϕ analytic in a domain D containing $q(\Delta)$ with $\phi(w) \neq 0$ when $w \in q(\Delta)$. Set $Q(z) = z\,q'(z)\phi\,(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that*

(i) *Q is starlike in Δ*

(ii)

$$\operatorname{Re}\left(\frac{z\,h'(z)}{Q(z)}\right) = \operatorname{Re}\left(\frac{\theta'(q(z))}{\phi(q(z))} + \frac{z\,Q'(z)}{Q(z)}\right) > 0, \quad z \in \Delta.$$

If $p(z)$ is analytic in Δ with $p(0) = q(0)$, $p(\Delta) \subset D$ and

$$\theta(p(z)) + z\,p'(z)\phi\,(p(z)) \prec \theta(q(z)) + z\,q'(z)\phi\,(q(z)) = h(z), \tag{1.3}$$

then $p(z) \prec q(z)$. The function q is the best dominant of (1.3).

Lemma 1.3 [**1**]. *Let $\Omega \subset \mathbf{C}$. Suppose that $\psi : \mathbf{C}^2 \to \mathbf{C}$ satisfies the condition $\psi(ix, y) \notin \Omega$ when x is real and $y \leq -(1 + x^2)/2$. If p is analytic in Δ, with $p(0) = 1$ and $\psi(p(z), z\,p'(z)) \in \Omega$ for $z \in \Delta$, then $\operatorname{Re} p(z) > 0$ in Δ.*

Lemma 1.3 is a special case of a result due to Miller and Mocanu in [**1**, Theorem 1].

2. Main results.

Theorem 2.1. *Let $k > 1$ and $\alpha > -1$. Suppose that p is analytic in Δ, $p(z) \neq 0$ in Δ, $p(0) = 1$ and satisfies the condition*

$$\frac{z\,p'(z)}{p(z)} + \alpha\, p(z) \prec \frac{k\alpha - z}{k + z}, \quad z \in \Delta, \tag{2.1}$$

or equivalently

$$\left| \frac{z\,p'(z)}{p(z)} + \alpha\,(p(z) - 1) - \frac{\alpha + 1}{k^2 - 1} \right| < \frac{k(\alpha + 1)}{k^2 - 1}, \quad z \in \Delta. \tag{2.2}$$

Then

$$p(z) \prec \frac{k}{k + z}, \qquad i.e. \quad \left| p(z) - \frac{k^2}{k^2 - 1} \right| < \frac{k}{k^2 - 1}, \quad z \in \Delta,$$

and $k/(k + z)$ is the best dominant of (2.1).

Proof. Choose

$$q(z) = \frac{k}{k + z} \quad \text{and} \quad \phi(w) = \frac{1}{w}.$$

Then q is a convex univalent function with $q(0) = 1$. Further

$$q(\Delta) = \left\{ w \in \mathbf{C} : \left| w - \frac{k^2}{k^2 - 1} \right| < \frac{k}{k^2 - 1} \right\},$$

ϕ is analytic in $\mathbf{C} \setminus \{0\} \supset q(\Delta)$ and $\phi(w) \neq 0$ when $w \in q(\Delta)$. Furthermore

$$Q(z) = z\,q'(z)\phi\,(q(z)) = \frac{z\,q'(z)}{q(z)} = -\frac{z}{k + z}$$

which is convex and in particular, Q is starlike in Δ. Define

$$\theta(w) = \alpha w, \quad h(z) = \theta(q(z)) + Q(z) = \frac{k\alpha - z}{k + z}.$$

It is easy to see that h is convex univalent in Δ, and

$$\frac{z\,h'(z)}{Q(z)} = (1+\alpha)\,\frac{k}{k+z}$$

so that

$$\operatorname{Re}\frac{z\,h'(z)}{Q(z)} > (1+\alpha)\,\frac{k}{k-1} > 0, \quad z \in \Delta.$$

