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A TELESCOPING PRINCIPLE FOR OSCILLATION OF SECOND ORDER DIFFERENTIAL EQUATIONS ON A TIME SCALE

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ABSTRACT. We establish a telescoping principle for oscillation of the second order scalar differential equation $(p(t)x^{\Delta}(t))^{\Delta} + q(t)x(\sigma(t)) = 0$ on a time scale and use it to obtain several new conditions for oscillation. Our results extend, unify, and modify the telescoping principles for oscillation of second order differential equations and difference equations, respectively, and provide a new tool for the investigation of oscillation on time scales. We illustrate the results obtained by several examples, none of which may be handled by known oscillation criteria.

1. Introduction. In this paper, we study the self-adjoint second order scalar equation

$$(p(t)x^{\Delta}(t))^{\Delta} + q(t)x(\sigma(t)) = 0 \tag{E}$$

on a time scale $\mathbf{T}$, that is, on a nonempty closed subset $\mathbf{T}$ of $\mathbf{R}$, the set of real numbers. Without loss of generality we assume throughout that $0 \in \mathbf{T}$ and $\sup \mathbf{T} = \infty$ since we are interested in extending oscillation and nonoscillation criteria for the corresponding differential and difference equations, namely

$$(p(t)x'(t))' + q(t)x(t) = 0 \tag{1.1}$$

with $\mathbf{T}$ the interval $[0, \infty)$, and

$$\Delta(p_n \Delta x_n) + q_n x_{n+1} = 0 \tag{1.2}$$

with $\mathbf{T} = \mathbf{N}_0$, the set of nonnegative integers.

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Numerous oscillation and nonoscillation criteria have been established for equations (1.1) and (1.2), see, for example, [**3–5, 8, 9, 11–21**] and the references therein. Many of the criteria involve the integral of $q(t)$ or the sum of q_n and hence require information concerning the equation on the whole set $[0, \infty)$ or $\mathbf{N}$. However, from the Sturm separation theorem, it is clear that oscillation may be regarded as an interval property. For instance, if there exists a sequence of subsets $[a_i, b_i]$ of $[0, \infty)$, $a_i \to \infty$ as $i \to \infty$, such that for each i there is a nontrivial solution of equation (1.1) which has at least two zeros in $[a_i, b_i]$, then every solution of equation (1.1) is oscillatory, no matter what the behavior of the coefficients is, in equation (1.1), on the remaining parts of $[0, \infty)$. The same property holds for equation (1.2).

Kwong and Zettl [**15**] applied this idea to oscillation of equation (1.1) and established a powerful telescoping principle that allows one to trim off the "troublesome" parts of $\int_0^t q(t)\,dt$ and apply any known criterion to the "good" parts. Kong and Zettl [**14**] obtained a parallel telescoping principle for equation (1.2) and applied it to obtain many new oscillation results for difference equations. The main purpose of this paper is to generalize the telescoping principles for equations (1.1) and (1.2) to time scales, and apply it to obtain new results on the oscillation of (E). Our work will unify and modify the work in [**14**] and [**15**], and will cover many existing criteria for oscillation, including the continuous case, discrete case, and the general case on time scales. Because of the generality, a number of additional technical arguments will be involved in the proofs.

A solution $x = x(t)$ of (E) is said to be nonoscillatory on $\mathbf{T}$ if there exists $\tau \in \mathbf{T}$ such that $x(t)x(\sigma(t)) > 0$ for $t > \tau$. Otherwise, it is oscillatory. It is well known that either all solutions of (E) are oscillatory or none are, so the equation (E) may be classified as oscillatory or nonoscillatory.

For the convenience of the reader, we recall the following concepts related to time scales.

Definition 1.1. Let $\mathbf{T}$ be a closed subset of $\mathbf{R}$ with the inherited Euclidean topology. Define the forward jump operator σ and the backward jump operator ρ, by

$$\sigma(t) = \inf\{\tau \in \mathbf{T} \mid \tau > t\} \in \mathbf{T} \quad \text{and} \quad \rho(t) = \sup\{\tau \in \mathbf{T} \mid \tau < t\} \in \mathbf{T}$$

for all $t \in \mathbf{T}$ with $t < \sup \mathbf{T}$ and $t > \inf \mathbf{T}$, respectively. If $\sup \mathbf{T} < \infty$, we define $\sigma(\sup \mathbf{T}) = \sup \mathbf{T}$. Similarly, if $\inf \mathbf{T} > -\infty$, we define $\rho(\inf \mathbf{T}) = \inf \mathbf{T}$. If $\sigma(t) > t$, t is said to be right-scattered; otherwise, it is right-dense. If $\rho(t) < t$, t is said to be left-scattered; otherwise, it is left-dense. We also define $\sigma^k(t) = \sigma(\sigma^{k-1}(t))$ for $t \in \mathbf{T}$ and $k = 2, 3, \dots$. Finally, the graininess function $\mu : \mathbf{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$.

Definition 1.2. For $f : \mathbf{T} \to \mathbf{R}$ and $t \in \mathbf{T}$, (if $t = \sup \mathbf{T}$, assume t is not left-scattered), define the Δ-derivative $f^{\Delta}(t)$ of $f(t)$ to be the number, provided it exists, with the property that, for any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \le \varepsilon |\sigma(t) - s|$$

for all $s \in U$. For $n > 1$, the nth Δ-derivative of $f(t)$ is defined by $f^{\Delta^n}(t) := (f^{\Delta^{n-1}})^{\Delta}(t)$.

We say that f is Δ-differentiable on $\mathbf{T}$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbf{T}$.

It can be shown that if $f : \mathbf{T} \to \mathbf{R}$ is continuous at $t \in \mathbf{T}$ and t is right-scattered, then

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

Note that if $\mathbf{T} = \mathbf{Z}$, the set of integers, then

$$f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t).$$

If $t \in \mathbf{T}$ is right-dense and $f : \mathbf{T} \to \mathbf{R}$ is differentiable at t, then

$$f^{\Delta}(t) = f'(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

Definition 1.3. Let $f : \mathbf{T} \to \mathbf{R}$ be a function. We say that f is rd-continuous if it is continuous at each right-dense point in $\mathbf{T}$ and $\lim_{s \to t^-} f(s)$ exists as a finite number for all left-dense points $t \in \mathbf{T}$.

Definition 1.4. If $F^{\Delta}(t) = f(t)$, then we define the integral of f on $[a,b] \cap \mathbf{T}$ by

$$\int_a^b f(\tau)\Delta\tau = F(b) - F(a);$$

and $\int_a^\infty f(\tau)\Delta\tau = \lim_{b\to\infty} \int_a^b f(\tau)\Delta\tau$.

It is well known that if f is rd-continuous on $[a,b]\cap\mathbf{T}$, then $\int_a^b f(\tau)\Delta\tau$ exists. For related results on the calculus on time scales, see [**1, 2, 7, 10**] and the references therein. In particular, we also have the following formula involving the graininess function which is valid for all points at which $f^{\Delta}(t)$ exists:

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t). \tag{1.3}$$

We assume throughout that $p \in C^1_{rd}(\mathbf{T},\mathbf{R})$ with $p(t) > 0$ and $q \in C_{rd}(\mathbf{T},\mathbf{R})$, where $C_{rd}(\mathbf{T},\mathbf{R})$ denotes the set of rd-continuous functions $f : \mathbf{T} \to \mathbf{R}$ and $C^1_{rd}(\mathbf{T},\mathbf{R})$ denotes the set of differentiable functions $f : \mathbf{T} \to \mathbf{R}$ whose derivative are rd-continuous. Our approach to the oscillation problems of (E) is based largely on the application of the Riccati equation. If $x(t)$ is a solution of (E) with $x(t)x(\sigma(t)) > 0$ for $t \in [t_1, t_2] \cap \mathbf{T}$, we let

$$u(t) = -\frac{p(t)x^{\Delta}(t)}{x(t)}. \tag{1.4}$$

Then for $t \in [t_1, t_2] \cap \mathbf{T}$, $u = u(t)$ satisfies the Riccati equation:

$$u^{\Delta}(t) = \frac{u^2(t)}{-\mu(t)u(t) + p(t)} + q(t). \tag{R}$$

If $t \in [t_1, t_2] \cap \mathbf{T}$ is right-scattered in $\mathbf{T}$, then from (1.3) and (R), we have

$$\begin{aligned} u(\sigma(t)) &= \mu(t)\Big[\frac{u^2(t)}{-\mu(t)u(t) + p(t)} + q(t)\Big] + u(t) \\ &= \frac{p(t)u(t)}{-\mu(t)u(t) + p(t)} + \mu(t)q(t). \end{aligned} \tag{1.5}$$

Note that (1.5) also holds for the case that t is right dense in $\mathbf{T}$.

In the following, we let $\mathbf{N}$ be the set of natural numbers and assume

$$J = \left(\bigcup_{i=1}^{\infty} J_i\right) \bigcap \mathbf{T}, \quad J_i = (a_i, \sigma(b_i)), \quad i \in \mathbf{N}, \tag{1.6}$$

where a_i, $b_i \in \mathbf{T}$, $i \in \mathbf{N}$, satisfy $0 < a_i < b_i < a_{i+1}$. We call J an "interval shrinking" set in $\mathbf{T}$. The set J will be used to introduce the following "shrinking" transformation on the time scale $\mathbf{T}$. That is, we define the time scale $\widehat{\mathbf{T}}$ by:

(1.7)

$$\widehat{\mathbf{T}} := \left\{s \in \mathbf{T} : s \le a_1\right\} \bigcup \left\{ \bigcup_{j=1}^{\infty} \left\{ s = t - \sum_{i=1}^{j} (\sigma(b_i) - \sigma(a_i)) : t \in \mathbf{T},\ \sigma(b_j) \le t \le a_{j+1} \right\} \right\},$$

and an interval shrinking transformation $\mathcal{T} : \mathbf{T} \to \widehat{\mathbf{T}}$

(1.8)

$$s = \mathcal{T}t = \begin{cases} t & t \in (-\infty, a_1] \bigcap \mathbf{T}, \\ \mathcal{T}a_j & t \in (a_j, \sigma(b_j)) \bigcap \mathbf{T}, \\ t - \sum_{i=1}^{j} (\sigma(b_i) - \sigma(a_i)) & t \in [\sigma(b_j), a_{j+1}] \bigcap \mathbf{T}, \end{cases} \quad j \in \mathbf{N}.$$

For $s \in \widehat{\mathbf{T}}$ we let

$$\mathcal{T}^{-1}s = \inf\{t \in \mathbf{T} : \mathcal{T}t = s\}.$$

2. Telescoping principle. In this section we establish the basic telescoping principle for oscillation. The following lemma plays a key role in the proofs.

