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EXISTENCE OF SOLUTIONS FOR THE BAROTROPIC-VORTICITY EQUATION IN AN UNBOUNDED DOMAIN

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ABSTRACT. In this paper we consider the two-dimensional barotropic-vorticity equation in the first quadrant, and using a rearrangement variational principle, prove it has a solution. The solution represents a steady localized topographic ideal flow. The data given are the behavior of the flow at infinity, the rearrangement class of the vorticity field and the height of the localized seamount.

1. Introduction. In this paper we prove existence of solutions for the following barotropic-vorticity equation

$$[\psi, \omega + h] = 0, \tag{1.1}$$

satisfying

$$-\Delta\psi \in \mathcal{F} + h, \tag{1.2}$$

where $[\cdot,\cdot]$ denotes the Jacobian. Here $\mathcal{F}$ denotes a class of rearrangements of a given function, and h is some fixed non-negative function, see the next section for precise definitions. Equation (1.1) is the governing equation describing the flow of an ideal fluid with ψ representing the stream function, ω the vorticity and h the height of the bottom topography. In the present work we are assuming (1.1) to hold in a *non-symmetric* planar domain, the first quadrant Π_+. Since the domain is unbounded, we require some asymptotic condition to be satisfied; namely, we assume $\psi \to \lambda x_1 x_2$ at infinity ($\lambda x_1 x_2$ represents the stream function of an irrotational flow). We also assume that ω belongs to the class of rearrangements of a given function. Similar problems in *symmetric* domains have been considered but the methods are not applicable in our situation. We derive new estimates in order to overcome this lack of symmetry.

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A weak formulation of (1.1), in Π_+, is given by

$$\int_{\Pi_+} (\omega + h)[\psi, u] = 0, \tag{1.3}$$

for all $u \in C_0^\infty(\Pi_+)$, where the functions ψ, ω and h are related by the inclusion (1.2), see for example [**11, 12**]. By a weak solution of (1.1) we mean a pair (ψ, ω) that satisfies (1.3) for every $u \in C_0^\infty(\Pi_+)$, and (1.2). To prove existence of solutions to (1.1) we employ the variational principle developed extensively by Burton [**4**] suitable for optimization problems where the admissible set is either the set of rearrangements of a given integrable function (the unconstrained case) or the intersection of that with an affine subspace of finite codimension (the constrained case). The main results of this paper are Theorems 1 and 2 which are stated in the next section.

Similar existence results for the Euler's equation (when $h = 0$) have recently been the focus of many authors, the reader could refer to [**5–8, 13**].

2. Notation, definitions and the statement of the main results. Throughout the paper p is a real number in $(2, +\infty)$. For any number $q \geq 1$, q^* denotes the conjugate of q; that is, $1/q + 1/q^* = 1$. If $A \subseteq \mathbf{R}^2$ is measurable, then $|A|$ denotes the two-dimensional Lebesgue measure of A. The upper and the right half planes are designated by Π_u and Π_r, respectively, and the first quadrant by Π_+. We write points in $\mathbf{R}^2$ as $x = (x_1, x_2)$, $y = (y_1, y_2)$, etc. For $\xi > 0$ we define

$$\Pi_+(\xi) = \{x \in \Pi_+ \mid x_1 < \xi\ ,\ x_2 < \xi\}\,.$$

The ball centered at x with radius R is denoted $B_R(x)$; in case the center is the origin we simply write B_R. In this paper we denote the Green's functions for $-\Delta$ with homogeneous Dirichlet boundary conditions on Π_u, Π_r and Π_+ by G_u, G_r and G_+, respectively, and these functions are given as follows

$$\begin{aligned}
G_u(x,y) &= \frac{1}{2\pi}\,\log\frac{|x-\bar{y}|}{|x-y|}, && x, y \in \Pi_u, \quad x \neq y,\\
G_r(x,y) &= \frac{1}{2\pi}\,\log\frac{|x-\underline{y}|}{|x-y|}, && x, y \in \Pi_r, \quad x \neq y,\\
G_+(x,y) &= \frac{1}{2\pi}\,\log\frac{|x-\bar{y}||x-\underline{y}|}{|x-y||x-\underline{\bar{y}}|}, && x, y \in \Pi_+, \quad x \neq y,
\end{aligned}$$