By Lemma 1.2, (2.1) implies that $p(z) \prec k/(k+z)$. Moreover, since

$$\left(\frac{k\alpha - z}{k+z} - \alpha\right) - \frac{\alpha+1}{k^2-1} = -\frac{k(1+\alpha)}{k^2-1}\left(\frac{1+kz}{k+z}\right)$$

and since, for $k > 1$,

$$\left|\frac{1+kz}{k+z}\right| < 1, \quad z \in \Delta,$$

we observe that (2.1) and (2.2) are equivalent. □

Corollary 2.2. *Let $k > 1$, $\alpha > -1$ and $f \in \mathcal{A}$. Then*

$$\left|1 - \alpha + \frac{zf''(z)}{f'(z)} + (\alpha - 1)\,\frac{zf'(z)}{f(z)} - \frac{\alpha+1}{k^2-1}\right| < \frac{k(\alpha+1)}{k^2-1}$$
$$\Longrightarrow \quad \left|\frac{zf'(z)}{f(z)} - \frac{k^2}{k^2-1}\right| < \frac{k}{k^2-1}.$$

Proof. We choose $p(z) = zf'(z)/f(z)$ in Theorem 2.1. □

In the case $\alpha = -1$, (2.1) is equivalent to

$$\frac{z\,p'(z)}{p(z)} - p(z) = -1$$

which gives $p(z) \equiv 1$. The following result extends this case with a different conclusion.

Theorem 2.3. *Let p be analytic in Δ, $p(z) \neq 0$ in Δ, $p(0) = 1$, $\alpha > -1/2$ and*

$$\left|\frac{z\,p'(z)}{p(z)} + \alpha\,(p(z)-1)\right| < \alpha + 1.$$

Then $\mathrm{Re}\, p(z) > 0$ *in* Δ.

Proof. Define $\psi(r,s) = sr^{-1} + \alpha\,(r-1)$. By Lemma 1.3, it suffices to show that

$$|\psi(ix,y)| \geq \alpha + 1$$

whenever x and y are real and $y \leq -(1+x^2)/2$. It follows that

$$|\psi(ix,y)|^2 = \left|\frac{y}{ix} + \alpha\,(ix-1)\right|^2 = \alpha^2 + \frac{1}{x^2}\,(\alpha x^2 - y)^2.$$

Since

$$\alpha x^2 - y \geq \alpha x^2 + \frac{1+x^2}{2} = \frac{1+(1+2\alpha)x^2}{2},$$

we see that for $\alpha \geq -1/2$

$$|\psi(ix,y)|^2 \geq \alpha^2 + \frac{1}{4}\left(\frac{1}{x} + (1{+}2\alpha)x\right)^2 \geq \alpha^2 + \frac{1}{4}\,[4\,(1{+}2\alpha)] = (\alpha{+}1)^2$$

which shows that $|\psi(ix,y)| \geq \alpha + 1$ for real x, y with $y \leq -(1+x^2)/2$. The desired conclusion follows from Lemma 1.3. □

For $p(z) = zf'(z)/f(z)$, Theorem 2.3 gives the following result.

Corollary 2.4. *Let $f \in \mathcal{A}$ and $\alpha > -1/2$. Then*

$$\left|1 - \alpha + \frac{zf''(z)}{f'(z)} - (1-\alpha)\,\frac{zf'(z)}{f(z)}\right| < \alpha + 1 \quad \Longrightarrow \quad f \in \mathcal{S}^*. \tag{2.3}$$

(i) For $\alpha = 0$, Corollary 2.4, shows that

$$f \in \mathcal{A} \quad \text{and} \quad \left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| < 1 \quad \Longrightarrow \quad f \in \mathcal{S}^*.$$

This implication can also be obtained as a special case of the following subordination result if we choose $p(z) = zf'(z)/f(z)$:

$$\frac{z\,p'(z)}{p(z)} \prec \frac{z}{(1-z)^2} \quad \Longrightarrow \quad p(z) \prec \frac{1+z}{1-z}.$$

We refer to Theorem 2.7 for an extension of this result.