Lemma 2.1. *A solution $x(t)$ of* (E) *satisfies $x(t)x(\sigma(t)) > 0$ for $t \in [t_1, t_2] \cap \mathbf{T}$ if and only if the corresponding solution $u(t)$ of the Riccati equation* (R) *satisfies $\mu(t)u(t) < p(t)$ for $t \in [t_1, t_2] \cap \mathbf{T}$.*

Proof. Since $u(t) = -p(t)x^{\Delta}(t)/x(t)$, for $t \in [t_1, t_2] \cap \mathbf{T}$,

$$\begin{aligned}-\mu(t)u(t) + p(t) &= \frac{\mu(t)p(t)x^{\Delta}(t)}{x(t)} + p(t)\\ &= p(t)\left(\frac{\mu(t)x^{\Delta}(t)}{x(t)} + 1\right)\\ &= p(t)\frac{x(\sigma(t))}{x(t)}.\end{aligned}$$

Therefore, $\mu(t)u(t) < p(t)$ if and only if $x(t)x(\sigma(t)) > 0$ for $t \in [t_1, t_2] \cap \mathbf{T}$. $\square$

Let $\widehat{\mathbf{T}}$ be defined by (1.7) and consider the telescoped equation of (E)

$$(\widehat{\mathrm{E}}) \qquad (\hat{p}(s)y^{\Delta}(s))^{\Delta} + \hat{q}(s)y(\hat{\sigma}(s)) = 0, \quad s \in \widehat{\mathbf{T}},$$

where $\hat{p}(s) = p(t)$, $\hat{q}(s) = q(t)$ for $t = \mathcal{T}^{-1}s$, and where $\hat{\sigma}$ denotes the forward jump operator in $\widehat{\mathbf{T}}$.

Our first result is a type of comparison theorem which, loosely speaking, says that if a certain solution $y = y(s)$ of the telescoped equation $(\widehat{\mathrm{E}})$ has a zero in $\widehat{\mathbf{T}}$, then a corresponding solution of the original equation (E) has a zero in $\mathbf{T}$.

Theorem 2.1. *Assume*

$$\int_{\sigma(a_i)}^{\sigma(b_i)} q(t)\Delta t \geq 0 \quad \text{for all } i \in \mathbf{N}, \tag{2.1}$$

and let $c \in \widehat{\mathbf{T}}$ such that $c > 0$. Suppose that y is a solution of $(\widehat{\mathrm{E}})$ with $y(s)y(\hat{\sigma}(s)) > 0$ for $s \in [0, c) \cap \hat{\mathbf{T}}$ and $y(c)y(\hat{\sigma}(c)) \leq 0$. Let x be a solution of (E) *with $x(0) \neq 0$, $p(0)x^{\Delta}(0)/x(0) \leq \hat{p}(0)y^{\Delta}(0)/y(0)$. Then there exists $d \leq \mathcal{T}^{-1}c$ such that $x(d)x(\sigma(d)) \leq 0$. More precisely, if $c \leq \mathcal{T}a_i$, then there exists $d \leq a_i$ such that $x(d)x(\sigma(d)) \leq 0$.*

Proof. In this proof, by $u \not< v$ we mean either $u \geq v$ or u does not exist. The proof is by induction (with respect to the location of the point $c \in \widehat{\mathbf{T}}$). Assume the conclusion is not true. Then $u(t)$ defined by

(1.4) satisfies the Riccati equation (R) and (1.5), and $\mu(t)u(t) < p(t)$ for $t \in [0, \mathcal{T}^{-1}c]$. For $s \in \widehat{\mathbf{T}}$, let $v(s) = -\hat{p}(s)y^{\Delta}(s)/y(s)$ and $\hat{\mu}(s) = \mu(t)$ for $t = \mathcal{T}^{-1}s$. Then it follows that v satisfies the Riccati equation

$$v^{\Delta}(s) = \frac{v^2(s)}{-\hat{\mu}(s)v(s) + \hat{p}(s)} + \hat{q}(s) \tag{2.2}$$

and

$$v(\hat{\sigma}(s)) = \frac{\hat{p}(s)v(s)}{-\hat{\mu}(s)u(s) + \hat{p}(s)} + \hat{\mu}(s)\hat{q}(s). \tag{2.3}$$

By Lemma 2.1, $\hat{\mu}(s)v(s) < \hat{p}(s)$ for $s \in [0, c) \cap \hat{\mathbf{T}}$ and $\hat{\mu}(c)v(c) \not< \hat{p}(c)$.

(i) Assume $c \leq \mathcal{T}a_1 = a_1$. Then in this case it follows that $t = \mathcal{T}^{-1}s = s$ for $s \in [0, c]$, so we have $\hat{p}(t) = p(t)$, $\hat{q}(t) = q(t)$, and the Riccati equations (R) and (2.2) are the same on $[0, c] \cap \mathbf{T}$. We claim that

$$u(t) \geq v(t) \quad \text{for} \quad t \in [0, c] \cap \mathbf{T}. \tag{2.4}$$

For any $n \in \mathbf{N}$ sufficiently large, it follows from [**2**, Theorem 4.5] (existence-uniqueness theorem for the linear IVP), that the initial value problem

$$u_n^{\Delta}(t) = \frac{u_n^2(t)}{-\mu(t)u_n(t) + p(t)} + q(t) + \frac{1}{n}, \quad u_n(0) = u(0), \tag{2.5}$$

has a unique solution, say $u_n(t)$, on $t \in [0, c] \cap \mathbf{T}$. We first observe that $u_n(t) \to u(t)$ as $n \to \infty$ for $t \in [0, c] \cap \mathbf{T}$. We wish to show that, for all large $n \in \mathbf{N}$,

$$u_n(t) \geq v(t) \quad \text{for} \quad t \in [0, c] \cap \mathbf{T}. \tag{2.6}$$

If this is not the case, then since $u_n(0) \geq v(0)$, there will exist $t^* \in (0, c]$ such that $u_n(t^*) < v(t^*)$. Hence, there exists $t_* \in [0, t^*)$ such that

$$u_n(t_*) \geq v(t_*) \quad \text{and} \quad u_n(t) < v(t) \quad \text{for} \quad t \in (t_*, t^*] \cap \mathbf{T}. \tag{2.7}$$

If t_* is right-scattered in $\mathbf{T}$ (hence in $\widehat{\mathbf{T}}$), then from (2.5)

$$u_n(\sigma(t_*)) = \frac{p(t_*)u_n(t_*)}{-\mu(t_*)u_n(t_*) + p(t_*)} + \mu(t_*)q(t_*) + \frac{\mu(t_*)}{n}. \tag{2.8}$$

Now, since for any p, $\mu > 0$, the function $f(u) := pu/(-\mu u + p)$ is increasing if we compare (2.3) and (2.8), we obtain a contradiction to (2.7).

If t_* is right-dense in $\mathbf{T}$ (hence in $\widehat{\mathbf{T}}$), then $u_n(t_*) = v(t_*)$ and $u_n(t) < v(t)$ for $t \in (t_*, t^*] \cap \mathbf{T}$. From (2.2) and (2.5), $u_n^{\Delta}(t_*) > v^{\Delta}(t_*)$, so there exists $\bar{t} \in (t_*, t^*] \cap \mathbf{T}$ such that $u_n(\bar{t}) > v(\bar{t})$, contradicting the definition of t_*. Hence (2.6) holds.

Therefore, from $u_n(t) \to u(t)$ as $n \to \infty$ for $t \in [0, c] \cap \mathbf{T}$, we obtain (2.4), and so letting $t = c$ in (2.4), we have

$$\mu(c)u(c) \geq \hat{\mu}(c)v(c) \not< \hat{p}(c) = p(c). \tag{2.9}$$

This implies that $\mu(c)u(c) \not< p(c)$, contradicting the assumption.

(ii) Assume $\mathcal{T}a_1 < c \leq \mathcal{T}a_2$, then arguing as in the first part above, we see that $u(\sigma(a_1)) \geq v(\hat{\sigma}(a_1)) = v(\hat{\sigma}(\mathcal{T}a_1))$. Integrating (R) from $\sigma(a_1)$ to $\sigma(b_1)$ and using (2.1), we obtain

$$u(\sigma(b_1)) - u(\sigma(a_1)) = \int_{\sigma(a_1)}^{\sigma(b_1)} \frac{u^2(t)\Delta(t)}{-\mu(t)u(t) + p(t)} + \int_{\sigma(a_1)}^{\sigma(b_1)} q(t)\Delta t \geq 0.$$

Hence $u(\sigma(b_1)) \geq u(\sigma(a_1)) \geq v(\hat{\sigma}(\mathcal{T}a_1))$. Now since $\mathcal{T}\sigma(b_1) = \hat{\sigma}(\mathcal{T}a_1)$, it follows that $u(t)$ and $v(s)$ satisfy the same Riccati equation for $\sigma(b_1) \leq t \leq \mathcal{T}^{-1}c$ and $\mathcal{T}\sigma(b_1) \leq s \leq c$, respectively. As before, we see that

$$\mu(\mathcal{T}^{-1}c)u(\mathcal{T}^{-1}c) \geq \hat{\mu}(c)v(c) \neq \hat{p}(c) = p(\mathcal{T}^{-1}c).$$

Again, this implies that $\mu(\mathcal{T}^{-1}c)u(\mathcal{T}^{-1}c) \not< p(\mathcal{T}^{-1}c)$, contradicting the assumption. The proof of the induction step from i to $i + 1$ is similar and hence is omitted. □

Theorem 2.2 (telescoping principle). *Under the conditions and with the notation of Theorem* 2.1, *if the telescoped equation* $(\widehat{\mathrm{E}})$ *is oscillatory, then* (E) *is oscillatory.*

Proof. Let $x(t)$ be a solution of (E) with $x(0) \neq 0$, and let $y_1(s)$ be a solution of $(\widehat{\mathrm{E}})$ satisfying $y_1(0) \neq 0$ and $p(0)x^{\Delta}(0)/x(0) \leq \hat{p}(0)y_1^{\Delta}(0)/y_1(0)$. Since $y_1(s)$ is oscillatory, there exists a smallest $c_1 > 0$ in $\widehat{\mathbf{T}}$ such that $y_1(s)y_1(\hat{\sigma}(s)) > 0$ for $s \in [0, c_1) \cap \widehat{\mathbf{T}}$ and $y_1(c_1)y_1(\hat{\sigma}(c_1)) \leq 0$. By Theorem 2.1, there exists $d_1 \leq \mathcal{T}^{-1}c_1$ in $\mathbf{T}$ such that $x(d_1)x(\sigma(d_1)) \leq 0$. Now, we will work on the solution $x(t)$ for $t > d_1$. Let $e_1 \in \mathbf{T}$ with $e_1 \geq d_1$ satisfy $x(e_1) \neq 0$. Let $y_2(s)$ be a solution of $(\widehat{\mathrm{E}})$ satisfying $y_2(e_1) \neq 0$ and $p(e_1)x^{\Delta}(e_1)/x(e_1) \leq \hat{p}(\mathcal{T}e_1)y_2^{\Delta}(\mathcal{T}e_1)/y_2(\mathcal{T}e_1)$. Proceeding as before, we show that there exists $d_2 \in \mathbf{T}$ with $d_2 > d_1$ such that $x(d_2)x(\sigma(d_2)) \leq 0$. Continuing this process leads to the conclusion that x is oscillatory and therefore the equation (E) is oscillatory. □

This principle can be applied to get many new examples of oscillatory equations. We use a process which is the reverse of the construction in Theorem 2.2. Start with any known oscillatory equation $(\widehat{\mathrm{E}})$. Choose a sequence of numbers $s_i \in \hat{\mathbf{T}}$ such that $s_i \to \infty$. Cut $\widehat{\mathbf{T}}$ at each s_i and pull the two halves of $\widehat{\mathbf{T}}$ apart to form a gap of arbitrary finite length. Now fill the gap with an arbitrary new (bounded) time scale and define a positive p and any q whose integral over the new time scale is nonnegative. Do this at each s_i and relabel the so-constructed new coefficient functions by $p(t)$ and $q(t)$. Then the obtained equation (E) is oscillatory.