where $\bar{x}$ and $\underline{x}$ denote the reflections with respect to x_1-axis and x_2-axis, respectively. Note that Green's functions are non-negative. For measurable functions f on $\mathbf{R}^2$, we define

$$\begin{aligned} T_u f(x) &= \int_{\Pi_u} G_u(x,y) f(y)\, dy, \\ T_r f(x) &= \int_{\Pi_r} G_r(x,y) f(y)\, dy, \\ T_+ f(x) &= \int_{\Pi_+} G_+(x,y) f(y)\, dy, \end{aligned}$$

when the integrals exist. The strong support of a measurable function f, denoted $\operatorname{supp}(f)$, is defined by

$$\operatorname{supp}(f) = \{x \mid f(x) > 0\}.$$

Let us fix $f_0 \in L^p(\Pi_+)$ which is a non-negative, non-trivial function with compact support and assume $|\operatorname{supp}(f_0)| = \pi a^2$, for some $a > 0$. Moreover, we suppose that $\|f_0\|_1 = 1$. The measurable function f is called a rearrangement of f_0 whenever

$$|\{x \in \Pi_+ \mid f(x) > \alpha\}| = |\{x \in \Pi_+ \mid f_0(x) > \alpha\}|,$$

for every $\alpha \in \mathbf{R}$. It is known that if f is a rearrangement of f_0, then $\|f\|_q = \|f_0\|_q$, $q \geq 1$. By $\mathcal{F}$ we denote the set of rearrangements of f_0 on Π_+ which have compact support. By $\mathcal{F}(\xi)$ we denote the subset of $\mathcal{F}$ comprising functions vanishing outside $\Pi_+(\xi)$. For a measurable function f on Π_+, we define the energy functional

$$E_\lambda(f) = \frac{1}{2}\int_{\Pi_+} f T_+ f + \int_{\Pi_+} \eta f - \lambda \int_{\Pi_+} x_1 x_2 f, \tag{2.1}$$

whenever the integrals exist, where λ is a positive fixed number and $\eta = T_+ h$, for a fixed function $h \in L^p(\Pi_+)$ which is a non-negative function and has compact support. We consider the following maximization problem

$$P_\lambda \ : \ \sup_{f \in \mathcal{F}} E_\lambda(f), \tag{2.2}$$

whose set of solutions is denoted Σ_λ. Similarly, for $\xi \geq \sqrt{\pi}\, a$ we define $P_{\lambda,\xi}$ as follows

$$P_{\lambda,\xi} \ : \ \sup_{f\in\mathcal{F}(\xi)} E_\lambda(f), \tag{2.3}$$

and $\Sigma_{\lambda,\xi}$ is defined similarly to Σ_λ.

We are now ready to state the main results of this paper.

Theorem 1. *There exists $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0)$, P_λ has a solution. Moreover if $f_\lambda \in \Sigma_\lambda$ and $\psi_\lambda = T_+ f_\lambda + \eta - \lambda x_1 x_2$, then ψ_λ satisfies the following semi-linear elliptic partial differential equation*

$$-\Delta\psi_\lambda = \varphi_\lambda \circ \psi_\lambda + h, \tag{2.4}$$

almost everywhere in Π_+, where φ_λ is an increasing function unknown a priori. Here "$\circ$" denotes composition of functions.

Theorem 2. *Let λ_0 be as in Theorem* 1. *Let $\lambda \in (0, \lambda_0)$ and suppose $f_\lambda \in \Sigma_\lambda$. Set $\psi = T_+ f_\lambda$ and $\omega = f_\lambda$, then (ψ, ω) is a weak solution of* (1.1) *in* Π_+.