(ii) For $\alpha = 1$, (2.3) gives

$$f \in \mathcal{A} \quad \text{and} \quad \left|\frac{zf''(z)}{f'(z)}\right| < 2 \quad \Longrightarrow \quad f \in \mathcal{S}^*.$$

(iii) For $\alpha \neq 1$, Corollary 2.4 is equivalent to

$$f \in \mathcal{A} \quad \text{and} \quad \left|1 + \frac{1}{1-\alpha}\,\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| < \frac{\alpha+1}{|1-\alpha|} \quad \Longrightarrow \quad f \in \mathcal{S}^*$$

so that this holds for $\alpha \in [-1/2, \infty) \setminus \{1\}$. If we let $\alpha' = 1/(1-\alpha)$, then by a simple calculation we deduce the following: For $\alpha' \in (-\infty, 0] \cup [2/3, \infty)$, we have

$$\left|1 + \alpha'\,\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| < \beta \quad \Longrightarrow \quad f \in \mathcal{S}^*,$$

where $\beta = (2 - (1/\alpha'))\,|\alpha'|$.

Thus, it is interesting to raise the following:

Problem 2.5. For a given $\alpha' \in (0, 2/3)$, does there exist a best value of $\beta > 0$ such that the condition

$$\left|1 + \alpha'\,\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| < \beta$$

implies that f is starlike or univalent in Δ?

In our next result we provide an affirmative answer when $\alpha' = 1/2$. However, the counterpart of Theorem 2.6 in the sense of Problem 2.5 for $\alpha' \in (0, 2/3) \setminus \{1/2\}$ remains open.

Theorem 2.6. *Let $f \in \mathcal{A}$ satisfy the condition*

$$\left|1 + \frac{zf''(z)}{2f'(z)} - \frac{zf'(z)}{f(z)}\right| < \beta, \quad z \in \Delta. \tag{2.4}$$

(i) *If $\beta \approx 0.426\ldots$ is the solution of the equation $\beta e^{2\beta} = 1$, then f is univalent in Δ.*

(ii) *If β is the solution of the equation $4\beta e^{2\beta} = -|f''(0)| + \sqrt{8 - |f''(0)|^2}$, then $f \in \mathcal{S}^*$. In particular, if $f''(0) = 0$ and if $\beta \approx 0.3507\ldots$ is the solution of the equation $2\beta e^{2\beta} = \sqrt{2}$, then $f \in \mathcal{S}^*$.*

Proof. Let

$$p(z) = \frac{z^2 f'(z)}{f^2(z)} = \frac{z}{f(z)} - z\left(\frac{z}{f(z)}\right)'.$$

Then a simple calculation shows that

$$\begin{aligned} z\,p'(z) &= z\left(\frac{z^2 f'(z)}{f^2(z)}\right)' = -z^2\left(\frac{z}{f(z)}\right)'' \\ &= 2\,\frac{z^2 f'(z)}{f^2(z)}\left[1 + \frac{zf''(z)}{2f'(z)} - \frac{zf'(z)}{f(z)}\right] \end{aligned}$$

and therefore, by (2.4), we have

$$\frac{z\,p'(z)}{p(z)} = 2\left(1 + \frac{zf''(z)}{2f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec 2\beta z.$$

Because

$$\frac{z\,p'(z)}{p(z)} \prec 2\beta z \quad \Longrightarrow \quad p(z) \prec e^{2\beta z},$$

we deduce that $|p(z)| \leq e^{2\beta \operatorname{Re} z} < e^{2\beta}$ for $z \in \Delta$. The above subordination relation is a consequence of Lemma 1.2, if we choose $q(z) = e^{2\beta z}$, $\phi(z) = 1/z$ and $\theta(z) = 0$. Therefore, by the hypothesis and the last subordination implication, it follows that

$$\left|-z^2\left(\frac{z}{f(z)}\right)''\right| = |p(z)|\left|\frac{z\,p'(z)}{p(z)}\right| = |z\,p'(z)| < 2\beta e^{2\beta} = 2. \tag{2.5}$$