The telescoping principle is also useful in extending various known oscillation criteria. It implies that any sufficient conditions for oscillation need only to be verified on "intervals", i.e., on $\mathbf{T} \setminus J$ where J is defined by (1.6), while on J, $p(t)$ and $q(t)$ can be arbitrary rd-continuous functions as long as $p(t) > 0$ and $q(t)$ has a nonnegative integral over each interval $(\sigma(a_i), \sigma(b_i))$ of J.

However, due to the jumping behavior of functions on time scales, the above telescoping principle may not always be easy to apply. In particular, if $\int_{\sigma(a_i)}^{\sigma(b_i)} q(t)\Delta t \neq 0$ for some $i \in \mathbf{N}$, then $\int_0^t q(l)\Delta l \not\equiv \int_0^{\mathcal{T}t} \hat{q}(l)\Delta l$. To overcome this difficulty, in the next section, we introduce a modified telescoping principle in which $\int_0^t q(l)\Delta l$ remains unchanged after the telescoping process.

3. Modified telescoping principle. We say that the Riccati equation (R) is nonoscillatory provided the corresponding equation (E) is nonoscillatory. Hence, by Lemma 2.1, (R) is nonoscillatory means that for any solution $u(t)$ of (R), there exists $t_1 \in \mathbf{T}$ such that $\mu(t)u(t) < p(t)$ for $t \in [t_1, \infty) \cap \mathbf{T}$.

To establish a modified telescoping principle we need to obtain a perturbed equation of (R). Lemma 3.1 and Remarks 3.1 and 3.2 will explain the ideas.

Lemma 3.1. *Assume the Riccati equation* (R) *is nonoscillatory on* $\mathbf{T}$, *and let* $u(t)$ *be a solution with* $\mu(t)u(t) < p(t)$, $t \in [t_1, \infty) \cap \mathbf{T}$, *for some* $t_1 \in \mathbf{T}$. *Let* $c \geq t_1$ *be right-scattered in* $\mathbf{T}$, *and* k *the real number. We define the perturbation* $\widetilde{\mathbf{T}}$ *of* $\mathbf{T}$ *at* c *by*

$$\widetilde{\mathbf{T}} = \big\{\tau \in (-\infty, c] : \tau \in \mathbf{T}\big\} \bigcup \big\{\tau \in (c, \infty) : \tau - \mu(c) \in \mathbf{T}\big\}. \tag{3.1}$$

Consider the perturbed equation of (R) *given by:*

$$w^{\Delta}(\tau) = \frac{w^2(\tau)}{-\tilde{\mu}(\tau)w(\tau) + \tilde{p}(\tau)} + \tilde{q}(\tau), \quad \tau \in \widetilde{\mathbf{T}}, \tag{3.2}$$

where $\tilde{p}(t) = p(t)$, $\tilde{q}(t) = q(t)$, $\tilde{\mu}(t) = \mu(t)$ *for* $t < c$; $\tilde{p}(t) = p(t - \mu(c))$, $\tilde{q}(t) = q(t - \mu(c))$, $\tilde{\mu}(t) = \mu(t - \mu(c))$ *for* $t > \sigma(c)$; *and* $\tilde{q}(c) = q(c) + k$, $\tilde{q}(\sigma(c)) = -k$, $\tilde{\mu}(c) = \mu(c)$, $\tilde{\mu}(\sigma(c)) = \mu(c)$. *If* $\tilde{p}(\sigma(c))$ *is sufficiently large, then there exists* $\tilde{p}(c) > p(c)$ *such that* (3.2) *is nonoscillatory. Furthermore,* $\tilde{p}(c) \to p(c)$ *as* $\tilde{p}(\sigma(c)) \to \infty$.

Proof. Let $w(t) = u(t)$ for $t \leq c$ and $w(t) = u(t - \mu(c))$ for $t > \sigma(c)$ in $\widetilde{\mathbf{T}}$. We need only choose $\tilde{p}(c)$ corresponding to $\tilde{p}(\sigma(c))$ and define $w(\sigma(c))$ accordingly such that $w(t)$, $t \in \widetilde{\mathbf{T}}$, is a solution of (3.2) satisfying

$$\tilde{\mu}(c)w(c) < \tilde{p}(c) \quad \text{and} \quad \tilde{\mu}(\sigma(c))w(\sigma(c)) < \tilde{p}(\sigma(c)).$$

Note that since c is right-scattered in $\mathbf{T}$, both c and $\sigma(c)$ are right-scattered in $\tilde{\mathbf{T}}$, and we have $\tilde{\mu}(\sigma(c)) = \tilde{\mu}(c) = \mu(c)$, $w(c) = u(c)$, and $u(\sigma(c)) = w(\sigma(c) + \mu(c))$. From (1.5) and the right-scattered form of (3.2), we get, after some manipulation,

$$u(\sigma(c)) = \frac{p(c)u(c)}{-\mu(c)u(c)+p(c)} + \mu(c)q(c) \tag{3.3}$$

$$w(\sigma(c)) = \frac{\tilde{p}(c)u(c)}{-\mu(c)u(c)+\tilde{p}(c)} + \mu(c)(q(c)+k) \tag{3.4}$$

and

$$u(\sigma(c)) = \frac{\tilde{p}(\sigma(c))w(\sigma(c))}{-\mu(c)w(\sigma(c))+\tilde{p}(\sigma(c))} - \mu(c)k. \tag{3.5}$$

Now if $\tilde{p}(\sigma(c))$ is sufficiently large, then the solution $w(\sigma(c))$ of (3.5) as a function of $\tilde{p}(\sigma(c))$ exists and we obtain

$$w(\sigma(c)) = \frac{(u(\sigma(c))+\mu(c)k)\,\tilde{p}(\sigma(c))}{\mu(c)\,(u(\sigma(c))+\mu(c)k)+\tilde{p}(\sigma(c))} \le u(\sigma(c))+\mu(c)k.$$

Also

$$w(\sigma(c)) \longrightarrow u(\sigma(c))+\mu(c)k \quad \text{as} \quad \tilde{p}(\sigma(c)) \longrightarrow \infty.$$

It is easy to see that, for any $u \in \mathbf{R}$ and $\mu > 0$, the function $g(p) := pu/(-\mu u+p)$ is continuous and decreasing for $p > \max\{0, \mu u\}$. Thus, from (3.3) and (3.4), for a sufficiently large $\tilde{p}(\sigma(c))$, we can find $\tilde{p}(c) > p(c) > \mu(c)u(c) = \tilde{\mu}(c)w(c)$ satisfying (3.4) and $\tilde{p}(c) \to p(c)$ as $\tilde{p}(\sigma(c)) \to \infty$. $\square$

Remark 3.1. Assume (E) is nonoscillatory on $\mathbf{T}$, and let $u(t)$ be a solution of the Riccati equation (R) with $\mu(t)u(t) < p(t)$, $t \in [t_1, \infty)$ for some $t_1 \in \mathbf{T}$.

(i) Let $c \ge t_1$ be right-dense in $\mathbf{T}$. Then $\widetilde{\mathbf{T}}$ given by (3.1) is the same as $\mathbf{T}$. In this case, we define the perturbed equation (3.2) of (R) at c to be the same as (R) in this case.

(ii) Let $\{c_i\}_{i=1}^{\infty}$ be a sequence in $\mathbf{T}$ such that $t_1 \le c_1 < c_2 < \cdots$. We define the perturbation $\widetilde{\mathbf{T}}$ of $\mathbf{T}$ with respect to $\{c_i\}_{i=1}^{\infty}$ by

$$\widetilde{\mathbf{T}} = \big\{\tau \in \mathbf{T} : \tau \le \tilde{c}_1\big\} \bigcup \Bigg\{ \bigcup_{i=1}^{\infty} \Big\{\tau \in (\tilde{c}_i, \tilde{c}_{i+1}] : \tau - \sum_{j=1}^{i} \mu(c_j) \in \mathbf{T}\Big\}\Bigg\}, \tag{3.6}$$

where $\tilde{c}_i = c_i + \sum_{j=1}^{i-1} \mu(c_j)$, $i \in \mathbf{N}$, with $\sum_{j=1}^{0} \mu(c_j) = 0$.

Using the same idea as in Lemma 3.1, we obtain a perturbed nonoscillatory equation of (R) on $\widetilde{\mathbf{T}}$, defined by

$$(\widetilde{\mathrm{R}}) \qquad w^{\Delta}(\tau) = \frac{w^2(\tau)}{-\tilde{\mu}(\tau)w(\tau) + \tilde{p}(\tau)} + \tilde{q}(\tau)$$

in the following way:

(a) If, for some i, c_i is right-dense in $\mathbf{T}$, then $(\widetilde{\mathrm{R}})$ for $\tau \in [\tilde{c}_i, \tilde{c}_{i+1}) \cap \widetilde{\mathbf{T}}$ is the same as (R) for $t \in [c_i, c_{i+1}) \cap \mathbf{T}$;

(b) If, for some i, c_i is right-scattered in $\mathbf{T}$, then $(\widetilde{\mathrm{R}})$ for $\tau \in [\tilde{\sigma}^2(\tilde{c}_i), \tilde{c}_{i+1}) \cap \widetilde{\mathbf{T}}$ is the same as (R) for $t \in [\sigma^2(c_i), c_{i+1}) \cap \mathbf{T}$, and for some $k_i \in \mathbf{R}$, $\tilde{q}(\tilde{c}_i) = q(c_i) + k_i$, $\tilde{q}(\tilde{\sigma}(\tilde{c}_i)) = -k_i$, $\tilde{p}(\tilde{c}_i)$ is sufficiently close to $p(c_i)$, and $\tilde{p}(\tilde{\sigma}(\tilde{c}_i))$ is sufficiently large, where $\tilde{\sigma}$ is the forward jump operator in $\widetilde{\mathbf{T}}$. Note that the arbitrariness of k_i allows some flexibility in applications.

In the following, we use the notation

$$r(t) = \int_0^t q(l)\Delta l, \quad t \in \mathbf{T}, \tag{3.7}$$

$$\hat{r}(s) = \int_0^s \hat{q}(l)\Delta l, \quad s \in \widehat{\mathbf{T}}, \tag{3.8}$$

and

$$\tilde{r}(\tau) = \int_0^\tau \tilde{q}(l)\Delta l, \quad \tau \in \widetilde{\mathbf{T}}. \tag{3.9}$$

Remark 3.2. In Remark 3.1 (ii), we let $c_i = a_i$, $i \in \mathbf{N}$, where a_i are defined as in (1.6), and consider the perturbed equation $(\widetilde{\mathrm{R}})$ of (R) with respect to $\{a_i\}_{i=0}^{\infty}$. Then it is easy to see that, for right scattered a_i, we can always choose suitable k_is in Remark 3.1 (ii) such that the telescoped equation of $(\widetilde{\mathrm{R}})$ based on J satisfies the condition $\hat{r}(s) = r(\mathcal{T}^{-1}(s))$ for all $s \in \hat{\mathbf{T}}$.