The proofs of Theorems 1 and 2 will be presented in the next section, but since a number of preliminaries are required, we digress at this stage to explain the strategy, which is proposed by Benjamin [**3**]. The first step is to prove the existence of a maximizer for E_λ relative to the rearrangements of f_0 defined on $\Pi_+(\xi)$. We then use the variational principle developed by Burton, which is particularly suitable to show solvability of $P_{\lambda,\xi}$. The second step is to show that increasing the size of the box $\Pi_+(\xi)$ indefinitely does not affect the maximizer; that is, the support of the maximizer does not touch the boundary of the box if it is large enough.

3. Properties of the operator T_+. In this section we present some lemmas which are crucial in our analysis.

Lemma 1. *Let $f \in L^p(\Pi_+)$ be a function with compact support. Then $T_+ f \in C^1(\overline{\Pi}_+)$ and*

$$-\Delta T_+ f = f,$$

almost everywhere in Π_+.

Proof. From Lemma 3 in [**3**] we have

$$-\Delta T_u f = f \quad \text{in} \quad D'(\Pi_u), \tag{3.1}$$

in the sense of distributions. Since $T_+ f(x) = T_u f(x) - T_u f(\underline{x})$ for all $x \in \mathbf{R}^2$, by [**2**] we deduce that $T_+ f \in W^{2,p}_{\mathrm{loc}}(\mathbf{R}^2)$. By the Sobolev embedding theorem [**1**], we infer $T_+ f \in C^1(\overline{\Pi}_+)$. □

Lemma 2. *Let $f \in L^p(\Pi_+)$ be a function with compact support. Then*

(i) $|\nabla T_+ f(x)| \le C\|f\|_p$,

(ii) $|T_+ f(x)| \le C \min\{x_1, x_2\}\|f\|_p$,

for every $x \in \Pi_+$, where C depends on $|\operatorname{supp}(f)|$ and p.

Proof. (i) follows from [**4**]. To prove (ii) we fix $x \in \Pi_+$ and apply the mean value theorem to obtain

$$|T_+ f(x)| = |T_+ f(x) - T_+ f(x_1, 0)| \le x_2 |\nabla T_+ f(\hat{x})|,$$

where $\hat{x}$ is a point on the segment joining x to $(x_1, 0)$. Whence from (i) we deduce that $|T_+ f(x)| \le C x_2 \|f\|_p$. Similarly, one can show $|T_+ f(x)| \le C x_1 \|f\|_p$. From these two inequalities we readily infer (ii). □

Lemma 3. *Let U be an open and bounded subset of Π_+. Then, for any $q \ge 1$, $T_+ : L^p(U) \to L^q(U)$ is a linear compact operator, in the sense that if $\{f_n\}$ is a sequence of functions, bounded in $L^p(\Pi_+)$ and vanishing outside U, then the sequence $\{T_+ f_n|_U\}$ has a subsequence converging in the q-norm.*

Proof. From Lemma 2, it follows that $T_+ : L^p(U) \to W^{1,2}(U)$ is bounded. Now by applying the Sobolev embedding theorem we derive the compactness of T_+. $\square$

Lemma 4. *Let $f \in L^p(\Pi_+)$ be a function with compact support. Then*

(i) $\nabla T_+ f(x) = O(|x|^{-2}),\ T_+ f(x) = O(|x|^{-1})$ *as* $|x| \to +\infty$.

(ii) $\int_{\Pi_+} f T_+ f > 0$.

Proof. Let us observe that, for every $x \in \mathbf{R}^2$, $T_+ f(x) = T_u f(x) - T_u f(\underline{x})$. Thus, (i) is an immediate consequence of Lemma 7 in [**5**]. To prove (ii), let us first recall from Lemma 1 that $-\Delta T_+ f = f$, almost everywhere in Π_+. We let $\Omega(R) = B_R \cap \Pi_+$, so the boundary of $\Omega(R)$ is Lipschitz. On the other hand we have $T_+ f \in C^1(\overline{\Pi}_+)$, hence we can apply the weak divergence theorem, see, for example [**9**], to obtain

$$-\int_{\Omega(R)} f\, T_+ f + \int_{\Omega(R)} |\nabla T_+ f|^2 = \int_{\partial\Omega(R)} (T_+ f)\,(\partial_{\vec{n}} T_+ f)\, d\sigma, \tag{3.2}$$