By (2.5) and the Schwarz lemma, we get

$$\left| -z^2 \left(\frac{z}{f(z)} \right)'' \right| \leq 2|z|^2$$

which implies that

$$\left| \left(\frac{z}{f(z)} \right)'' \right| \leq 2, \quad z \in \Delta,$$

and in particular, f is univalent, see [**5**]. Part (i) follows. Part (ii) is a consequence of [**5**, Example 1.11] and so we omit the details. □

In [**8**, Remark 4.4.3], the following result was obtained as a special case (see also [**4**]) of a general result: if F is analytic in $|z| < 1$, $F(z)F'(z)/z \neq 0$ in $|z| < 1$, then for $\operatorname{Re} \alpha > 0$, we have

$$\left| (\alpha - 1) \frac{z F'(z)}{F(z)} + \frac{z F''(z)}{F'(z)} \right| < 1 \quad \Longrightarrow \quad F \in \mathcal{S}^*. \tag{2.6}$$

If we put $p(z) = z F'(z)/F(z)$ in (2.6), then the last implication takes the form

$$\left| \frac{z p'(z)}{p(z)} + \alpha p(z) - 1 \right| < 1 \quad \Longrightarrow \quad \operatorname{Re} p(z) > 0. \tag{2.7}$$

For α real, we extend this result in the following form.

Theorem 2.7. *Let $p(z)$ be analytic in Δ, $p(z) \neq 0$ in Δ, $p(0) = 1$ and $\alpha \geq -1/2$. Then*

$$\frac{z p'(z)}{p(z)} + \alpha p(z) \prec \alpha \frac{1+z}{1-z} + \frac{2z}{1-z^2} \quad \Longrightarrow \quad p(z) \prec \frac{1+z}{1-z}, \quad z \in \Delta, \tag{2.8}$$

and $(1+z)/(1-z)$ is the best dominant.

Proof. Let $q(z) = (1+z)/(1-z)$. Then q is convex univalent in Δ and $q(\Delta) = \{w : \operatorname{Re} w > 0\}$. Let $\theta(w) = \alpha w$ and $\phi(w) = 1/w$. Then

$\theta(w)$ and $\phi(w)$ are analytic in $\mathbf{C} \setminus \{0\} \supset q(\Delta)$ and $\phi(w) \neq 0$ when $w \in q(\Delta)$. Now

$$Q(z) = z\,q'(z)\phi\,(q(z)) = \frac{z\,q'(z)}{q(z)} = \frac{2z}{1-z^2}$$

is starlike in Δ. Further,

$$\begin{aligned} h(z) &= \theta(q(z)) + Q(z) = \alpha\,q(z) + \frac{z\,q'(z)}{q(z)} \\ &= \alpha\,\frac{1+z}{1-z} + \frac{2z}{1-z^2} = \frac{1+2\alpha}{1-z} - \frac{1}{1+z} - \alpha \end{aligned}$$

and

$$z\,h'(z) = \frac{z}{(1+z)^2} + (1+2\alpha)\,\frac{z}{(1-z)^2}$$

so that

$$\frac{z\,h'(z)}{Q(z)} = \frac{1}{2}\left(\frac{1-z}{1+z}\right) + \frac{1+2\alpha}{2}\left(\frac{1+z}{1-z}\right).$$

Therefore, for $1+2\alpha \geq 0$, we have $\mathrm{Re}\,(z\,h'(z)/Q(z)) > 0$ and thus all the conditions of Lemma 1.2 are satisfied and the theorem is proved. □