Theorem 3.1 (modified telescoping principle). *Let J and $\mathcal{T}$ be defined by* (1.6) *and* (1.8)*, respectively. Assume the Riccati equation* (R) *is nonoscillatory. Then the modified telescoped equation*

$$(\widehat{\mathrm{R}}) \qquad v^{\Delta}(s) = \frac{v^2(s)}{-\hat{\mu}(s)v(s) + \hat{p}(s)} + \hat{q}(s), \quad s \in \widehat{\mathbf{T}},$$

is also nonoscillatory where $\hat{p}(s) = p(t)$, $\hat{q}(s) = q(t)$ for $t = \mathcal{T}^{-1}s$ and $t \neq a_i$, $\hat{p}(\mathcal{T}a_i)$ is sufficiently close to $p(a_i)$, $i \in \mathbf{N}$, and $\hat{r}(s) = r(\mathcal{T}^{-1}s)$, $\hat{\mu}(s) = \mu(\mathcal{T}^{-1}s)$ for $s \in \hat{\mathbf{T}}$. In particular, $\hat{p}(\mathcal{T}a_i) = p(a_i)$, $\hat{q}(\mathcal{T}a_i) = q(a_i)$ if a_i is right-dense for some $i \in \mathbf{N}$.

Proof. Since (R) is nonoscillatory, without loss of generality, we may assume $\mu(t)u(t) < p(t)$, $t \in [0, \infty) \cap \mathbf{T}$. Let $\widetilde{\mathbf{T}}$ defined by (3.6) be the perturbation of $\mathbf{T}$ with respect to $\{a_i\}_{i=1}^{\infty}$, and consider the perturbed nonoscillatory equation of (R) on $\widetilde{\mathbf{T}}$ defined by $(\widetilde{\mathrm{R}})$ such that, if a_i is right-scattered in $\mathbf{T}$, then $\tilde{p}(\tilde{\sigma}(\tilde{a_i}))$ is sufficiently large, $\tilde{p}(\tilde{a_i}) = \hat{p}(\mathcal{T}a_i)$, and k_i is chosen so that $\tilde{r}(\tilde{\sigma}(\tilde{a}_i)) = r(\sigma(b_i))$. Note that in $\widetilde{\mathbf{T}}$, J becomes $\tilde{J} = (\cup_{i=1}^{\infty}(\tilde{a}_i, \tilde{\sigma}(\tilde{b}_i))) \cap \widetilde{\mathbf{T}}$, where $\tilde{a}_i = a_i + \sum_{j=1}^{i-1}\mu(a_j)$ and $\tilde{b}_i = b_i + \sum_{j=1}^{i}\mu(a_j)$. Let $\tilde{J}$ be the interval shrinking set in $\widetilde{\mathbf{T}}$, and based on $\tilde{J}$ define $\overline{\mathbf{T}}$ and the interval shrinking transformation $\widetilde{\mathcal{T}} : \widetilde{\mathbf{T}} \to \overline{\mathbf{T}}$ in the same way as in (1.7) and (1.8), where a_i and $\sigma(b_i)$, $i \in \mathbf{N}$, are replaced by $\tilde{a}_i$ and $\tilde{\sigma}(\tilde{b}_i)$, respectively. Consider the telescoped equation of $(\widetilde{\mathrm{R}})$

$$\bar{w}^{\Delta}(r) = \frac{\bar{w}^2(r)}{-\bar{\mu}(r)\bar{w}(r) + \bar{p}(r)} + \bar{q}(r), \quad r \in \overline{\mathbf{T}}, \tag{3.10}$$

where $\bar{p}(r) = \tilde{p}(\tau)$, $\bar{q}(r) = \tilde{q}(\tau)$ and $\bar{\mu}(r) = \tilde{\mu}(\tau)$ for $\tau = \widetilde{\mathcal{T}}^{-1}r$.

Note that $\int_{\tilde{\sigma}(\tilde{a}_i)}^{\tilde{\sigma}(\tilde{b}_i)} \tilde{q}(l)\Delta l = 0$ for all $i \in \mathbf{N}$ regardless of whether or not a_i is right scattered or right-dense in $\mathbf{T}$ and equation $(\widetilde{\mathrm{R}})$ is nonoscillatory. By Theorem 2.2, equation (3.10) is nonoscillatory. Comparing J and $\tilde{J}$, we see that $\hat{p}(s) = \bar{p}(r)$, $\bar{q}(s) = \hat{q}(r)$ for $s \neq \mathcal{T}a_i$ and $r \neq \widetilde{\mathcal{T}}(\tilde{a}_i)$; $\hat{\mu}(s) = \bar{\mu}(r)$ for $s \in \widehat{\mathbf{T}}$ and $r \in \overline{\mathbf{T}}$; $\hat{p}(\mathcal{T}a_i) = \tilde{p}(\tilde{a}_i) = \bar{p}(\widetilde{\mathcal{T}}(\tilde{a}_i))$; and $\hat{q}(\mathcal{T}a_i) = \bar{q}(\widetilde{\mathcal{T}}(\tilde{a}_i)$ since $\hat{r}(\mathcal{T}\sigma(b_i)) = \bar{r}(\tilde{\sigma}\widetilde{\mathcal{T}}(\tilde{a}_i))$, where $\bar{r}(r) = \int_0^r \bar{q}_n(l)\Delta l$. This implies that $(\widehat{\mathrm{R}})$ is nonoscillatory and completes the proof. $\square$

Remark 3.3. Theorem 3.1 covers the telescoping principle for differential equations in [**15**] and improves the one for difference equations in [**14**] since the "adjusted sequence" used there for applications, see [**14**, p. 1052], is no longer needed.

4. Extensions of known oscillation criteria. Now we are ready to establish extensions of known oscillation criteria using the modified telescoping principle. We shall content ourselves with several examples to show how it works. The following results are taken from [**2, 6**].

Result 4.1 [**2**, Theorem 4.64]. Assume

$$\int_0^\infty \frac{1}{p(t)}\Delta t = \int_0^\infty q(t)\Delta t = \infty.$$

Then (E) is oscillatory.

Result 4.2 [**2**, Corollary 4.50]. Assume that there exist a strictly increasing sequence $\{t_k\}_{k=1}^\infty \subset \mathbf{T}$ with $\lim_{k\to\infty} t_k = \infty$ and two positive real numbers K and M such that $\mu(t_k) \geq K$, $p(t_k) \leq M$ for $k \in \mathbf{N}$, and

$$\lim_{k\to\infty} \int_{t_1}^{t_k} q(t)\Delta t = \infty.$$

Then (E) is oscillatory.

Result 4.3 [**6**, Corollary 7]. Assume that there exists a strictly increasing sequence $\{t_k\}_{k=1}^\infty \subset \mathbf{T}$ with $\lim_{k\to\infty} t_k = \infty$ such that $\mu(t_k) > 0$ for $k \in \mathbf{N}$. Suppose further that

$$\limsup_{k\to\infty} \left(\int_0^{t_k} q(t)\Delta t - \frac{p(t_k)}{\mu(t_k)} \right) = \infty.$$

Then (E) is oscillatory.

We shall now indicate how to extend the above results using Theorem 3.1. In the following, given the set

$$
\begin{gathered}
I = \left(\bigcup_{i=1}^{\infty} [n_i, m_i] \right) \bigcap \mathbf{T}, \\
n_i,\ m_i \in \mathbf{T}, \quad 0 < n_i \le m_i < n_{i+1}, \quad i \in \mathbf{N}, \\
\text{with} \quad n_i \to \infty \text{ as } i \to \infty.
\end{gathered} \tag{4.1}
$$

Theorem 4.1. *Let I be defined by* (4.1) *and suppose that*

$$
\sum_{i=1}^{\infty} \int_{n_i}^{\sigma(m_i)} \frac{1}{p(t)}\, \Delta t = \sum_{i=1}^{\infty} \int_{n_i}^{\sigma(m_i)} q(t) \Delta t = \infty. \tag{4.2}
$$

Then (E) *is oscillatory.*

Proof. Assume the contrary. Then the associated Riccati equation $Ru = 0$ is nonoscillatory. Let $J = \mathbf{T} \setminus I$. Then J can be written in the form of (1.6). Let $\widehat{\mathbf{T}}$ and $\mathcal{T}$ be defined as in (1.7) and (1.8), respectively. By Theorem 3.1, the telescoped equation $(\widehat{\mathrm{R}})$ is also nonoscillatory, where $\hat{p}(s) = p(t)$, $\hat{q}(s) = q(t)$ for $t = \mathcal{T}^{-1}s$, $t \neq a_i$, $\hat{p}(\mathcal{T}a_i)$ is sufficiently close to $p(a_i)$, $i = 1, \dots, n$, and $\hat{r}(s) = r(\mathcal{T}^{-1}s)$, $\hat{\mu}(s) = \mu(\mathcal{T}^{-1}s)$ for $s \in \widehat{\mathbf{T}}$.

Choose $\hat{p}(\mathcal{T}a_i)$ sufficiently close to $p(a_i)$ so that $\hat{p}(\mathcal{T}a_i) > 0$ and $\int_0^{\infty} \Delta s / \hat{p}(s) = \infty$. This is possible because of (4.2) and the fact that we can choose $\hat{p}(s)$ to satisfy

$$
\left| \sum_{i=1}^{\infty} \int_{[n_i, \sigma(m_i)] \bigcap [0, v]} \frac{\Delta t}{p(t)} - \int_0^{\mathcal{T}v} \frac{\Delta s}{\hat{p}(s)} \right| < 1 \quad \text{for any} \quad v \in \mathbf{R}^+.
$$

Also $\int_0^{\infty} \hat{q}(s) \Delta s = \infty$ since $\hat{r}(s) = r(\mathcal{T}^{-1}s)$ and $\sum_{i=1}^{\infty} \int_{n_i}^{\sigma(m_i)} q(t) \Delta t = \infty$. But then by Result 4.1, if follows that the equation $(\widehat{\mathrm{E}})$ is oscillatory, a contradiction to the fact that $(\widehat{\mathrm{R}})$ is nonoscillatory. This completes the proof. $\square$

Theorem 4.2. *Let I be defined by* (4.1). *Assume there exists a strictly increasing sequence $\{t_k\}_{k=1}^{\infty} \subset I$ with $\lim_{k\to\infty} t_k = \infty$ and two positive real numbers K and M such that $\mu(t_k) \geq K$, $p(t_k) \leq M$ for $k \in \mathbf{N}$, and*

$$\lim_{k\to\infty} \sum_{i=1}^{\infty} \int_{[n_i,\sigma(m_i)] \bigcap [t_1,t_k]} q(t)\Delta t = \infty. \tag{4.3}$$

Then (E) *is oscillatory.*

Proof. Assume the contrary. Then the Riccati equation $Ru = 0$ is nonoscillatory. Let $J = \mathbf{T} \setminus I$. Then J can be written in the form of (1.6). Let $\widehat{\mathbf{T}}$ and $\mathcal{T}$ be defined as in (1.7) and (1.8), respectively. By Theorem 3.1, the telescoped equation $(\widehat{\mathrm{R}})$ is also nonoscillatory, where $\hat{p}(s) = p(t)$, $\hat{q}(s) = q(t)$ for $t = \mathcal{T}^{-1}s$, $t \neq a_i$, $\hat{p}(\mathcal{T}a_i)$ is sufficiently close to $p(a_i)$, $i = 1, \dots, n$, and $\hat{r}(s) = r(\mathcal{T}^{-1}s)$, $\hat{\mu}(s) = \mu(\mathcal{T}^{-1}s)$ for $s \in \widehat{\mathbf{T}}$.