where $\vec{n}$ denotes the unit outward normal vector to $\partial\Omega(R)$. Now from (i) we have $\lim_{R\to\infty} \int_{\partial\Omega(R)} (T_+ f)(\partial_{\vec{n}} T_+ f)\, d\sigma = 0$. Moreover, since $\int_{\Pi_+} f\, T_+ f$ is finite and $|\nabla T_+ f|^2$ is bounded in Π_+ we can apply the Lebesgue dominated convergence theorem to conclude

$$\lim_{R\to\infty} \int_{\Omega(R)} f\, T_+ f = \int_{\Pi_+} f\, T_+ f$$

$$\lim_{R\to\infty} \int_{\Omega(R)} |\nabla T_+ f|^2 = \int_{\Pi_+} |\nabla T_+ f|^2.$$

Then, from (3.2) we derive

$$\int_{\Pi_+} f\, T_+ f = \int_{\Pi_+} |\nabla T_+ f|^2.$$

Lemma 5. *We have*

$$\lim_{\xi\to\infty} \sup_{f \in \mathcal{F}(\xi)} \int_{\Pi_+} f\, T_+ f = \infty.$$

Proof. Let us fix $(t,t) \in \mathbf{R}^2$ and denote the Schwarz-rearrangement of f_0, about (t,t), by ${f_t}^*$ which is spherically decreasing and vanishes outside $B_a(t)$, the ball centered at (t,t) with radius a. There exist $\beta > 0$ and $0 < b < a$ such that for all x with $|x| < b$ we have ${f_0}^* \geq \beta$. Clearly we can assume $t \geq 3a$. Now consider $x \in B_a(t)$, $y \in B_b(t)$. Thus

$$\begin{aligned} &|x - \bar{y}| \geq 2t - 2a, \quad && |x - \underline{y}| \geq 2t - 2a, \\ &|x - y| \leq 2a, && |x - \underline{\bar{y}}| \leq 2\sqrt{2}\,t + 2a. \end{aligned}$$

Therefore

$$T_+ f_t^*(x) \geq \frac{\beta}{2\pi} \int_{B_b(t)} \log \frac{(2t-2a)^2}{2a(2\sqrt{2}\,t + 2a)}\, dy = \frac{\beta b^2}{2} \log \frac{(t-a)^2}{a(\sqrt{2}\,t + a)}.$$

Hence

$$\int_{\Pi_+} f_t^* T_+ f_t^* \geq \frac{\pi \beta^2 b^4}{2} \log \frac{(t-a)^2}{a(\sqrt{2}\,t + a)},$$

from which the conclusion follows. $\square$

4. Main results. In this section we prove Theorems 1 and 2. However we need some more lemmas that are given below. We begin by proving existence of a maximizer for the energy functional E_λ relative to $\mathcal{F}(\xi)$. To do this we need the following result from Burton's theory [**4**].

Lemma 6. *Let $q \geq 1$ and q^* its conjugate. Let $g \in L^{q^*}(\Pi_+(\xi))$ and $T : L^q(\Pi_+(\xi)) \to L^{q^*}(\Pi_+(\xi))$, $\xi \geq \sqrt{\pi}\, a$, be a compact strictly positive symmetric linear operator. Define*

$$\Phi(f) = \frac{1}{2} \int_{\Pi_+} f\, T f - \int_{\Pi_+} g f,$$

for $f \in L^q(\Pi_+(\xi))$. Then Φ attains its supremum on $\mathcal{F}(\xi)$, and if $\hat{f}$ is a maximizer then $\hat{f} = \varphi \circ (T\hat{f} - g)$ almost everywhere in $\Pi_+(\xi)$ for some increasing function φ.

The following lemma is a straightforward result from the symmetry of G_+ and Lemmas 3, 4 and 6.