For $\alpha \geq -1/2$, the function h defined by

$$h(z) = \alpha\,\frac{1+z}{1-z} + \frac{2z}{1-z^2},$$

maps the unit disc Δ conformally onto the complex plane with slits along the half-lines $\mathrm{Re}\,w = 0$, $|\mathrm{Im}\,w| \geq \sqrt{1+2\alpha}$. Suppose that $\alpha \in [-1/2, \infty) \setminus \{1\}$. Then (2.8) is equivalent to

$$\frac{z\,p'(z)}{|1-\alpha|p(z)} + \frac{\alpha\,(p(z)-1)}{|1-\alpha|} \prec \frac{h(z)-\alpha}{|1-\alpha|} = H(z) \quad \Longrightarrow \quad p(z) \prec \frac{1+z}{1-z}$$

where

$$H(\Delta) = \mathbf{C} \setminus \left\{ w \in \mathbf{C} : \mathrm{Re}\,w = -\frac{\alpha}{|1-\alpha|},\ |\mathrm{Im}\,w| \geq \frac{\sqrt{1+2\alpha}}{|1-\alpha|} \right\}.$$

Now, let $\alpha' = 1/(1-\alpha)$. Then a simple calculation gives the following.

Corollary 2.8. *Let $f \in \mathcal{A}$ and $\alpha' \in (-\infty, 0) \cup [2/3, \infty)$. Then we have*

$$1 + \alpha' \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec G(z) \quad \Longrightarrow \quad f \in \mathcal{S}^*,$$

where G is the conformal mapping of the unit disc Δ with $G(0) = 1$ and

$$G(\Delta) = \mathbf{C} \Bigg\backslash \left\{ w \in \mathbf{C} : \operatorname{Re} w = \frac{(1-\alpha')|\alpha'|}{\alpha'}, \right. \\ \left. |\operatorname{Im} w| \geq |\alpha'|\sqrt{3 - 2/\alpha'} = \sqrt{3\alpha'^2 - 2\alpha'} \right\}.$$

Problem 2.9. Find the counterpart of Corollary 2.8 (as in Problem 2.5) if $\alpha \in (0, 2/3)$.

For example if $\alpha' \in [2/3, 1)$, then Corollary 2.8 gives

$$\operatorname{Re}\,(G_{\alpha'}(f)) < 1 - \alpha' \quad \Longrightarrow \quad f \in \mathcal{S}^*,$$

where

$$G_{\alpha'} f(z) = 1 + \alpha' \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}.$$

In particular, if we suppose $\alpha' = 1/2$, then this condition becomes

$$\operatorname{Re}\left(1 + \frac{1}{2}\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) < \beta, \quad z \in \Delta,$$

where $\beta = 1/2$. In Corollary 2.11 we show that if $\beta = 1/4$, then the function satisfying the last inequality implies that f is univalent in Δ. On the other hand, it can be shown that a function f satisfying this condition does not necessarily imply that $f \in \mathcal{S}^*$. For instance, consider the function f_n defined by

$$f_n(z) = \frac{z}{1 + (s - t/n)e^{i\theta} z + (t/n)e^{-i\lambda} z^{n+1}}$$

where $t \in (0, 1)$, $0 < \varepsilon < t$, $s = \sqrt{1 - (t-\varepsilon)^2}$, $\theta = \arccos(-s)$, $\lambda = \arccos t$ and n is a positive integer such that $ns - t > 0$. As shown in [**3**], $f_n \notin \mathcal{S}^*$ for sufficiently large n. Let N be one such large value of n with the property that $f_N \notin \mathcal{S}^*$. Then

$$\frac{z^2 f_N'(z)}{f_N^2(z)} - 1 = -z^2\left(\frac{1}{f_N'(z)} - \frac{1}{z}\right)' = -te^{-i\lambda}z^{N+1}$$

showing that $f_N \in \mathcal{S}$ because $\mathcal{U}(t) \subset \mathcal{U}(1) \subset \mathcal{S}$ for $0 < t < 1$. Further, we find that

$$1 + \frac{1}{2}\,\frac{zf_N''(z)}{f_N'(z)} - \frac{zf_N'(z)}{f_N(z)} = \frac{N+1}{2}\left[1 - \frac{1}{1 - te^{-i\lambda}z^{N+1}}\right] \tag{2.9}$$

and therefore, for $0 < t \le 1/N$, we have

$$\mathrm{Re}\left\{1 + \frac{1}{2}\,\frac{zf_N''(z)}{f_N'(z)} - \frac{zf_N'(z)}{f_N(z)}\right\} < \frac{N+1}{2}\left[1 - \frac{1}{t+1}\right] = \frac{(N+1)t}{2(1+t)} \le \frac{1}{2}.$$