Choose $\hat{p}(\mathcal{T}a_i)$ so close to $p(a_i)$ that $p(\mathcal{T}a_i) \leq M + 1$, $i \in \mathbf{N}$. Set $s_k = \mathcal{T}t_k$; then $s_k \to \infty$ as $k \to \infty$, $\hat{\mu}(s_k) \geq K$, $p(t_k) \leq M + 1$ for all $k \in \mathbf{N}$. From (4.3), we also have

$$\lim_{k\to\infty} \int_{s_1}^{s_k} \hat{q}(s)\Delta s = \infty.$$

By Result 4.2, $(\widehat{\mathrm{E}})$ is oscillatory, which is a contradiction, since $(\widehat{\mathrm{R}})$ is nonoscillatory. This contradiction completes the proof. $\square$

Corollary 4.1. *Assume that there exists a strictly increasing sequence $\{t_k\}_{k=1}^{\infty} \subset I$ with $\lim_{k\to\infty} t_k = \infty$ and two positive real numbers K and M such that $\mu(t_k) \geq K$, $p(t_k) \leq M$ for $k \in \mathbf{N}$ and*

$$\sum_{k=1}^{\infty} q(t_k)\mu(t_k) = \infty.$$

Then (E) *is oscillatory.*

Proof. This follows from Theorem 4.2 where $I = \cup_{k=1}^{\infty}\{t_k\}$. $\square$

Theorem 4.3. *Let I be defined by* (4.1)*. Assume that there exists a strictly increasing sequence $\{t_k\}_{k=1}^{\infty} \subset \mathbf{T}$ with $\lim_{k\to\infty} t_k = \infty$ and $\mu(t_k) > 0$ for $k \in \mathbf{N}$. Suppose further that*

$$\limsup_{k\to\infty}\left(\sum_{i=1}^{\infty}\int_{[n_i,\sigma(m_i)]\bigcap[0,t_k)} q(t)\Delta t - \frac{p(t_k)}{\mu(t_k)}\right) = \infty.$$

Then (E) *is oscillatory.*

Proof. Assume the contrary. Then as in Theorem 4.1, the associated Riccati equation $Ru = 0$ is nonoscillatory. Let $J = \mathbf{T} \setminus I$. Then J can be written in the form of (1.6). Let $\widehat{\mathbf{T}}$ and $\mathcal{T}$ be defined as in (1.7) and (1.8), respectively. By Theorem 3.1, the telescoped equation $(\widehat{\mathrm{R}})$ is also nonoscillatory, where $\hat{p}(s) = p(t)$, $\hat{q}(s) = q(t)$ for $t = \mathcal{T}^{-1}s$, $t \neq a_i$, $\hat{p}(\mathcal{T}a_i)$ is sufficiently close to $p(a_i)$, $i = 1, \dots, n$, and $\hat{r}(s) = r(\mathcal{T}^{-1}s)$, $\hat{\mu}(s) = \mu(\mathcal{T}^{-1}s)$ for $s \in \widehat{\mathbf{T}}$.

Set $s_k = \mathcal{T}t_k$, then $s_k \to \infty$ as $k \to \infty$ and $\hat{\mu}(s_k) > 0$. Also,

$$\limsup_{k\to\infty}\left(\int_0^{s_k} \hat{q}(s)\Delta s - \frac{\hat{p}(s_k)}{\hat{\mu}\,(s_k)}\right) = \infty.$$

But then by Result 4.3, $(\widehat{\mathrm{E}})$ is oscillatory. This contradiction completes the proof. ◻

Corollary 4.2. *Assume that there exists a strictly increasing sequence $\{t_k\}_{k=1}^{\infty} \subset \mathbf{T}$ with $\lim_{k\to\infty} t_k = \infty$ and $\mu(t_k) > 0$ for $k \in \mathbf{N}$. Suppose further that*

$$\limsup_{k\to\infty}\left(\sum_{i=1}^{k-1} q(t_k)\mu(t_k) - \frac{p(t_k)}{\mu(t_k)}\right) = \infty.$$

Then (E) *is oscillatory.*

Proof. This follows from Theorem 4.3 where $I = \cup_{k=1}^{\infty}\{t_k\}$. ◻

5. More oscillation criteria. In this section we obtain some new oscillation criteria by more sophisticated applications of the telescoping principle. Our results cover and improve Theorem 3 to Corollary 5 in [**15**] for differential equations since the condition $p(t) \equiv 1$ needed there is not required here, cover and simplify Theorem 5.2 to Corollary 5.8 in [**14**] for difference equations, and generalize and improve the Olech-Opial-Wazewski type criteria for oscillation for differential equations in [**17**].

Lemma 5.1. *Let* $\mathbf{T}_1 = \cup\{[t, \sigma(t)] : t \in \mathbf{T}$ *is right scattered in* $\mathbf{T}\}$ *and* $\mathbf{T}_2 = \{t \in \mathbf{T} : t$ *is right-dense in* $\mathbf{T}\}$*. Assume that a function* $u(t) \in C_{rd}(\mathbf{T}, \mathbf{R})$ *has the property that there exist* $\alpha > 0$ *and* $p(t) > 0$ *for* $t \in \mathbf{T}$ *such that*

(5.1)
$$u(t) \geq \alpha + \int_0^t \frac{u^2(l)\Delta l}{-\mu(l)u(l) + p(l)} \quad \text{and} \quad \mu(t)u(t) < p(t) \text{ for } t < T,$$

where $T \in [0, \infty]$*, and*

$$\alpha^2 \sum_{l \in \mathbf{T}_1} \frac{\mu(\sigma(l))\mu(l)}{p(\sigma(l))(-\alpha\mu(l) + p(l))} + \alpha \int_{\mathbf{T}_2} \frac{\Delta l}{p(l)} > 1. \tag{5.2}$$

Then $T < \infty$.

Proof. Assume the contrary. Then $\mu(t)u(t) < p(t)$ for all $t \in \mathbf{T}$. Consider the problem

$$v(t) = \alpha + \int_0^t \frac{v^2(l)\Delta l}{-\mu(l)v(l) + p(l)}, \quad v(0) = \alpha. \tag{5.3}$$

Comparing (5.1) and (5.3), we get that $\alpha \leq v(t) \leq u(t)$ and $\mu(t)v(t) < p(t)$ for $t \in \mathbf{T}$, and

$$v^\Delta(t) = \frac{v^2(t)}{-\mu(t)v(t) + p(t)} \quad \text{for} \quad t \in \mathbf{T}. \tag{5.4}$$

Hence for $t \in \mathbf{T}_1$,

$$\begin{aligned}\left(\frac{1}{v(t)}\right)^\Delta &= -\frac{v^\Delta(t)}{v(t)v(\sigma(t))} = -\frac{v(t)}{v(\sigma(t))(-\mu(t)v(t) + p(t))}\\ &\leq -\frac{\alpha\mu(\sigma(t))}{p(\sigma(t))(-\alpha\mu(t) + p(t))},\end{aligned}$$

and for $t \in \mathbf{T}_2$

$$\left(\frac{1}{v(t)}\right)^{\Delta} = -\frac{v^{\Delta}(t)}{v^2(t)} = -\frac{1}{p(t)}.$$

Combining the above we see that, for $t \in \mathbf{T}$,

$$\begin{aligned}
\frac{1}{v(t)} - \frac{1}{v(0)} &= \int_0^t \left(\frac{1}{v(l)}\right)^{\Delta} \Delta l \\
&= \int_{[0,t] \bigcap \mathbf{T}_1} \left(\frac{1}{v(l)}\right)^{\Delta} \Delta l + \int_{[0,t] \bigcap \mathbf{T}_2} \left(\frac{1}{v(l)}\right)^{\Delta} \Delta l \\
&\le -\alpha \sum_{l \in [0,t] \bigcap \mathbf{T}_1} \frac{\mu(\sigma(l))\mu(l)}{p(\sigma(l))(-\alpha\mu(l) + p(l))} \\
&\quad - \int_{[0,t] \bigcap \mathbf{T}_2} \frac{\Delta l}{p(l)}.
\end{aligned}$$

Therefore, for $t \in \mathbf{T}$,

$$\begin{aligned}
\frac{1}{v(t)} &< \frac{1}{\alpha} - \alpha \sum_{l \in [0,t] \bigcap \mathbf{T}_1} \frac{\mu(\sigma(l))\mu(l)}{p(\sigma(l))(-\alpha\mu(l) + p(l))} - \int_{[0,t] \bigcap \mathbf{T}_2} \frac{\Delta l}{p(l)} \\
&= \frac{1}{\alpha}\left(1 - \alpha^2 \sum_{l \in [0,t] \bigcap \mathbf{T}_1} \frac{\mu(\sigma(l))\mu(l)}{p(\sigma(l))(-\alpha\mu(l) + p(l))} - \alpha \int_{[0,t] \bigcap \mathbf{T}_2} \frac{\Delta l}{p(l)}\right).
\end{aligned}$$

By (5.2), there exists $T_1 \in \mathbf{T}$ such that $v(t) \le 0$ for $t \ge T_1$. This contradicts the fact that $v(t) \ge \alpha > 0$ and completes the proof. □

Theorem 5.1. *Let $r(t)$ be defined by (3.7). For any $\lambda \in \mathbf{R}$, define $J(\lambda) = \{t \in \mathbf{T} : r(t) < \lambda\}$ and $\bar{J}(\lambda) = \{t \in \mathbf{T} : r(t) \ge \lambda\}$. Let $\bar{J}_1(\lambda) = \cup\{[t, \sigma(t)] : t \in \bar{J}(\lambda)$ is right-scattered in $\mathbf{T}\}$ and $\bar{J}_2(\lambda) = \{t \in \bar{J}(\lambda) : t$ is right dense in $\mathbf{T}\}$. Assume that there exists an increasing sequence of positive numbers*

$$\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots, \quad \lambda_k \to \infty \text{ as } k \to \infty$$

such that, for any $k \in \mathbf{N}$, either

(i) *there exists* $t_k \in \bar{J}_1(\lambda_k)$ *satisfying* $p(t_k) \le \mu(t_k)\lambda_k$, *or*

(ii) $p(t) > \mu(t)\lambda_k$ *for all* $t \in \bar{J}_1(\lambda_k)$ *and there exists* $c \in (0,1)$ *such that*

$$\lambda_k^2 \sum_{t\in \bar{J}_1(\lambda_k)} \frac{\mu(\sigma^*(t))\mu(t)}{p(\sigma^*(t))(-c\lambda_k\mu(t)+p(t))} + \lambda_k \int_{\bar{J}_2(\lambda_k)} \frac{\Delta t}{p(t)} > 1, \tag{5.5}$$

where $\sigma^*(s) = \min\{t \in \bar{J}(\lambda_k) : t \ge \sigma(s)\}$.