Lemma 7. *Suppose* $\lambda > 0$ *and* $\xi \geq \sqrt{\pi}\, a$. *Then problem* $P_{\lambda,\xi}$ *is solvable. Moreover, if* $f_{\lambda,\xi} \in \Sigma_{\lambda,\xi}$, *then*

$$f_{\lambda,\xi} = \varphi_{\lambda,\xi} \circ (T_+ f_{\lambda,\xi} + \eta - \lambda\, x_1 x_2), \tag{4.1}$$

almost everywhere in $\Pi_+(\xi)$ *for some increasing function* $\varphi_{\lambda,\xi}$.

Lemma 8. *Let* $\lambda > 0$. *Then there exists* $R(\lambda) > 0$ *such that*

$$T_+ f(x) + \eta(x) - \lambda\, x_1 x_2 \leq 0, \quad |x| \geq R(\lambda), \quad f \in \mathcal{F}.$$

Proof. Let us fix $x \in \Pi_+$ and $f \in \mathcal{F}$. According to Lemma 2 there exists $M > 0$, independent of f, such that

$$T_+ f(x) + \eta(x) \leq M \min\{x_1, x_2\}.$$

Thus,

$$T_+ f(x) + \eta(x) - \lambda\, x_1 x_2 \leq \min\{x_1, x_2\}(M - \lambda \max\{x_1, x_2\}).$$

Hence if we assume $|x| \geq M/\lambda$, then

$$T_+ f(x) + \eta(x) - \lambda\, x_1 x_2 \leq 0.$$

Therefore the result follows for $R(\lambda) = M/\lambda$.

Lemma 9. *There exist* $\lambda_0 > 0$ *and* $\xi_0 \geq \sqrt{\pi}\, a$ *such that when* $0 < \lambda < \lambda_0$, $\xi \geq \xi_0$ *and* $f_{\lambda,\xi}$ *is a maximizer of* E_λ *relative to* $\mathcal{F}(\xi)$, *then*

$$|\mathrm{supp}(\psi_{\lambda,\xi}) \cap \Pi_+(\xi)| \geq \pi a^2,$$

where $\psi_{\lambda,\xi}(x) = T_+ f_{\lambda,\xi}(x) + \eta(x) - \lambda\, x_1 x_2$.

Proof. Let us fix $\alpha > 0$. From Lemma 5 there are $\lambda_0 > 0$, $\xi_0 \geq \sqrt{\pi} a$ such that

$$\sup_{f \in \mathcal{F}(\xi)} E_\lambda(f) \geq \alpha, \quad 0 < \lambda < \lambda_0, \quad \xi \geq \xi_0. \tag{4.2}$$

Next we set $\alpha = 3aC(\|f_0\|_p + 2\|h\|_p)$, where C is the constant in Lemma 2. Notice that C is independent of λ and ξ; it merely depends on $|\mathrm{supp}\,(\mathrm{h})|$, a and p. From (4.2) we have

$$E_\lambda(f_{\lambda,\xi}) \geq 3aC(\|f_0\|_p + 2\|h\|_p), \quad 0 < \lambda < \lambda_0, \quad \xi \geq \xi_0, \tag{4.3}$$

where $f_{\lambda,\xi}$ denotes a maximizer of E_λ relative to $\mathcal{F}(\xi)$. Also note that

$$E_\lambda(f_{\lambda,\xi}) \leq \sup_{\Pi_+(\xi)} \left(\frac{1}{2}\, T_+ f_{\lambda,\xi} + \eta - \lambda\, x_1 x_2 \right), \tag{4.4}$$

for $0 < \lambda \leq \lambda_0$ and $\xi \geq \xi_0$. Hence, from (4.3), we have

$$\sup_{\Pi_+(\xi)} \left(\frac{1}{2}\, T_+ f_{\lambda,\xi}(x) + \eta(x) - \lambda\, x_1 x_2 \right) \geq 3aC(\|f_0\|_p + 2\|h\|_p). \tag{4.5}$$

Since the function $(1/2)\, T_+ f_{\lambda,\xi}(x) + \eta(x) - \lambda\, x_1 x_2$ is continuous on $\overline{\Pi}_+(\xi)$, it attains its maximum at $(r_1(\lambda,\xi), r_2(\lambda,\xi)) \equiv (r_1, r_2)$. Whence, by Lemma 2,

$$\frac{1}{2}\, T_+ f_{\lambda,\xi}(r_1, r_2) + \eta(r_1, r_2) \leq \frac{1}{2}\, C(\|f_0\|_p + 2\|h\|_p) \min\{r_1, r_2\}.$$