Furthermore, from (2.9), it follows that

$$\begin{aligned}\left|1 + \frac{1}{2}\,\frac{zf_N''(z)}{f_N'(z)} - \frac{zf_N'(z)}{f_N(z)}\right| = \frac{N+1}{2}\left|\frac{te^{-i\lambda}z^{N+1}}{1 - te^{-i\lambda}z^{N+1}}\right| &< \frac{(N+1)}{2}\,\frac{t}{1-t}\\ &\le \frac{1}{2} \quad \text{if} \quad 0 < t \le \frac{1}{N+3}.\end{aligned}$$

This observation shows that there exists a function $f_N \in \mathcal{S} \setminus \mathcal{S}^*$ such that

$$\left|1 + \frac{1}{2}\,\frac{zf_N''(z)}{f_N'(z)} - \frac{zf_N'(z)}{f_N(z)}\right| < \frac{1}{2}, \quad z \in \Delta.$$

This observation also motivates Problem 2.5.

Theorem 2.10. *Let p be analytic in Δ, $p(z) \neq 0$ in Δ, $p(0) = 1$, $\alpha > -1/4$ and*

$$\frac{z\,p'(z)}{p(z)} + \alpha\,(p(z) - 1) \prec -\alpha\,z - \frac{z}{1-z}, \quad z \in \Delta. \tag{2.10}$$

Then $p(z) \prec 1 - z$ and this is the best dominant of (2.10).

Proof. Choose $q(z) = 1 - z$ so that $q(\Delta) = \{w : |w-1| < 1\}$. With the same choices of $\theta(w) = \alpha w$ and $\phi(w) = 1/w$, we get

$$Q(z) = -\frac{z}{1-z} = 1 - \frac{1}{1-z}.$$

We observe that Q is convex univalent and $\operatorname{Re} Q(z) < 1/2$ in Δ. Now

$$h(z) = \theta(q(z)) + Q(z) = \alpha\,(1-z) + 1 - \frac{1}{1-z}$$

and

$$z\,h'(z) = \alpha\,z - \frac{z}{(1-z)^2}$$

so that

$$\frac{z\,h'(z)}{Q(z)} = \alpha\,(1-z) + \frac{1}{1-z}.$$

Clearly, Re $(z\,h'(z)/Q(z)) > 1/2$ if $\alpha > 0$, and Re $(z\,h'(z)/Q(z)) > 2\alpha + 1/2$ if $\alpha < 0$. The desired conclusion follows from Lemma 1.2. □

Corollary 2.11. *If $f \in \mathcal{A}$ satisfies the condition*

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{2f'(z)} - \frac{zf'(z)}{f(z)}\right) < \frac{1}{4}, \quad z \in \Delta,$$

then f is univalent in Δ.

Proof. Let $(z/f(z))^2\, f'(z) = p(z)$. Then

$$\frac{z\,p'(z)}{p(z)} = 2\left(1 - \frac{zf'(z)}{f(z)}\right) + \frac{zf''(z)}{f'(z)}$$

and therefore, by Theorem 2.10 with $\alpha = 0$, it follows that

$$\frac{zf''(z)}{2f'(z)} - \frac{zf'(z)}{f(z)} \prec -1 - \frac{z}{2(1-z)} \quad \Longrightarrow \quad \left(\frac{z}{f(z)}\right)^2 f'(z) \prec 1-z.$$

Since $\mathrm{Re}\,(z/(1-z)) > -1/2$, the above subordination relation is equivalent to

$$\mathrm{Re}\left(\frac{zf''(z)}{2f'(z)} - \frac{zf'(z)}{f(z)}\right) < -\frac{3}{4} \quad \Longrightarrow \quad f \in \mathcal{U}(1),$$

and the desired conclusion follows as $\mathcal{U}(1) \subset \mathcal{S}$. □

More generally, we can easily prove the following which we state without proof.