Then (E) *is oscillatory.*

Proof. Assume the contrary, and without loss of generality, assume the associated Riccati equation (R) has a solution $u \in C_{rd}^1(\mathbf{T},\mathbf{R})$ satisfying $\mu(t)u(t) < p(t)$, $t \in \mathbf{T}$. An integration of the Riccati equation gives

$$u(t) = u(0) + r(t) + \int_0^t \frac{u^2(l)\Delta l}{-\mu(l)u(l)+p(l)}. \tag{5.6}$$

By passing to a subsequence if necessary, we may assume $\lambda_1 > \max\{1, -u(0)/(1-c)\}$. In the following, we first show that for, any $k \in \mathbf{N}$,

$$\lambda_k^2 \int_{\bar{J}_1(\lambda_k)\bigcup \bar{J}_2(\lambda_k)} \frac{\Delta t}{-c\mu(t)\lambda_k + p(t)} \ge \frac{1}{k}. \tag{5.7}$$

Assume (i) holds for some $k \in \mathbf{N}$. Then we observe that $p(t_k) \le \mu(t_k)\lambda_k$ implies that $\mu(t_k) > 0$ since $p(t_k) > 0$. It is easy to see that, for $t \in \bar{J}(\lambda_k)$,

$$\lambda_k + u(0) \ge \lambda_k - \lambda_1(1-c) \ge \lambda_k - \lambda_k(1-c) = c\lambda_k. \tag{5.8}$$

From (5.6),

$$p(t) > \mu(t)u(t) \ge \mu(t)(\lambda_k + u(0)) \ge c\mu(t)\lambda_k \quad \text{for all } t \in \bar{J}(\lambda_k).$$

On the other hand, $p(t_k) \le \mu(t_k)\lambda_k$ implies that

$$p(t_k) \le k\mu(t_k)\lambda_k^2 + c\mu(t_k)\lambda_k,$$

and hence

$$\frac{\lambda_k^2 \mu(t_k)}{-c\mu(t_k)\lambda_k + p(t_k)} \geq 1/k.$$

As a result,

$$\lambda_k^2 \int_{\bar{J}_1(\lambda_k) \bigcup \bar{J}_2(\lambda_k)} \frac{\Delta t}{-c\mu(t)\lambda_k + p(t)} \geq \lambda_k^2 \int_{t_k}^{\sigma(t_k)} \frac{\Delta t}{-c\mu(t)\lambda_k + p(t)}$$
$$= \frac{\lambda_k^2 \mu(t_k)}{-c\mu(t)\lambda_k + p(t_k)} \geq \frac{1}{k},$$

i.e., (5.7) holds.

Assume (ii) holds for some $k \in \mathbf{N}$. Then $p(t) > \mu(t)\lambda_k$ for all $t \in \bar{J}_1(\lambda_k)$ implies that $\mu(t)/p(t) < 1/\lambda_k \leq 1$ for all $t \in \bar{J}_1(\lambda_k)$. So

$$\lambda_k^2 \int_{\bar{J}_1(\lambda_k) \bigcup \bar{J}_2(\lambda_k)} \frac{\Delta t}{-c\mu(t)\lambda_k + p(t)}$$
$$= \lambda_k^2 \sum_{t \in \bar{J}_1(\lambda_k)} \frac{\mu(t)}{-c\lambda_k \mu(t) + p(t)} + \lambda_k^2 \int\limits_{\bar{J}_2(\lambda_k)} \frac{\Delta t}{p(t)}$$
$$\geq \lambda_k^2 \sum_{t \in \bar{J}_1(\lambda_k)} \frac{\mu(\sigma^*(t))\mu(t)}{p(\sigma^*(t))(-c\lambda_k \mu(t) + p(t))} + \lambda_k \int_{\bar{J}_2(\lambda_k)} \frac{\Delta t}{p(t)} > 1.$$

Hence (5.7) also holds.

Now, we show that

$$\int_0^\infty \frac{u^2(t)\Delta t}{-\mu(t)u(t) + p(t)} = \infty. \tag{5.9}$$

From (5.6) and (5.8) we have that, for $t \in \bar{J}(\lambda_1)$,

$$u(t) \geq c\lambda_1 + \int_0^t \frac{u^2(l)\Delta l}{-\mu(l)u(l) + p(l)} \geq c\lambda_1.$$

From (5.7), there exists $l_1 \in \mathbf{T}$ such that $l_1 > 0$ and

$$\lambda_1^2 \int_{\bar{J}_1(\lambda_1) \bigcup \bar{J}_2(\lambda_1) \bigcap [0,l_1]} \frac{\Delta t}{-c\mu(t)\lambda_1 + p(t)} \geq c.$$

Then

$$\begin{aligned}\int_0^{l_1} \frac{u^2(t)\Delta t}{-\mu(t)u(t)+p(t)} &\geq \int_{\bar{J}_1(\lambda_1)\bigcup \bar{J}_2(\lambda_1)\bigcap[0,l_1]} \frac{u^2(t)\Delta t}{-\mu(t)u(t)+p(t)} \\ &\geq c^2\lambda_1^2 \int_{\bar{J}_1(\lambda_1)\bigcup \bar{J}_2(\lambda_1)\bigcap[0,l_1]} \frac{\Delta t}{-c\mu(t)\lambda_1+p(t)} \\ &\geq c^3.\end{aligned}$$

Next, we may assume that λ_2 is so large that $\lambda_2 > \max_{t\in[0,l_1]\cap\mathbf{T}} r(t)$. Then $\bar{J}(\lambda_2) \subset (l_1,\infty)\cap\mathbf{T}$. Repeating the above arguments, we obtain $l_2 \in \mathbf{T}$ such that $l_2 \geq \sigma(l_1)$ and

$$\lambda_2^2 \int_{\bar{J}_1(\lambda_2)\bigcup \bar{J}_2(\lambda_2)\bigcap[l_1,l_2]} \frac{\Delta t}{-c\mu(t)\lambda_2+p(t)} \geq \frac{c}{2}.$$

Hence,

$$\int_{l_1}^{l_2} \frac{u^2(t)\Delta t}{-\mu(t)u(t)+p(t)} \geq \frac{c^3}{2},$$

as before. In general, we obtain $l_n \in \mathbf{T}$, $n = 1,2,\dots$, such that $l_{n+1} \geq \sigma(l_n)$ and

$$\int_{l_n}^{l_{n+1}} \frac{u^2(t)\Delta t}{-\mu(t)u(t)+p(t)} \geq \frac{c^3}{n+1}.$$

As a result, we obtain (5.9).

Since $\min \bar{J}(\lambda_k) \to \infty$ as $k\to\infty$, we can choose $k_1 \in \mathbf{N}$ large enough such that $c_1 = \min \bar{J}(\lambda_{k_1})$ satisfies

$$u(0) + \int_0^{c_1} \frac{u^2(t)\Delta t}{-\mu(t)u(t)+p(t)} > 0. \tag{5.10}$$

There are two cases to be considered for k_1.

Case 1. Assumption (i) holds for $k = k_1$. From (5.6) and (5.10), $u(t) > r(t) \geq \lambda_{k_1}$ for $t \in \bar{J}(\lambda_{k_1})$. Hence $\mu(t_{k_1})u(t_{k_1}) > \mu(t_{k_1})\lambda_{k_1} \geq p(t_{k_1})$, contradicting the assumption.

Case 2. Assumption (ii) holds for $k = k_1$. Let $\bar{J}(\lambda_{k_1}) = (\cup_{i=1}^{n}[c_i, d_i]) \cap \mathbf{T}$, $n \le \infty$. Rewrite (5.6) as

$$u(t) = r(t) + \left(u(0) + \int_0^{c_1} \frac{u^2(l)\Delta l}{-\mu(l)u(l) + p(l)}\right) + \int_{c_1}^{t} \frac{u^2(l)\Delta l}{-\mu(l)u(l) + p(l)}.$$

Consider the telescoped equation on $\widehat{\mathbf{T}}$ obtained by cutting off $J(\lambda_{k_1})$ from $\mathbf{T}$

(5.11)
$$v(s) = \hat{r}(s) + \left(u(0) + \int_0^{c_1} \frac{u^2(l)\Delta l}{-\mu(l)u(l) + p(l)}\right) + \int_0^{s} \frac{v^2(l)\Delta l}{-\hat{\mu}(l)v(l) + \hat{p}(l)}$$

where $s = \mathcal{T}t$ is defined by (1.8) according to $J(\lambda_{k_1})$, $\hat{r}(s) = r(\mathcal{T}^{-1}s)$, $\hat{\mu}(s) = \mu(\mathcal{T}^{-1}s)$, $\hat{p}(s) = p(t)$ for $t \neq d_i$, $\hat{p}(\mathcal{T}d_i)$ is sufficiently close to $p(d_i)$, $i = 1, \dots, n$. From (5.5) we have that

$$\lambda_{k_1}^2 \sum_{l \in \hat{\mathbf{T}}_1} \frac{\hat{\mu}(\hat{\sigma}(l))\hat{\mu}(l)}{\hat{p}(\hat{\sigma}(l))(-c\lambda\hat{\mu}(l) + \hat{p}(l))} + \lambda_{k_1} \int_{\hat{\mathbf{T}}_2} \frac{\Delta l}{\hat{p}(l)} > 1 \tag{5.12}$$

where $\widehat{\mathbf{T}}_1 = \cup\{[s, \sigma(s)] : s \in \hat{\mathbf{T}}$ is right-scattered in $\widehat{\mathbf{T}}\}$ and $\widehat{\mathbf{T}}_2 = \{s \in \hat{\mathbf{T}} : s$ is right-dense in $\widehat{\mathbf{T}}\}$. By Theorem 3.1, (5.11) is nonoscillatory, i.e., $\hat{\mu}(s)v(s) < \hat{p}(s)$ for $s \in \hat{\mathbf{T}}$. However, from (5.10) and (5.11) we see that

$$v(s) > \lambda_{k_1} + \int_0^{s} \frac{v^2(l)\Delta l}{-\hat{\mu}(l)u(l) + \hat{p}(l)}.$$

This, together with (5.12), contradicts Lemma 5.1 and completes the proof. $\square$

Corollary 5.1. *Let* $\bar{J}_2(\lambda)$, $\lambda \in \mathbf{R}$, *be defined as in Theorem* 5.1. *Assume that, for any* $\lambda \in \mathbf{R}$,

$$\int_{\bar{J}_2(\lambda)} \frac{\Delta t}{p(t)} > \frac{1}{\lambda}.$$

Then (E) *is oscillatory.*

Proof. This follows from Theorem 5.1, (i) and (ii). $\square$

Corollary 5.2. *Let $r(t)$ be defined by* (3.7). *Assume for a strictly increasing sequence $\{t_k\}_{k=1}^{\infty} \subset \mathbf{T}$*

$$\limsup_{t\to\infty} r(t) = \lim_{k\to\infty} r(t_k) = \infty,$$

and $p(t_k) \le \mu(t_k) r(t_k)$ for all $k \in \mathbf{N}$. Then (E) *is oscillatory.*

Proof. This follows from Theorem 5.1 (i). $\square$

Corollary 5.3. *Let $\bar{J}(\lambda)$ and $\sigma^*(t)$ be defined as in Theorem* 5.1 *and d be any positive number less than $(1+\sqrt{5})/2$. Assume that, for all $k \in \mathbf{N}$, there exists $t_k \in \bar{J}(\lambda_k)$ such that*

$$p(t_k) \le d\mu(t_k)\lambda_k \quad \text{and} \quad p(\sigma^*(t_k)) \le d\mu(\sigma^*(t_k))\lambda_k.$$