Therefore from (4.5) we infer $\min\{r_1, r_2\} > 2a$. Now fix $0 < \lambda < \lambda_0$, $\xi \geq \xi_0$ and set

$$S = \{x \in \Pi_+ |\ x_1 < r_1,\ x_2 < r_2\} \cap B_{2a}(r_1, r_2).$$

Observe that $S \subseteq \overline{\Pi}_+(\xi)$. Consider $x \in S$, hence

$$T_+ f_{\lambda,\xi}(x) + \eta(x) - \lambda\, x_1 x_2 \geq \frac{1}{2}\, T_+ f_{\lambda,\xi}(x) + \eta(x) - \lambda\, r_1 r_2. \tag{4.6}$$

By an application of the mean value theorem in conjunction with Lemma 2,

$$|T_+ f_{\lambda,\xi}(x) - T_+ f_{\lambda,\xi}(r_1, r_2)| \leq 2aC(\|f_0\|_p).$$
$$|\eta(x) - \eta(r_1, r_2)| \leq 2aC\|h\|_p.$$

Thus from (4.5) and (4.6)

$$
\begin{aligned}
(4.7)\quad & T_+ f_{\lambda,\xi}(x) + \eta(x) - \lambda\, x_1 x_2 \\
& \geq \frac{1}{2} T_+ f_{\lambda,\xi}(r_1, r_2) + \eta(r_1, r_2) - \lambda\, r_1 r_2 - aC(\|f_0\|_p + 2\|h\|_p) \\
& \geq 3aC(\|f_0\|_p + 2\|h\|_p) - aC(\|f_0\|_p + 2\|h\|_p) \\
& = 2aC(\|f_0)\|_p + 2\|h\|_p).
\end{aligned}
$$

Therefore $S \subseteq \operatorname{supp}(\psi_{\lambda,\xi})$ apart from a set of zero measure. Hence

$$|\operatorname{supp}(\psi_{\lambda,\xi}) \cap \Pi_+(\xi)| \geq |S| \geq \pi a^2, \quad 0 < \lambda \leq \lambda_0, \quad \xi \geq \xi_0.$$

This completes the proof of the lemma. □

Proof of Theorem 1. Let λ_0, ξ_0 be as in Lemma 9 and fix $\lambda < \lambda_0$. So by Lemma 8 there exists positive $R(\lambda)$ such that

$$(4.8)\quad T_+ f(x) + \eta(x) - \lambda\, x_1 x_2 \leq 0, \quad x \in \Pi_+ \setminus \Pi_+(R(\lambda)), \quad f \in \mathcal{F}.$$

Let us set $\xi(\lambda) = \max\{\xi_0, R(\lambda)\}$. Fix $\xi \geq \xi(\lambda)$; hence, from Lemma 7 it follows that $\Sigma_{\lambda,\xi}$ is non-empty. Consider $f_{\lambda,\xi} \in \Sigma_{\lambda,\xi}$, and apply Lemma 7 to find an increasing function $\varphi_{\lambda,\xi}$ such that

$$(4.9)\qquad f_{\lambda,\xi} = \varphi_{\lambda,\xi} \circ (T_+ f_{\lambda,\xi} + \eta - \lambda\, x_1 x_2),$$

for almost every $x \in \Pi_+(\xi(\lambda))$. Notice that we can assume φ_λ is non-negative. Since $f_{\lambda,\xi}$ is an increasing function of $T_+ f_{\lambda.\xi} + \eta - \lambda\, x_1 x_2$ on $\Pi_+(\xi)$, there exists a constant γ_λ such that

$$(4.10)\qquad \operatorname{supp}(f_{\lambda,\xi}) = (T_+ f_{\lambda,\xi} + \eta - \lambda\, x_1 x_2)^{-1}(\gamma_\lambda, \infty),$$

apart from a set of measure zero. Note that $\xi \geq \xi_0$, hence by Lemma 9 we have

$$(4.11)\qquad |(T_+ f_{\lambda,\xi} + \eta - \lambda\, x_1 x_2)^{-1}(0, \infty)| \geq \pi a^2.$$