Corollary 2.12. *If $f \in \mathcal{A}$ satisfies the condition*

$$1 + \frac{zf''(z)}{2f'(z)} - \frac{zf'(z)}{f(z)} \prec \frac{kz}{2(1+kz)}, \quad z \in \Delta, \tag{2.11}$$

for $k \in (0,1]$, then we have

$$\frac{z^2 f'(z)}{f^2(z)} \prec 1 + kz, \quad z \in \Delta.$$

For $k \in (0,1)$, (2.11) is equivalent to

$$\left|1 + \frac{zf''(z)}{2f'(z)} - \frac{zf'(z)}{f(z)} - \frac{k^2}{2(1-k^2)}\right| < \frac{k}{2(1-k^2)}, \quad z \in \Delta,$$

and for $k = 1$, this gives Corollary 2.11. If we apply Lemma 1.1, one can obtain a number of results for certain $k < 1$ implying that f is in $\mathcal{S}^*$ or $\mathcal{R}$, respectively. For example if $f \in \mathcal{A}$ with $f''(0) = 0$, then f satisfying the subordination condition (2.11) implies that $f \in \mathcal{S}^*$ whenever $0 < k \le 1/\sqrt{2}$ and $f \in \mathcal{R}$ whenever $0 < k \le 1/2$.

REFERENCES

1. S.S. Miller and P.T. Mocanu, *Differential subordinations and inequalities in the complex plane*, J. Differential Equations **67** (1987), 199–211.

2. ———, *Differential subordinations: Theory and applications*, Marcel Dekker, No. 225, New York, Basel, 2000, p. 459.

3. N. Mochizwki and T. Sano, *Some conditions for the strong starlikeness of holomorphic functions*, Interdiscip. Inform. Sci. **3** (1997), 87–90.

4. M. Obradović, *Starlikeness of a certain integral transforms*, in *Computational methods and function theory* (CMFT '97) (N. Papamichael, St. Ruscheweyh and E.B. Saff, eds.), World Scientific Publ. Co., Singapore, 1997, pp. 431–436.

5. M. Obradović and S. Ponnusamy, *New criteria and distortion theorems for univalent functions*, Complex Variables Theory Appl. **44** (2001), 173–191.

6. M. Obradović, S. Ponnusamy, V. Singh and P. Vasundhra, *Univalency, starlikeness and convexity applied to certain classes of rational functions*, Analysis **22** (2002), 225–242.

7. S. Ozaki and M. Nunokawa, *The Schwarzian derivative and univalent functions*, Proc. Amer. Math. Soc. **33** (1972), 392–394.

8. S. Ponnusamy, *Some applications of differential subordination and convolution techniques to univalent function theory*, Ph.D Thesis, Indian Institute of Technology, Kanpur, India, 1988.

9. V. Singh, *On a class of univalent functions*, Internat. J. Math. Math. Sci. **23** (2000), 855–858.

Department of Mathematics, Faculty of Technology and Metallurgy, 4 Karnegijeva Street, 11000 Belgrade, Serbia and Montenegro
E-mail address: obrad@elab.tmf.bg.ac.yu

Department of Mathematics, Indian Institute of Technology Madras, Chennai - 600 036, India
E-mail address: samy@iitm.ac.in

Department of Mathematics, Indian Institute of Technology Madras, Chennai - 600 036, India
E-mail address: vasu2kk@yahoo.com

3A/95, Azad Nagar, Kanpur-208 002, India