Then (E) *is oscillatory.*

Proof. Choose $c \in (0,1)$ such that $(c+\sqrt{c^2+4})/2 > d$. For $k \in \mathbf{N}$, if assumption (i) of Theorem 5.1 is not satisfied, then

$$\begin{aligned} 0 &\le p(\sigma^*(t_k))(-c\lambda_k\mu(t_k) + p(t_k)) \\ &\le \mu(t_k)\mu(\sigma^*(t_k))d(-c+d)\lambda_k^2 \\ &< \frac{1}{4}\,\mu(t_k)\mu(\sigma^*(t_k))(c+\sqrt{c^2+4})(-c+\sqrt{c^2+4})\lambda_k^2 \\ &= \mu(t_k)\mu(\sigma^*(t_k))\lambda_k^2. \end{aligned}$$

Hence

$$\frac{\mu(t_k)\mu(\sigma^*(t_k))\lambda_k^2}{p(\sigma^*(t_k))(-c\lambda_k\mu(t_k) + p(t_k))} > 1.$$

This implies (5.5). Therefore assumption (ii) of Theorem 5.1 is satisfied. $\square$

Theorem 5.2. *Let $r(t)$ be defined by* (3.7), *$J(\lambda)$, $\bar{J}(\lambda)$, $\bar{J}_1(\lambda)$, and $\bar{J}_2(\lambda)$ as in Theorem 5.1. Assume that $\mu(t)$ is bounded on $\mathbf{T}$, and there exists $\lambda \in \mathbf{R}$ such that*

$$\sum_{t\in\bar{J}_1(\lambda)} \frac{\mu(\sigma^*(t))\mu(t)}{p(\sigma^*(t))p(t)} + \int_{\bar{J}_2(\lambda)} \frac{\Delta t}{p(t)} = \infty \tag{5.13}$$

where $\sigma^*(s) = \min\{t \in \bar{J}(\lambda) : t \geq \sigma(s)\}$, *and*

$$\int_{J(\lambda)} \frac{(\lambda - r(t))^2 \Delta t}{\mu(t)(\lambda - r(t)) + p(t)} = \infty. \tag{5.14}$$

Then (E) *is oscillatory.*

Proof. Assume the contrary, and without loss of generality, suppose the associated Riccati equation (R) has a solution $u \in C^1_{rd}(\mathbf{T}, \mathbf{R})$ satisfying $\mu(t)u(t) < p(t)$, $t \in \mathbf{T}$. There are two possible cases according to whether or not

$$u(0) + \int_0^\infty \frac{\mu^2(t)\Delta t}{-\mu(t)u(t) + p(t)} > -\lambda.$$

In the former case, since $1/p$ is continuous on $\bar{J}(\lambda)$, (5.13) implies that $\sup \bar{J}(\lambda) = \infty$. So we can choose $t_1 \in \bar{J}(\lambda)$ such that

$$u(0) + \int_0^{t_1} \frac{u^2(t)\Delta t}{-\mu(t)u(t) + p(t)} \geq -\lambda + \alpha$$

for some $\alpha > 0$. Rewrite the Riccati equation (R) as

$$u(t) = r(t) + \left(u(0) + \int_0^{t_1} \frac{u^2(l)\Delta l}{-\mu(l)u(l) + p(l)}\right) + \int_{t_1}^t \frac{u^2(l)\Delta l}{-\mu(l)u(l) + p(l)}. \tag{5.15}$$

It follows that for $t \in \bar{J}(\lambda)$

$$u(t) \geq \alpha + \int_{t_1}^t \frac{u^2(l)\Delta l}{-\mu(l)u(l) + p(l)}, \tag{5.16}$$

and hence $u(t) \geq \alpha$ for $t \in \bar{J}(\lambda) \cap [t_1, \infty)$. If there exists $t^* \in \bar{J}(\lambda) \cap [t_1, \infty)$ such that $p(t^*) \leq \mu(t^*)\alpha$, then $\mu(t^*)u(t^*) \geq p(t^*)$, contradicting the assumption. Thus, $p(t) > \mu(t)\alpha$ for all $t \in \bar{J}(\lambda) \cap [t_1, \infty)$. As in the proof of Theorem 5.1, by using Theorem 3.1, we may assume without loss of generality that (5.16) holds for all $t \geq t_1$, and (5.13) can be replaced by

$$\sum_{t \in \mathbf{T}_1} \frac{\mu(\sigma(t))\mu(t)}{p(\sigma(t))p(t)} + \int_{\mathbf{T}_2} \frac{\Delta t}{p(t)} = \infty$$

where $\mathbf{T}_1 = \cup\{[t, \sigma(t)] : t \in \mathbf{T} \cap [t_1, \infty) \text{ is right-scattered in } \mathbf{T}\}$ and $\mathbf{T}_2 = \{t \in \mathbf{T} \cap [t_1, \infty) : t \text{ is right-dense in } \mathbf{T}\}$. This implies that

$$\sum_{t \in \mathbf{T}_1} \frac{\mu(\sigma(t))\mu(t)}{p(\sigma(t))p(t)} = \infty \quad \text{or} \quad \int_{\mathbf{T}_2} \frac{\Delta t}{p(t)} = \infty.$$

Hence,

$$\alpha^2 \sum_{t \in \mathbf{T}_1} \frac{\mu(\sigma(t))\mu(t)}{p(\sigma(t))(-\alpha\mu(t) + p(t))} + \alpha \int_{\mathbf{T}_2} \frac{\Delta t}{p(t)} > 1. \tag{5.17}$$

Now, from (5.16), (5.17) and $\mu(t)u(t) < p(t)$, $t \in \mathbf{T}$, we get a contradiction to Lemma 5.1.

In the remaining case,

$$u(0) + \int_0^\infty \frac{u^2(t)\Delta t}{-\mu(t)u(t) + p(t)} \le -\lambda. \tag{5.18}$$

It follows from (5.15) that $u(t) \le r(t) - \lambda < 0$ for $t \in J(\lambda)$. Since the function $f(u) = u^2/(-\mu u + p)$ is decreasing for $u < 0$ and $p > 0$, we see that, for $t \in J(\lambda)$,

$$\frac{u^2(t)}{-\mu(t)u(t) + p(t)} \ge \frac{(r(t) - \lambda)^2}{-\mu(t)(r(t) - \lambda) + p(t)}.$$

From (5.14)

$$\begin{aligned}\int_0^\infty \frac{u^2(t)\Delta t}{-\mu(t)u(t) + p(t)} &\ge \int_{J(\lambda)} \frac{u^2(t)\Delta t}{-\mu(t)u(t) + p(t)} \\ &\ge \int_{J(\lambda)} \frac{(r(t) - \lambda)^2\Delta t}{-\mu(t)(r(t) - \lambda) + p(t)} = \infty.\end{aligned}$$

This contradicts (5.18) and completes the proof. □

The following definition is needed in the next lemmas.

Definition 5.1. For any set $E \subset \mathbf{T}$, let $E_1 = \cup\{[t, \sigma(t)] : t \in E \text{ is right-scattered in } \mathbf{T}\}$ and $E_2 = \{t \in E : t \text{ is right-dense in } \mathbf{T}\}$. Then we define a measure of E by

$$m\{E\} = \sum_{t \in E_1} \mu(t)\mu(\sigma^*(t)) + \operatorname{mess} E_2$$

where $\sigma^*(t) = \min\{\tau \in E : \tau \geq \sigma(t)\}$ and $\operatorname{mess} E_2$ is the Lebesgue measure of E_2.

Corollary 5.4. *Let $r(t)$ be defined by* (3.7), $J(\lambda)$, $\bar{J}(\lambda)$, $\bar{J}_1(\lambda)$ *and* $\bar{J}_2(\lambda)$ *as in Theorem* 5.1. *Assume that $1/p(t)$ and $\mu(t)$ are bounded on* $\mathbf{T}$, *and that there exists $\lambda \in \mathbf{R}$ such that $m\{\bar{J}(\lambda)\} = \infty$ and*

$$\int_{J(\lambda)} (\lambda - r(t))^2 \Delta t = \infty. \tag{5.19}$$

Then (E) *is oscillatory.*

Proof. It is easy to see that $m\{\bar{J}(\lambda)\} = \infty$ implies that (5.13) holds. Assume that $p(t) \leq M$ and $\mu(t) \leq M$ for $t \in \mathbf{T}$ and $M > 0$. If there exists a sequence $\{t_k\}_{k=1}^{\infty} \subset J(\lambda)$ such that

$$(\lambda - r(t_k))\sqrt{\mu(t_k)} \longrightarrow \infty \quad \text{as} \quad k \to \infty,$$

then

$$\sum_{k=1}^{\infty} \frac{(\lambda - r(t_k))^2 \mu(t_k)}{(\lambda - r(t_k))\mu(t_k) + p(t_k)} \geq \sum_{k=1}^{\infty} \frac{(\lambda - r(t_k))^2 \mu(t_k)}{\sqrt{M}(\lambda - r(t_k))\sqrt{\mu(t_k)} + M} = \infty.$$

Hence (5.14) holds. Otherwise, there exists $N > 0$ such that $\mu(t)(\lambda - r(t)) + p(t) \leq N$ for all $t \in J(\lambda)$. By (5.19),

$$\int_{J(\lambda)} \frac{(\lambda - r(t))^2 \Delta t}{\mu(t)(\lambda - r(t)) + p(t)} \geq \frac{1}{N} \int_{J(\lambda)} (\lambda - r(t))^2 \Delta t = \infty.$$

Hence (5.14) also holds. By Theorem 5.2, (E) is oscillatory. □

Corollary 5.5. *Let $r(t)$ be defined by* (3.7), $J(\lambda)$, $\bar{J}(\lambda)$, $\bar{J}_1(\lambda)$, *and* $\bar{J}_2(\lambda)$ *as in Theorem* 5.1. *Assume that $p(t)$ and $\mu(t)$ are bounded on* $\mathbf{T}$. *Suppose further that there exist two real numbers λ_1 and λ_2 with $\lambda_1 < \lambda_2$ such that $m\{J(\lambda_1)\} = m\{\bar{J}(\lambda_2)\} = \infty$. Then* (E) *is oscillatory.*

Proof. Choose $\bar{\lambda} \in (\lambda_1, \lambda_2)$. Note that $\bar{J}(\lambda_2) \subset \bar{J}(\bar{\lambda})$, we have that $m\{\bar{J}(\bar{\lambda})\} = \infty$. Assume $\mu(t) \leq M$ for $t \in \mathbf{T}$. Then $J(\bar{\lambda}) \supset J(\lambda_1)$

implies that

$$\begin{aligned}\int_{J(\bar\lambda)}(\bar\lambda - r(t))^2\Delta t &\geq \int_{J(\lambda_1)}(\bar\lambda - r(t))^2\Delta t\\ &\geq (\bar\lambda-\lambda_1)^2\Big(\sum_{t\in J_1(\lambda_1)}\mu(t) + \operatorname{mess} J_2(\lambda_1)\Big)\\ &\geq (\bar\lambda-\lambda_1)^2\left(\frac{1}{M}\sum_{t\in J_1(\lambda_1)}\mu(t)\mu(\sigma^*(t)) + \operatorname{mess} J_2(\lambda_1)\right) = \infty\end{aligned}$$

since

$$m\{J(\lambda_1)\} = \sum_{t\in J_1(\lambda_1)}\mu(t)\mu(\sigma^*(t)) + \operatorname{mess} J_2(\lambda_1) = \infty.$$

Therefore, the assumptions of Corollary 5.4 are satisfied with $\bar\lambda$ in place of λ. $\square$

The last result provides the Olech-Opial-Wazewski type criteria for oscillation, see [**17**] for details. The following definitions for approximate limits are used in the statement.