Whence $\gamma_\lambda \geq 0$ and this implies $\operatorname{supp}(f_{\lambda,\xi}) \subseteq \operatorname{supp}(T_+ f_{\lambda,\xi} + \eta - \lambda\, x_1 x_2)$, apart from a set of measure zero in $\Pi_+(\xi)$. Thus by (4.8) and (4.10)

$$(4.12)\qquad \operatorname{supp}(f_{\lambda,\xi}) \subseteq \Pi_+(\xi(\lambda)), \quad \xi \geq \xi_0.$$

It now follows that $f_{\lambda,\xi(\lambda)} \in \Sigma_\lambda$; hence, P_λ is solvable for $0 < \lambda < \lambda_0$. To derive (2.4) consider $f_\lambda \in \Sigma_\lambda$. From (4.12), $\operatorname{supp}(f_\lambda) \subseteq \Pi_+(\xi(\lambda))$, except for a set of measure zero. Hence by Lemma 7 there exists an increasing function φ_λ such that

$$f_\lambda = \varphi_\lambda \circ (T_+ f_\lambda + \eta - \lambda\, x_1 x_2), \tag{4.13}$$

for almost every $x \in \Pi(\xi(\lambda))$. Now define

$$\varphi(t) = \begin{cases} \varphi_\lambda(t) & t \in \operatorname{dom}\varphi_\lambda,\ t \geq 0, \\ 0 & t < 0. \end{cases}$$

Clearly φ is increasing and

$$f_\lambda = \varphi \circ (T_+ f_\lambda + \eta - \lambda\, x_1 x_2),$$

almost everywhere in Π_+. Thus by applying Lemma 1, we obtain (2.4). This completes the proof.

Proof of Theorem 2. We must show

$$\int_{\Pi_+} (f_\lambda + h)\, [T_+ f_\lambda - \lambda\, x_1 x_2, u] = 0, \tag{4.14}$$

for every $u \in C_0^\infty(\Pi_+)$. Since $f_\lambda + h$ has compact support we infer existence of an open set $\Omega \subseteq \Pi_+$ such that $\operatorname{supp}(f_\lambda + h) \subseteq \Omega$. Therefore it suffices to prove (4.14) only for test functions $u \in C_0^\infty(\Omega)$. So let us fix $u \in C_0^\infty(\Omega)$, and denote by $g_t(x)$ the unique solution of the following Hamiltonian system

$$\frac{dz}{dt} = \nabla^\perp u(z),$$

satisfying the initial condition $z(0) = x \in \Omega$; where $\nabla^\perp = ((\partial/\partial x_2), -(\partial/\partial x_1))$. It is well known that the mapping $x \to f_t(x)$, $t \in [-\tau, \tau]$, τ small, defines a one-parameter family of measure preserving diffeomorphisms of Ω, see for example [**10**]. Now following [**11, 12**] we obtain

$$\begin{aligned} &E_\lambda(f_\lambda \circ g_t^{-1}) \\ &\quad = E_\lambda(f_\lambda) + t \int_\Pi (f_\lambda + h)[T_+ f_\lambda - \lambda\, x_1 x_2, u] + o(t), \end{aligned} \tag{4.15}$$

as $t \to 0^+$. Hence, if we set

$$\alpha(t) = E_\lambda(f_\lambda \circ g_t^{-1}),$$

for $t \in [-\tau, \tau]$, we infer from (4.15) that

$$\alpha'(0) = \int_\Pi (f_\lambda + h)[T_+ f_\lambda - \lambda\, x_1 x_2, u].$$

Moreover, since $f_\lambda \in \Sigma_\lambda$ and $f_\lambda \circ g_t^{-1} \in \mathcal{F}$, it follows that α has a global minimum at zero, whence $\alpha'(0) = 0$, from which (4.14) follows.

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