Definition 5.2. Let f be a real valued function on $\mathbf{T}$ and $-\infty \leq l, L \leq \infty$. Then we write $\lim\operatorname{app}\sup_{t\to\infty} f(t) = L$ in case

$$m\{t\in\mathbf{T} : f(t) > L_1\} = \infty \quad \text{for all} \quad L_1 < L$$

and

$$m\{t\in\mathbf{T} : f(t) > L_2\} < \infty \quad \text{for all} \quad L_2 > L.$$

Similarly, $\lim\operatorname{app}\inf_{t\to\infty} f(t) = l$ in case

$$m\{t\in\mathbf{T} : f(t) < l_1\} < \infty \quad \text{for all} \quad l_1 < l$$

and

$$m\{t\in\mathbf{T} : f(t) < l_2\} = \infty \quad \text{for all} \quad l_2 > l.$$

Corollary 5.6. *Let $r(t)$ be defined by* (3.7). *Assume $p(t)$ and $\mu(t)$ are bounded on $\mathbf{T}$, and either*

$$\lim\operatorname{app}\sup_{t\to\infty} r(t) = \infty \tag{5.20}$$

or

$$-\infty \le \text{lim app inf}_{t\to\infty} r(t) < \text{lim app sup}_{t\to\infty} r(t) < \infty. \tag{5.21}$$

Then (E) *is oscillatory.*

Proof. Assume (5.20) holds, and $p(t) \le M$ for $t \in \mathbf{T}$ and for some $M > 0$. Let $\{\lambda_k\}_{k=1}^{\infty} \subset [1,\infty)$ such that $\lambda_k \to \infty$ as $k \to \infty$. Then for any $k \in \mathbf{N}$, we have that $m\{\bar{J}(\lambda_k)\} = \infty$, i.e., either $\sum_{t\in\bar{J}_1(\lambda_k)} \mu(t)\mu(\sigma^*(t)) = \infty$ or mess $\bar{J}_2(\lambda_k) = \infty$. If assumption (i) of Theorem 5.1 is not satisfied, then

$$\begin{aligned}\lambda_k^2 \sum_{t\in\bar{J}_1(\lambda_k)} \frac{\mu(\sigma^*(t))\mu(t)}{p(\sigma^*(t))(-c\lambda_k\mu(t)+p(t))} + \lambda_k \int_{\bar{J}_2(\lambda_k)} \frac{\Delta t}{p(t)} \\ \ge \frac{\lambda_k^2}{M^2} \sum_{t\in\bar{J}_1(\lambda_k)} \mu(t)\mu(\sigma^*(t)) + \frac{\lambda_k}{M} \operatorname{mess} \bar{J}_2(\lambda_k) = \infty.\end{aligned}$$

Therefore, assumption (ii) of Theorem 5.1 is satisfied. The conclusion then follows from Theorem 5.1.

Assume (5.21) holds. Then choose λ_1 and λ_2 such that

$$\text{lim app inf}_{t\to\infty} r(t) < \lambda_1 < \lambda_2 < \text{lim app sup}_{t\to\infty} r(t).$$

Then the conclusion follows from Corollary 5.5. $\square$

6. Examples. In this section, we give several examples to illustrate our results. To the best of our knowledge, no previous criteria for oscillation can be applied to these examples. We note in particular that, in the first example, the graininess function $\mu(t)$ is unbounded while, in the second example, $\mu(t) \to 0$ as $t \to \infty$. The third example deals with the case when $\mathbf{T}$ is the union of closed intervals.

Example 6.1. Let $\mathbf{T} = \{a^k : k \in \mathbf{N}_0\}$, where $a > 1$. Let $p(t) \equiv 1$ for all $t \in \mathbf{T}$ and

$$q(t) = \begin{cases} 1 & t = a^{2k}, \\ -k & t = a^{2k+1}, \end{cases} \qquad k \in \mathbf{N}_0.$$

Then (E) is oscillatory.

In fact, for any $k \in \mathbf{N}_0$, $\sigma(a^k) = a^{k+1}$ and $\mu(a^k) = \sigma(a^k) - a^k = (a-1)a^k$. Let I be defined by (4.1) with $n_k = m_k = a^{2k}$ for $k \in \mathbf{N}$. Obviously,

$$\sum_{k=1}^{\infty} \int_{n_k}^{\sigma(m_k)} \frac{1}{p(t)} \, \Delta t = \sum_{k=1}^{\infty} \int_{a^{2k}}^{a^{2k+1}} \frac{1}{p(t)} \, \Delta t = \sum_{k=1}^{\infty} (a-1)a^{2k} = \infty,$$

and

$$\sum_{k=1}^{\infty} \int_{n_k}^{\sigma(m_k)} q(t) \Delta t = \sum_{k=1}^{\infty} \int_{a^{2k}}^{a^{2k+1}} q(t) \Delta t = \sum_{k=1}^{\infty} (a-1)a^{2k} = \infty.$$

It then follows from Theorem 4.1 that (E) is oscillatory. Note that $\int_0^\infty q(t)\Delta t = -\infty$ in this case.

In the next two examples, for any $\lambda \in \mathbf{R}$, we let $J(\lambda)$, $\bar{J}(\lambda)$, $\bar{J}_1(\lambda)$, and $\bar{J}_2(\lambda)$ be defined as in Theorem 5.1 and let the measure $m(\cdot)$ and $\sigma^*(t)$ be as in Definition 5.1.

Example 6.2. Let $\mathbf{T} = \{t_k : k \in \mathbf{N}\}$, where $t_k = \sum_{n=1}^{k} 1/\sqrt{k}$ for all $k \in \mathbf{N}$. Let p be any positive bounded function on $\mathbf{T}$, and define q to satisfy

$$\sum_{n=1}^{k} \frac{q(t_n)}{\sqrt{n+1}} = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ -1/\sqrt[4]{2k+1} & \text{if } k \text{ is even,} \end{cases} \qquad k \in \mathbf{N}_0.$$

Then (E) is oscillatory.

In fact, for any $t_k \in \mathbf{T}$, $\mu(t_k) = \sigma(t_k) - t_k = 1/\sqrt{k+1}$ and hence is bounded on $\mathbf{T}$. Clearly,

$$J(0) = \left\{ t_k \in \mathbf{T} : \int_0^{t_k} q(t)\Delta t < 0 \right\} = \{t_{2k} : k \in \mathbf{N}_0\},$$

and

$$\bar{J}(0) = \left\{ t_k \in \mathbf{T} : \int_0^{t_k} q(t)\Delta t \geq 0 \right\} = \{t_{2k+1} : k \in \mathbf{N}_0\}.$$

Therefore,

$$m(\bar{J}(0)) = \sum_{t\in\bar{J}(0)} \mu(t)\mu(\sigma^*(t)) = \sum_{k=1}^{\infty} \frac{1}{\sqrt{2k+2}}\,\frac{1}{\sqrt{2k+4}} = \infty,$$

and

$$\begin{aligned}\int_{J(0)} r^2(t)\Delta t &= \sum_{t\in J(0)} \Big(\int_0^t q(t)\Delta t\Big)^2 \mu(t)\\ &= \sum_{k=1}^{\infty} \frac{1}{\sqrt{2k+1}}\,\frac{1}{\sqrt{2k+1}} = \infty.\end{aligned}$$

It then follows from Corollary 5.4 with $\lambda = 0$ that (E) is oscillatory. Note that $\int_0^\infty q(t)\Delta t = 0$ in this case.

Example 6.3. Let $\mathbf{T} = \cup_{k=0}^{\infty}[2k, 2k+1]$. Let $p(t) \le t^2/c$, where $c \ge 128$, and

$$q(t) = \begin{cases} 2k+1 & t \in [4k, 4k+1], \\ -2(k+1) & t \in [4k+2, 4k+3], \end{cases} \qquad k \in \mathbf{N}_0.$$

Then (E) is oscillatory.

In fact, we may let $\lambda_k = k+1$, $k \in \mathbf{N}$. For any $k \in \mathbf{N}_0$, define $l_{4k+1} = 2k+1$ and $l_{4k+3} = -2(k+1)$; then $r(t) = \int_0^t q(l)\Delta l$ can be written as

$$r(t) = \begin{cases} 2\sum_{n=1}^{2k} l_{2n-1} + (2k+1)(t-4k) & t \in [4k, 4k+1], \\ 2\sum_{n=1}^{2k+1} l_{2n-1} - 2(k+1)(t-4k-2) & t \in [4k+2, 4k+3], \end{cases}$$
$$k \in \mathbf{N}_0.$$

From a direct computation we obtain that, for any $k \in \mathbf{N}_0$, $r(4k) = -2k$, $r(4k+1) = 1$, $r(4k+2) = 2(k+1)$, and $r(4k+3) = 0$. Set

$$E_k = \bigcup_{n=k}^{\infty} [4n+2, 4n+5/2], \quad k \in \mathbf{N}.$$

It is easy to see that, for $k \in \mathbf{N}$, $\bar{J}(\lambda_k) \supset E_k$; as a consequence, $\bar{J}_2(\lambda_k) \supset E_k$. Hence, for $k \in \mathbf{N}$,

$$\begin{aligned}\lambda_k \int_{\bar{J}_2(\lambda_k)} \frac{1}{p(t)}\,\Delta t &\geq (k+1)\int_{E_k} \frac{c}{t^2}\,dt = (k+1)c\sum_{n=k}^{\infty}\int_{4n+2}^{4n+(5/2)} \frac{1}{t^2}\,dt \\ &> \frac{(k+1)c}{2}\sum_{n=k}^{\infty}\frac{1}{(4n+4)^2} = \frac{(k+1)c}{32}\sum_{n=k}^{\infty}\frac{1}{(n+1)^2} \\ &> \frac{(k+1)c}{32}\sum_{n=k}^{2k+1}\frac{1}{(n+1)^2} > \frac{(k+1)^2c}{32(2k+2)^2} \\ &= \frac{(k+1)^2c}{128(k+1)^2} \geq 1.\end{aligned}$$

We observe that $r(t)$ is either 0 or 1 for all right-scattered points $t = 2k+1$, $k \in \mathbf{N}_0$. Therefore $\bar{J}_1(\lambda_k) = \varnothing$ for all $k \in \mathbf{N}$. Then the conclusion follows from Theorem 5.1 (ii). Note that, in this case, $\int^{\infty} p(l)\Delta l < \infty$,

$$\liminf_{t\to\infty}\int_0^t q(l)\Delta l = -\infty \quad \text{and} \quad \limsup_{t\to\infty}\int_0^t q(l)\Delta l = \infty.$$

Additional examples may readily be given to illustrate the oscillation criteria of the other results. We leave this to the interested reader.

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